Ch. 5

1. (i) $\lim_{x\to 1} \frac{x^2-1}{x+1} = \lim_{x\to 1}(x-1) = 0$.

(ii) $\lim_{x\to 2} \frac{x^3-8}{x-2} = \lim_{x\to 2} \frac{(x-2)(x^2+2x+4)}{x-2} = \lim_{x\to 2}(x^2+2x+4) = 12$.

(iii) $\lim_{x\to 3} \frac{x^3-8}{x-2} = \frac{19}{1} = 19$.

(iv) $\lim_{x\to y} \frac{x^n-y^n}{x-y} = \lim_{x\to y}(x^{n-1}+x^{n-2}y+\cdots+x^2y^{n-3}+xy^{n-2}+y^{n-1}) = ny^{n-1}$.

(v) $\lim_{y\to x} \frac{x^n-y^n}{x-y} = nx^{n-1}$.

(vi) $\lim_{h\to 0} \frac{\sqrt{a+h}-\sqrt{a}}{h} = \lim_{h\to 0} \frac{(\sqrt{a+h}-\sqrt{a})(\sqrt{a+h}+\sqrt{a})}{h(\sqrt{a+h}+\sqrt{a})} = \lim_{h\to 0} \frac{a+h-h}{h(\sqrt{a+h}+\sqrt{a})}$
$= \lim_{h\to 0} \frac{1}{\sqrt{a+h}+\sqrt{a}} = 1/2\sqrt{a}$.

2. (i) $\lim_{x\to 1} \frac{1-\sqrt{x}}{1-x} = \lim_{x\to 1} \frac{1-\sqrt{x}}{(1-\sqrt{x})(1+\sqrt{x})} = \lim_{x\to 1} \frac{1}{1+\sqrt{x}} = 1/2$.

(ii) $\lim_{x\to 0} \frac{1-\sqrt{1-x^2}}{x} = \lim_{x\to 0} \frac{(1-\sqrt{1-x^2})(1+\sqrt{1-x^2})}{x(1+\sqrt{1-x^2})} = \lim_{x\to 0} \frac{1-(1-x^2)}{x(1+\sqrt{1-x^2})}$
$= \lim_{x\to 0} \frac{x}{1+\sqrt{1-x^2}} = 0/2 = 0$.

(iii) Similar to (ii), except you get $= \lim_{x\to 0} \frac{1}{1+\sqrt{1-x^2}} = 1/2$ at the end.

3. See solutions in back of the book.

4. (i) All x that are not integers.

(ii) All x that are not integers.

(iii) All x that are not integers.

(v) All x different from 0 and different from $1/n$ for any integer n.

9. Suppose $\lim_{x\to a} f(x) = \ell$. Prove that $\lim_{h\to 0} f(a+h) = \ell$. To do this, take $\epsilon > 0$. Then, there exists $\delta > 0$ such that $0 < |x-a| < \delta$ implies $|f(x)-\ell| < \epsilon$. Now, if $0 < |h| < \delta$, $x = a+h$ satisfies $0 < |x-a| < \delta$, so $|f(x)-\ell| < \epsilon$, so $|f(a+h)-\ell| < \epsilon$. Believe it or not, this verifies the definition of what $\lim_{h\to 0} f(a+h) = \ell$ means!

12. (a) Let $\ell = \lim_{x\to a} f(x)$, $m = \lim_{x\to a} g(x)$. Suppose for a contradiction that $\ell > m$. Set $\epsilon = (\ell - m)/2 > 0$. Then, there exists $\delta_1 > 0$ such that $0 < |x-a| < \delta_1$ implies $|f(x)-\ell| < \epsilon$ and there exists $\delta_2 > 0$ such that $0 < |x-a| < \delta_2$ implies $|g(x)-\ell| < \epsilon$. Now take any x with $0 < |x-a| < \min(\delta_1,\delta_2)$. Then, we have in particular that $f(x) > \ell - \epsilon$ and $g(x) < m+\epsilon$. By assumption, $f(x) \leq g(x)$. Hence,

$$\ell - \epsilon < f(x) \leq g(x) < m + \epsilon.$$

Rearranging gives $\ell - m < 2\epsilon$ in other words, by the choice of ϵ,

$$\ell - m < \ell - m$$

which is a contradiction!

(b) Weaken the hypotheses to $f(x) < g(x)$ for all x. It is of course still true that $\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x)$.

