Math 261: Homework 7 solutions

Chapter 7

5. If f is continuous on $[a, b]$ and $f(x)$ is always rational, then f is constant.

Proof. Suppose for a contradiction that f is not constant. Then, we can find $x < y \in [a, b]$ such that $f(x) \neq f(y)$. Choose an irrational number m lying between $f(x)$ and $f(y)$. Then, by the intermediate value theorem, there exists $z \in [x, y]$ with $f(z) = m$. Hence, f takes an irrational value, contradicting the hypotheses.

6. We know $f(0)$ is either 1 or -1. Suppose that $f(0) = 1$ (the other case $f(0) = -1$ goes in a similar way). I'll show that $f(x) = \sqrt{1-x^2}$ for all $x \in [-1, 1]$. Well suppose not. Then for some $x \in [-1, 1]$, we have that $f(x) = -\sqrt{1-x^2} \neq \sqrt{1-x^2}$, hence $x \in (-1, 1)$ and $f(x) < 0$. Since $f(0) = 1$, the IVT implies there's some y in between x and 0, i.e. in $(-1, 1)$ too, such that $f(y) = 0$. But $f(y) = \pm\sqrt{1-y^2} \neq 0$, so this is a contradiction.

8. Suppose that $f(0) > 0$ (the other case $f(0) < 0$ is similar). I'll show first that $f(x) > 0$ for all x. Well, if not then there's some x with $f(x) < 0$, so by IVT since f is continuous, there's some x with $f(x) = 0$, a contradiction.

Since $g(0) = \pm f(0)$, we then either have that $g(0) > 0$ or $g(0) < 0$. In the former case, the argument in the previous paragraph shows that $g(x) > 0$ for all x. Hence since $g(x) = \pm f(x)$ for all x we actually have that $g(x) = f(x)$ for all x (else some $g(x)$ would be negative).

The case that $g(0) < 0$ is similar: you get that $g(x) < 0$ for all x hence that $g(x) = -f(x)$ for all x.

12. (a) We need to show that $f(x) = 1-x$ for some $x \in [0, 1]$ (since $y = 1-x$ is the equation of the dashed line). Certainly, $f(0) \leq 1$ and $f(1) \geq 0$ by the assumptions. Now set $g(x) = f(x) - 1 + x$. Then, $g(0) \leq 1 - 1 + 0 = 0$ and $g(1) \geq 0-1+1 = 0$. So, $g(0) \leq 0 \leq g(1)$. So by the intermediate value theorem, there exists $x \in [0, 1]$ with $g(x) = 0$. But then $f(x) - 1 + x = 0$ so $f(x) = 1 - x$ and we're done.

(b) Let $h(x) = f(x) - g(x)$.

Case one. $g(0) = 0, g(1) = 1$. Then, $h(0) = f(0) \geq 0$ and $h(1) = f(1) - 1 \leq 0$. So $h(0) \geq 0 \geq h(1)$, so by the intermediate value theorem, there exists $x \in [0, 1]$ such that $h(x) = 0$. But then, $f(x) - g(x) = 0$ so $f(x) = g(x)$ and we're done.

Case two. $g(0) = 1, g(1) = 0$. Then, $h(0) = f(0) - 1 \leq 0$ and $h(1) = f(1) \geq 0$. So $h(0) \leq 0 \leq h(1)$, so by the intermediate value theorem, there exists $x \in [0, 1]$ such that $h(x) = 0$. But then, $f(x) - g(x) = 0$ so $f(x) = g(x)$ and we're done.

Chapter 8

1.

(ii) 1 is the greatest, -1 is the least.

(iv) 0 is the least element, and the least upper bound is $\sqrt{2}$ which is not in the set.

(vi) Since $\{x \mid x^2 + x + 1 < 0\} = ((-1-\sqrt{5})/2, (-1+\sqrt{5})/2])$, the greatest lower bound is $(-1-\sqrt{5})/2$ and the least upper bound is $(-1+\sqrt{5})/2$; neither is in the set.

(viii) $1 - 1/2$ is the greatest element, and the greatest lower bound is -1 which is not in the set.

