Math 261: Midterm practise questions

1. (i) Say precisely what it means for the limit of $f(x)$ as x approaches a to be l (that is, $\lim_{x\to a} f(x) = l$).

(ii) Prove, using the definition you provided above, that if

$$f(x) = \begin{cases} -3x+6 & x \neq 3 \\ 0 & x = 3 \end{cases}$$

then

$$\lim_{x\to 3} f(x) = -3$$

(iii) Prove, using the definition you provided above, that if $f(x) = x^2$ then

$$\lim_{x\to 5} f(x) = 25.$$

Solution.

(i) For all $\epsilon > 0$, there exists $\delta > 0$ such that $0 < |x-a| < \delta$ implies $|f(x) - l| < \epsilon$.

(ii) Take $\epsilon > 0$. Set $\delta = \epsilon/3$. Then, let $0 < |x-3| < \delta$. In other words, $3 - \epsilon/3 < x < 3 + \epsilon/3$ and $x \neq 3$. Then, $-3(3+\epsilon/3)+6 < -3x+6 < -3(3-\epsilon/3)+6$ so $-3-\epsilon < -3x+6 < -3+\epsilon$. This says that $|f(x)-(-3)| < \epsilon$ which is what we wanted.

(iii) Take $\epsilon > 0$. Set $\delta = \min(\epsilon/11, 1)$. Then, if $0 < |x-5| < \delta$, we have that $f(x) - 25 = x^2 - 25 = (x-5)(x+5) = (x-5)(x-5+10)$. So,

$$|f(x) - 25| \leq |x-5|(|x-5|+10) < \delta(\delta+10) \leq \frac{\epsilon}{11}(1+10) = \epsilon.$$

This is what we needed.

2. (i) Define what it means to say the function $f(x)$ is continuous at a.

(ii) Prove (using ε and δ) that the function

$$f(x) = \begin{cases} x & x \in \mathbf{Q} \\ 0 & \text{else} \end{cases}$$

is continuous at 0.

Solution.

(i) It means $\lim_{x\to a} f(x) = f(a)$.

(ii) Take $\epsilon > 0$. Set $\delta = \epsilon$. Then, for $|x| < \delta$, we have that $|f(x)| = |x| < \delta$ for $x \in \mathbf{Q}$ or $|f(x)| = 0$ otherwise. Either way, $|f(x)| < \epsilon$. This proves that $\lim_{x\to 0} f(x) = 0$. But $f(0) = 0$ by definition. Hence, f is continuous at 0.

3. Answer the following true or false. If you can, give a very short justification for your answer.

(i) $f(x+y) = f(x) + f(y)$.

(ii) $[(g+h)\circ f](x) = g(f(x)) + h(f(x))$.

(iii) $|x+y| \le |x| + |y|$.

(iv) $|x-y| \le |x| - |y|$.

(v) $|x| - |y| \le |x-y|$.

(vi) If $f(a) = f(b)$ then $a = b$.

Solution.

(i) False. Consider $f(x) = x^2$. Then, $f(4) = 16 \neq f(2) + f(2)$!

(ii) True. By the definitions, $[(g+h)\circ f](x) = (g+h)(f(x)) = g(f(x)) + h(f(x))$.

(iii) True. We proved this in class.

(iv) False. Take $x = 2, y = 3$. Then LHS $= 1$, RHS $= -1$.

(v) True. Set $x' = x + y, y' = y$ in (iii), so $x = x' - y'$. Then, (iii) reads $|x'| \le |x' - y'| + |y'|$. Rearranging gives $|x'| - |y'| \le |x' - y'|$.

(vi) False. Take $f(x) = x^2$. Here, $f(1) = f(-1)$, but $1 \neq -1$.

4. Find the following limits.

(i)

$$\lim_{x\to\infty} \frac{2x^2 - \cos(x)}{x^2 + x + 1}$$

(ii)

$$\lim_{x\to a} \frac{x^3 - a^3}{x - a}.$$

(iii)

$$\lim_{x\to 4} \frac{\sqrt{x} - 2}{x - 4}$$

(iv)

$$\lim_{x\to\infty} (\sqrt{x^2 + 2x} - x).$$

Solution.