(c) But it is FALSE that $\lim_{x\to a} f(x) < \lim_{x\to a} g(x)$. For instance, let $f(x) = 1 - x$, $g(x) = 1 + x$ on the interval $[0, 1]$. Then, $f(x) < 1 < g(x)$ for all $x \in [0, 1]$. But $\lim_{x\to 0} f(x) = 1 = \lim_{x\to 0} g(x)$.

13. Let $\ell = \lim_{x\to a} f(x) = \lim_{x\to a} h(x)$. Then, for all $\epsilon > 0$ there exists $\delta_1 > 0$ such that $0 < |x - a| < \delta_1$ implies $|f(x) - \ell| < \epsilon$, and there exists $\delta_2 > 0$ such that $0 < |x - a| < \delta_2$ imlies $|g(x) - \ell| < \epsilon$. Set $\delta = \min(\delta_1, \delta_2)$. Then, $0 < |x - a| < \delta$ implies both that $\ell - \epsilon < f(x)$ and that $h(x) < \ell + \epsilon$. But then since $f(x) \leq g(x) \leq h(x)$ we get both that $\ell - \epsilon < g(x) < \ell + \epsilon$, i.e. $|g(x) - \ell| < \epsilon$. This proves that $\lim_{x\to a} g(x) = \ell$.

(PS. I call this the policeman lemma 'coz there's this guy stuck in between the policeman and where the policeman go...)

15.(i) $\lim_{x\to 0} \frac{\sin 2x}{x} = \lim_{2x\to 0} \frac{\sin 2x}{x} = \lim_{y\to 0} \frac{\sin y}{y/2} = 2 \lim_{y\to 0} \frac{\sin y}{y} = 2\alpha$.

(ii) $\lim_{x\to 0} \frac{\sin ax}{\sin bx} = (\lim_{x\to 0} \frac{\sin ax}{x})/(\lim_{x\to 0} \frac{\sin bx}{x}) = a\alpha/b\alpha = a/b$ (by a slight variation of (i)).

(iii) $\lim_{x\to 0} \frac{\sin^2 2x}{x} = \lim_{x\to 0} \frac{\sin 2x}{x} \lim_{x\to 0}(\sin 2x) = 2\alpha.0 = 0$.

24. Given $\epsilon > 0$, pick n with $1/n < \epsilon$. Let δ be the minimum distance from a to all points in $A_1, \ldots, A_n$ (except a itself if a is one of these points). Then $0 < |x - a| < \delta$ implies that x is not in $A_1, \ldots, A_n$, so $f(x) = 0$ or $1/m$ for $m > n$, so $|f(x)| < \epsilon$.

37. (a) Take $N > 1$. Need to choose $\delta > 0$ so that $0 < |x-3| < \delta$ implies $1/(x - 3)^2 > N$. Try $\delta = \sqrt{1/N}$. Then, if $0 < |x - 3| < \delta$, we have that $|x - 3| < \sqrt{1/N}$ so $(x - 3)^2 < 1/N$ so $1/(x - 3)^2 > N$. We're done.

(b) Take $N > 1$. As $\lim_{x\to a} = 0$, we can find $\delta > 0$ such that $0 < |x - a| < \delta$ implies $|g(x)| < \epsilon/N$. But then $f(x)/|g(x)| > \epsilon/(\epsilon/N) = N$. As required.

39. (i) ∞.

(ii) $x(1 + \sin^2 x) \geq 2x$ so $\lim_{x\to\infty} x(1 + \sin^2 x) = \infty$.

(iii) Does not exist (keeps fluctuating between big positive and zero as x gets bigger).

(iv) You need to know that $\lim_{x\to\infty} x \sin \frac{1}{x}$ exists and is positive to do this one: set $y = \frac{1}{x}$ then as $x \to \infty$ you have that $y \to 0^+$. So its the same as $\lim_{y\to 0^+} \frac{\sin y}{y} = 1$ (from class). Then given that, to do the problem you see that

$$\lim_{x\to\infty} x^2 \sin \frac{1}{x} = (\lim_{x\to\infty} x \sin \frac{1}{x})(\lim_{x\to\infty} x) = \infty.$$

(v) This is 1. Proof.

$$\begin{aligned}
\lim_{x\to\infty} \sqrt{x^2+2x} - x &= \lim_{x\to\infty} \frac{(\sqrt{x^2+2x}-x)(\sqrt{x^2+2x}+x)}{(\sqrt{x^2+2x}+x)} \\
&= \lim_{x\to\infty} \frac{2x}{\sqrt{x^2+2x}+x} \\
&= \lim_{x\to\infty} \frac{2}{\sqrt{1+2/x}+1} = \frac{2}{1+1} = 1.
\end{aligned}$$

(vi) ∞

(vii) 0.