6. (a) Suppose not, say $f(a) \neq 0$ for some a. Let $\epsilon = |f(a)|$. By the definition of continuity, there exists $\delta > 0$ such that $|x - a| < \delta$ implies $|f(x) - f(a)| < |f(a)|$. By the definition of density there exists such an x belonging to the set A, i.e. with $f(x) = 0$. But then, $|f(a)| = |f(x) - f(a)| < |f(a)|$ which is a contradiction.

(b) Apply (a) to the continuous function $h(x) = f(x) - g(x)$.

(c) Like in (b) you can reduce to the case that $g(x) = 0$, i.e. you know $f(x) \geq 0$ for all $x \in A$ and want to prove that $f(x) \geq 0$ for all $x \in \mathbb{R}$. Suppose not, say $f(a) < 0$ for some a. Arguing as in (a) with $\epsilon = -f(a)$, you then get an $x \in A$ such that $|f(x) - f(a)| < \epsilon$. But then $0 \leq f(x) < f(a) + \epsilon = 0$ which is the contradiction.

The answer to the final statement is "NO", e.g. take $A = \mathbb{R} \setminus \{0\}$ and $f(x) = x^2$.

7. Let $c = f(1)$. We'll show $f(x) = cx$ in steps.

(1) $f(x) = cx$ for all $x \in \mathbb{N}$. Well, $f(n) = f(1 + \cdots + 1) = f(1) + \cdots + f(1) = nc$.

(2) $f(0) = 0$. Well, $f(0) + f(0) = f(0 + 0) = f(0)$, so subtracting $f(0)$ from both sides gives $f(0) = 0$.

(3) $f(x) = cx$ for all $x \in \mathbb{N}$. Well, for $n \in \mathbb{N}$, $f(n - n) = f(n) + f(-n) = f(0) = 0$. Hence by (1), $cn + f(-n) = 0$, so $f(-n) = -cn$ which is all that was left to prove after (1) and (2).

(4) $f(x) = cx$ for all $x \in \mathbb{Q}$. Well, say $x = m/n$ for $m \in \mathbb{Z}, n \in \mathbb{N}$. Then, $f(nx) = nf(x) = f(m) = cm$. So $f(x) = cm/n$.

(5) $f(x) = cx$ for all $x \in \mathbb{R}$. Since $\mathbb{Q}$ is dense and the functions $f(x)$ and cx are continuous, this follows from (4) and 6(b).

14. (a) Let $A = \{a_n \mid n \in \mathbb{N}\}$ and $B = \{b_n \mid n \in \mathbb{N}\}$. Both are non-empty sets of real numbers, and A is bded above (by b_1 say) while B is bded below (by a_1 say). Hence we can set $a = supA, b = infB$, which both exist by the least upper bound axiom for real numbers.

Claim. $a \leq b$.

Proof. Suppose for a contradiction that $a > b$ and let $\epsilon = a - b > 0$. Since a is the least upper bound of A, there exists some $a_n \in A$ with $a - \epsilon/2 < a_n < a$ (else $a - \epsilon/2$ would be a smaller upper bound for A). Similarly, there exists some $b_m \in B$ with $b < b_m < b + \epsilon/2$. Setting $t = \max(m, n)$, we then have that $a - \epsilon/2 < a_t \leq b_t < b + \epsilon/2$. Hence, $a - \epsilon/2 < b + \epsilon/2$ hence $a - b < \epsilon$. But this is a contradiction since $a - b = \epsilon$. The claim is proved.

So the interval $[a, b]$ is non-empty. Take any $x \in [a, b]$. Then, $x \geq a_n$ for all n, since a is an upper bound for A, and $x \leq b_n$ for all n since b is a lower bound for B. Hence, $x \in [a_n, b_n]$ for all n, i.e. x lies in every I_n.

(b) Try $I_n = (0, 1/n)$. Suppose x lies in all the intervals I_n. Then, $0 < x < 1/n$ for all n. So $x \neq 0$ so we can choose n sufficiently large so that $nx > 1$, i.e. $x > 1/n$, a contradiction.