Practise!
(Solutions : see web...)

Fall 2000

Honors Calculus I First Midterm

Name: ______________________

1	2	3	4	TOT.

Answer ALL questions. Each question is worth TEN points. Show all your work and try to be as CLEAR as possible in explaining proofs – that way, I can give some credit even for wrong answers.

1. Describe the *domains* of the following functions (work in degrees when talking about angles for sin):

(a) $\sin\left(\frac{1}{x}\right)$. $\mathbb{R} - \{0\}$

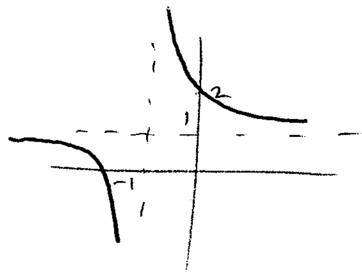
(b) $\sqrt{(1-x^2)}$. $[-1, 1]$

(c) $\sqrt{\left(\frac{1}{x+1}\right)}$. $(-1, \infty)$.

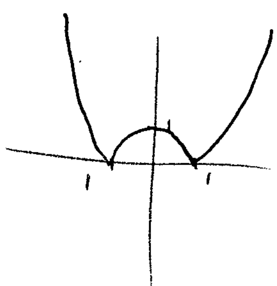
(d) $\sqrt{\sin x}$. All x such that $360k \leq x \leq 360k+180$ for some $k \in \mathbb{Z}$

2. Sketch the graphs of the following functions:

(a) $f(x) = \frac{1}{x+1} + 1$.

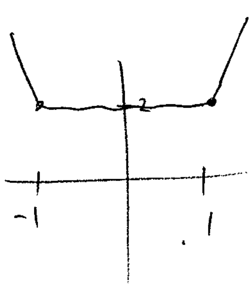


(b) $f(x) = |x^2 - 1|$.



$$\begin{cases} 2x & x \geq 1 \\ 2 & -1 \leq x \leq 1 \\ -2x & x \leq -1 \end{cases}$$

(c) $f(x) = |x-1| + |x+1|$.



Using your answer to (c), or otherwise, find all numbers x for which $|x - 1| + |x+1| = 2$.

$-1 \leq x \leq 1$

3. Use mathematical induction to prove that

$$1 + 2 + 4 + 8 + 16 + \cdots + 2^n = 2^{n+1} - 1$$

for all $n \geq 1$.

Base case $n=1$ LHS: $1+2=3$ RHS: $2^2-1=3$ ✓

Induction step Assume true for $n=k$, i.e. $1+2+\dots+2^k = 2^{k+1}-1$

Add 2^{k+1} to both sides:

$$1+2+\dots+2^k+2^{k+1} = 2^{k+1}-1+2^{k+1}$$

$$= 2 \cdot 2^{k+1} - 1$$

$$= 2^{k+2} - 1$$

✓ What we want with $n=k+1$

$\therefore$ Done by PMI

4. (a) If $x \le y$ and neither x or y are zero, then $\frac{1}{x} \ge \frac{1}{y}$. True or False? Explain your answer.

False! eg $x=-2$, $y=1$ $\quad -2 \leqslant 1$

$-\frac{1}{2} \not\geq 1$ #

(b) For all real numbers a, b, $|a+b| \ge |a-b|$. True or False? Explain your answer.

False! eg $a=1, b=-1$

$|a+b|=0$

$|a-b|=2$

(c) [harder!] By factorizing $(x^5 - y^5)$ as $(x-y)\times$(something), prove carefully that if $x \neq y$, then

$$x^4 + x^3y + x^2y^2 + xy^3 + y^4 > 0.$$

$$(x^5 - y^5) = (x-y)(x^4 + x^3y + x^2y^2 + xy^3 + y^4) \quad \text{———} (*)$$

If $x \neq y$ then $x<y$ or $x>y$

<u>Case one</u> $x<y$. Then $x^5<y^5$ too

$\therefore$ $x-y$ is negative and x^5-y^5 is negative

$\therefore$ Using $(*)$ we must have that $x^4+x^3y+x^2y^2+xy^3+y^4$ is positive

<u>Case two</u> $x>y$. Then $x^5>y^5$ too

$\therefore$ $x-y$ and x^5-y^5 are positive

$\therefore$ Using $(*)$ we must have that $x^4+x^3y+x^2y^2+xy^3+y^4$ is positive

$\therefore$ $x^4+x^3y+x^2y^2+xy^3+y^4 > 0$ //