Math 261: Homework 3 solutions

Ch. 3

5.

(i) $P \circ s$. (ii) $s \circ P$. (iii) $s \circ S$. (iv) $S \circ s$. (v) $P \circ P$. (vi) $s \circ (P + P \circ S)$. (vii) $s \circ s \circ s \circ P \circ P \circ P \circ s$. (viii) $P \circ S \circ s + s \circ S + S \circ s \circ (S + s)$.

6. (a) $f_i(x) = \prod_{1 \le j \le n, j \ne i} \frac{(x - x_j)}{(x_i - x_j)}$.

(b) $f = \sum_{i=1}^n a_i f_i$

8. If $f(f(x)) = x$ then the *domains* of $f(f(x))$ and of x must certainly be the same. If $c \neq 0$ then $x = -d/c$ is not in the domain of $f(x)$, hence its not in the domain of $f(f(x))$ either. So if $c \neq 0$ there's no way $f(f(x)) = x$ for all x. This shows that $c = 0$. Hence $f(x) = ax/d + b/d$ (and $d \neq 0$ for sure or it wouldn't make sense). Now let's expand the equation $f(f(x)) = x$:

$$a(ax/d + b/d)/d + b/d = x$$

Hence

$$a^2x/d^2 + (a/d + 1)b/d = x$$

Hence

$$a^2x + (a + d)b = d^2x.$$

Hence $(a^2 - d^2)x + (a+d)b = 0$. If this is to be true for all x, the coefficients $a^2 - d^2$ and $(a + d)b$ must both be zero. Hence $a^2 = d^2$ and either $b = 0$ or $a + d = 0$. Hence either $b = c = 0$ and $a = \pm d \neq 0$ OR $c = 0, a = -d \neq 0$.

Here's my final answer: either $a = d \neq 0, b = c = 0$ or $a = -d \neq 0, c = 0$.

13. (a) Let $E(x) = \frac{1}{2}(f(x) + f(-x))$ and $O(x) = \frac{1}{2}(f(x) - f(-x))$. Then $E(x) = E(-x)$ and $O(x) = -O(-x)$. So E is even and O is odd, and:

$$f(x) = E(x) + O(x).$$

(b) Suppose $f(x) = E(x) + O(x)$ where E is even and O is odd. Then,

$$f(x) = E(x) + O(x),$$

and

$$f(-x) = E(-x) + O(-x) = E(x) - O(x)$$

using that E is even and O is odd. Adding gives

$$2E(x) = f(x) + f(-x),$$

so $E(x) = \frac{1}{2}(f(x) + f(-x))$. Similarly, subtracting the equations gives the formula for $O(x)$ in (i).

17. Just follow the steps! We're assuming $f(x+y) = f(x) + f(y)$ and $f(xy) = f(x)f(y)$, and f is not always zero.

(a) Since f is not always zero we can find x with $f(x) \neq 0$. Then, $f(x) = f(x.1) = f(x)f(1)$. Since $f(x) \neq 0$ we can cancel to get $f(1) = 1$. A similar argument using addition shows $f(0) = 0$.

(b) Consider $f(a) = f(\frac{a}{b}.b) = f(\frac{a}{b})f(b)$. So if I could show that $f(a) = a$ for a an integer, this would give $f(a/b) = a/b$, i.e. $f(x) = x$ for any rational x.

Well, $f(1) = 1$ by (a), so $f(2) = f(1) + f(1) = 1 + 1 = 2$. And so on, get $f(n) = n$ for n positive. Now $0 = f(0) = f(n-n) = f(n) + f(-n)$. So $f(-n) = -n$ so $f(n) = n$ for all $n \in \mathbb{Z}$.

(c) Suppose $x > 0$. Write $x = y^2$ for some y, i.e. $y = \sqrt{x}$ (don't worry that we don't know this is possible strictly!). Then, $f(x) = f(y^2) = f(y)f(y) = f(y)^2 \geq 0$. But it can't be zero: if $f(y) = 0$ then we'd get $1 = f(1) = f(y.\frac{1}{y}) = f(y)f(1/y) = 0$, a contradiction. So indeed $f(x) > 0$.

(d) Now if $x > y$ then $f(x) - f(y) = f(x-y) > 0$ by (c), so $f(x) > f(y)$.

(e) At last take any x and suppose for a contradiction that $f(x) \neq x$. Say $x < f(x)$. Pick a rational number y lying between x and $f(x)$. So $f(y) = y$ by (b). But f preserves inequalities by (d), so $x < y$ implies $f(x) < f(y) = y$, while $y < f(x)$ by choice of y, which is a contradiction. So must have that $f(x) = x$ FOR ALL x.

Ch. 4

1(iii) $(a - \epsilon, a + \epsilon)$.

(iv) $(-\sqrt(3/2), -\sqrt(1/2)) \cup (\sqrt(1/2), \sqrt(3/2))$.

(v) $[-2, 2]$.

4(i) a diamond passing through $(1,0), (0,1), (-1,0), (0,-1)$.

(ii) Draw the line $y = x - 1$ but only in the northeast quadrant. Now reflect in x and y-axes so you get something in all four quadrants.

(iii) A cross passing through $(1,1)$ and going in directions NE, NW, SW, SE.

(iv) Same as (iii).

(v) The origin only!

(vi) Either $x = 0$ or $y = 0$ so this is the x and the y axis in a cross.

(vii) You can write this as

$$(x-1)^2 + y^2 = 5$$

when you complete the square. So its a circle origin $(1,0)$ radius $\sqrt{5}$.

(viii) Either $x = y$ or $x = -y$. So its these two diagonal lines forming a cross.

8(a). We may as well move f down by b and g down by c, since such translation will not change the angle between the lines. So we just need to consider $f(x) = mx$ and $g(x) = nx$. Now take the triangle as in the hint, passing through 0, $(1, m)$ and $(1, n)$. The squares of the lengths of the sides are $(1 + m^2), (1 + n^2)$ and $(m - n)^2$. So according to pythagoras, for the triangle to be right angled, we need that $1 + m^2 + 1 + n^2 = (m - n)^2 = m^2 - 2mn + n^2$. So we need that $mn = -1$.

10(i) You know what $f(x) = x$ looks like and $f(x) = 1/x$. To add them together, for large x it'll look essentially like $f(x) = x$. For small x, it'll look essentially like $f(x) = 1/x$. Then in the middle, say $0.75 \leq x \leq 5$, it'll gradually curve from one shape to the other.

(iii) This is similar, except of course its symmetric in the y-axis this time. The gradual curve from one shape to the other will occur much more quickly this time, too...

14(i) Moved up by c.

(ii) Moved left by c.

(iii) y-axis scaled by c (if $c < 0$ the graph gets turned upsidedown).

(iv) x-axis scaled by c (if $c < 0$ the graph gets flipped in the y-axis).