Math 261: Homework 1 solutions

The important thing in writing down solutions is to try to write SENTENCES that make logical sense to someone else reading it. So pretend someone (who knows mathematical language and who can follow logical argument) has just read through the question you are trying to answer. What you write should make readable sense to them! Here is my attempt (you may disagree that it makes readable sense...)

Part II.

1. Suppose $a + b = 0$. Add $(-a)$ to both sides using (P2): get $(-a) + a + b = (-a) + 0 = -a$. Use (P3) to get $0 + b = -a$. Hence by (P2), $b = -a$. We're done.

2. Suppose b is another number which when multiplied by a gives 1. So $ab = 1$. Multiply both sides by a^{-1}, get $a^{-1}ab = a^{-1}$. Hence $b = a^{-1}$, i.e. a^{-1} is unique.

3. By question 1, $-a$ is the unique number which when added to a gives 0. So to show that $a.(-1) = -a$, we have to show that $a.(-1)$ gives zero when added to a too. But $a.(-1) + a = a.(-1) + a.1 = a.(-1 + 1)$ by (P9). This is $a.0$ by (P3). Which is zero by what we proved in class. We're done.

4. We're going to use question 2. First note that a/b is not zero: for if $a/b = 0$ then we'd get $a = a/b.a = 0.a = 0$, and we're told in the question that $a \neq 0$.

So now by question 2, $(a/b)^{-1}$ is the unique number which when multplied by a/b gives one. So to prove that $(a/b)^{-1} = (b/a)$, we need to check that

$$(b/a).(a/b) = 1$$

which is easy so I will skip it. Hence $(b/a) = (a/b)^{-1}$ by the uniqueness.

5. Using 4,

$$\frac{a/b}{c/d} = (a/b).(c/d)^{-1} = (a/b)(d/c) = ab^{-1}dc^{-1} = adb^{-1}c^{-1} = ad/bc.$$

Strictly speaking at the end I've used that $b^{-1}c^{-1} = (bc)^{-1}$. You prove this using the uniqueness from question 2.

6. To prove $ac < bc$, we need to show that $bc - ac \in P$. We are told $a > b$, which means that $a - b \in P$, and that $c < 0$, which means that $-c \in P$. So by (P12), $(a - b)(-c) \in P$. This is $-ac + bc = bc - ac$. That's what we wanted.

Part III. Questions from Chapter 1.

1. (iii) If $x^2 = y^2$ then $x^2 - y^2 = 0$ by (P3). Now, $(x-y)(x+y) = x^2 - y^2$ by (P9) and (P4,8). So we have shown that $(x-y)(x+y) = 0$. Now in class, we showed that if $ab = 0$ then either $a = 0$ or $b = 0$. Hence, either $x - y = 0$ or $x + y = 0$. Using (P3), we finally have that either $x = y$ or $x = -y$.

(iv) The right hand side equals

$$\begin{aligned}(x-y)(x^2+xy+y^2) &= (x-y)x^2 + (x-y)xy + (x-y)y^2 \\ &= x^3 - yx^2 + x^2y - yxy + xy^2 - y^3 \\ &= x^3 - y^3\end{aligned}$$

using (P9), commutativity and (P3) repeatedly.

4 (i) $4 - x < 3 - 2x$. Add $2x$ to both sides and use (P1),(P3) to see that $4 + x < 3$. Subtract 4 from both sides to see that $x < -1$.

(ii) $5 - x^2 < 8$. Add $x^2 - 5$ to both sides. Get $0 < x^2 + 3$. But $x^2 \geq 0$ whatever x. So this is always true. Answer: all x.

(iii) $5 - x^2 < -2$. Add $x^2 - 5$ to both sides. Get $0 < x^2 - 7$. So,

$$0 < (x - \sqrt{7})(x + \sqrt{7}).$$

Now if a product of two things is positive, they are either both positive or both negative. In the first case, $x > \sqrt{7}$ and $x > -\sqrt{7}$. So $x > \sqrt{7}$ here. In the second case, $x < \sqrt{7}$ and $x < -\sqrt{7}$. So $x < -\sqrt{7}$ here. Final answer: either $x > \sqrt{7}$ or $x < -\sqrt{7}$.