(i)

$$\begin{aligned}\lim_{x\to\infty} \frac{2x^2 - \cos(x)}{x^2 + x + 1} &= \lim_{x\to\infty} \frac{2x^2}{x^2 + x + 1} - \lim_{x\to\infty} \frac{\cos x}{x^2 + x + 1} \\ &= \lim_{x\to\infty} \frac{2(x^2 + x + 1) - 2x - 2}{x^2 + x + 1} - 0 = 2 - \lim_{x\to\infty} \frac{2x + 2}{x^2 + x + 1} = 2 - 0 = 2.\end{aligned}$$

(ii)

$$\lim_{x\to a} \frac{x^3 - a^3}{x - a} = \lim_{x\to a} \frac{(x-a)(x^2 + ax + a^2)}{x - a} = \lim_{x\to a} (x^2 + ax + a^2) = 3a^2.$$

(iii)

$$\lim_{x\to 4}\frac{\sqrt{x}-2}{x-4}=\lim_{x\to 4}\frac{\sqrt{x}-2}{(\sqrt{x}-2)(\sqrt{x}+2)}=\frac{1}{\lim_{x\to 4}\sqrt{x}+2}=\frac{1}{4}.$$

(iv)

$$\begin{aligned}\lim_{x\to\infty}(\sqrt{x^2+2x}-x) &= \lim_{x\to\infty}\frac{(\sqrt{x^2+2x}-x)(\sqrt{x^2+2x}+x)}{\sqrt{x^2+2x}+x}\\ &= \lim_{x\to\infty}\frac{x^2+2x-x^2}{\sqrt{x^2+2x}+x}=\lim_{x\to\infty}\frac{2}{(\sqrt{x^2+2x})/x+1}\\ &= \lim_{x\to\infty}\frac{2}{\sqrt{1+2/x}+1}=\frac{2}{1+1}=1.\end{aligned}$$

5. Find the following limits.

(i)

$$\lim_{x\to\infty}\frac{3x^2-x\sin(x)+x^2\sin(x)}{x^2+4x+3}$$

(ii)

$$\lim_{x\to a}\frac{\frac{1}{x}-\frac{1}{a}}{x-a}.$$

(iii)

$$\lim_{x\to 4}\frac{x^2-16}{x-4}$$

(iv)

$$\lim_{x\to\infty}(\sqrt{x^2+2x}-\sqrt{x^2+3x}).$$

Solution.

(i)

$$\begin{aligned}\lim_{x\to\infty}\frac{3x^2-x\sin(x)+x^2\sin(x)}{x^2+4x+3} &= \lim_{x\to\infty}\frac{(x^2+4x+3)(3+\sin x)-(4x+3)(3+\sin x)-x\sin x}{x^2+4x+3}\\ &= \lim_{x\to\infty}(3+\sin x)-\lim_{x\to\infty}\frac{(4x+3)(3+\sin x)-x\sin x}{x^2+4x+3}\\ &= \lim_{x\to\infty}(3+\sin x)-0\end{aligned}$$

which does not exist.

(ii)

$$\lim_{x\to a}\frac{\frac{1}{x}-\frac{1}{a}}{x-a}=\lim_{x\to a}\frac{\frac{a-x}{ax}}{x-a}=-\lim_{x\to a}\frac{1}{ax}=-\frac{1}{a^2}.$$

(iii)

$$\lim_{x\to 4}\frac{x^2-16}{x-4}=\lim_{x\to 4}\frac{(x-4)(x+4)}{x-4}\lim_{x\to 4}(x+4)=8.$$

(iv)

$$\begin{aligned}\lim_{x\to\infty}(\sqrt{x^2+2x}-\sqrt{x^2+3x}) &= \lim_{x\to\infty}\frac{-x}{\sqrt{x^2+2x}+\sqrt{x^2+3x}}\\ &= -\lim_{x\to\infty}\frac{1}{\sqrt{1+2/x}+\sqrt{1+3/x}}=-\frac{1}{2}.\end{aligned}$$

6.

Suppose that $f(x)>7$ for all x and that $\lim_{x\to 11}f(x)=\ell$. Prove that $\ell\geq 7$. Is it possible that $\ell=7$?

Solution.

Suppose for a contradiction that $\ell<7$. Let $\epsilon=7-\ell>0$. By the definition of limit, there exists $\delta>0$ such that $0<|x-11|<\delta$ implies $|f(x)-\ell|<\epsilon$. But that means that for such an x, $f(x)<\ell+\epsilon=7$ which contradicts the assumption that $f(x)>7$ for all x.

For the second part its possible. For example you could have

$$f(x)=\begin{cases}7+|x-11| & x\neq 11,\\ 8 & x=11.\end{cases}$$

7. Find a pair of successive integers so that $4x^3-3x^4+1$ has a zero between them. State the theorem that you are using.

Solution.

We use the Intermediate Value Theorem, which says that if f is continuous on $[a,b]$ and $f(a)>0$, $f(b)<0$ then there is an $x\in[a,b]$ so that $f(x)=0$.

In our case $f(1)=2>0$ and $f(2)=-15<0$. So there is a 0 between 1 and 2.

8. Give an example of a function that is continuous on (a,b), and bounded above on (a,b) but so that it does not have a maximum value on (a,b). Give the supremum of the values of the function on (a,b).

Solution.

One example is

$$(0,1)=(a,b),f(x)=x.$$

Then the supremum of the values is 1, but $f(x)$ is never 1 on $(0,1)$.