You might think this answer is too laborious. It is, I'm just showing how to use the axioms formally. If you see right away that $x^2 > 7$ means either $x > \sqrt{7}$ or $x < -\sqrt{7}$ without writing anything else, that is fine.

(iv) The product of two numbers is positive if and only if both are positive or both are negative. So $(x-1)(x-3) > 0$ if and only if *either* $x - 1 > 0$ and $x - 3 > 0$ *or* $x - 1 < 0$ and $x - 3 < 0$. So in the first case, $x > 1$ and $x > 3$, hence simply $x > 3$. In the second case, $x < 1, x < 3$ hence simply $x < 1$. So the answer is: either $x > 3$ or $x < 1$.

5(vi). Suppose $a > 1$. Then, $(a-1) \in P$. Since $1 \in P$, we have by (P11) that $a = (a-1) + 1$ is also in P. So by (P12) $a(a-1) \in P$. Thus, $a^2 - a \in P$, i.e. $a^2 > a$.

7. There are really three inequalities to be proved here.

(i) $a < \sqrt{ab}$. We know $a < b$ and $a \in P$. So I can multiply both sides by a to get $a^2 < ab$. Now take square roots to get $a < \sqrt{ab}$.

Yuk! What is taking square roots? Why does $0 < x < y$ imply $\sqrt{x} < \sqrt{y}$? To prove the latter thing, we know that either $\sqrt{x} < \sqrt{y}, \sqrt{x} = \sqrt{y}$ or

$\sqrt{x} > \sqrt{y}$ by trichotomy. Assuming $0 < x < y$, we need to rule out the last two possibilities. So suppose $\sqrt{x} = \sqrt{y}$; squaring both sides gives $x = y$ contrary to the hypothesis. Or if $\sqrt{x} > \sqrt{y}$, then multiplying both sides by $\sqrt{x}$, we get $x > \sqrt{xy}$. And multiplying both sides by $\sqrt{y}$, we get $\sqrt{xy} > y$. Hence, $x > y$, contrary to assumption.

(ii) $\frac{a+b}{2} < b$. To prove this, $a < b$ so $a + b < 2b$ so $\frac{a+b}{2} < b$.

(iii) Finally, we prove the middle inequality. Since $a < b$, $\sqrt{a} < \sqrt{b}$, so $\sqrt{b} - \sqrt{a} > 0$. So $(\sqrt{b} - \sqrt{a})^2 > 0$. So $(\sqrt{b} - \sqrt{a})^2 = b - 2\sqrt{ab} + a > 0$. So $a + b > 2\sqrt{ab}$. So $\frac{a+b}{2} > \sqrt{ab}$.

10. (i) and (iii) are in the back of the book. I'll do (ii) and (iv).

(ii) $|(|a + b|) - |a| - |b||$. Suppose both a, b are positive or zero. Then so is $a + b$. So it is $|a + b - a - b| = 0$.

Suppose a is positive or zero, b is negative and $a \geq -b$. Then $a + b$ is positive too. So its $|a + b - a + b| = 2|b|$.

Suppose a is positive or zero, b is negative and $a < -b$. Then $a + b$ is negative. So its $| - a - b - a + b = 2a$.

If a is negative and b is positive or zero that's more or less the same as the two cases just done.

Finally if a and b are both negative, so is $a + b$ and you get zero again.

(iv) Let's assume $a \geq 0$. Then its $a - |a - a| = a - 0 = a$. If $a < 0$ then $|a| = -a$ so its $a - |a - -a| = a - 2|a| = a + 2a = 3a$.

(If in doubt check for $a = 1$ and $a = -1$ for instance!)

11.

(i) $|x - 3| = 8$. Case one: $x \geq 3$. Then $|x - 3| = x - 3$ which equals 8 providing $x = 11$. Case two: $x < 3$. Then $|x - 3| = 3 - x$ which equals 9 providing $x = -5$. So either $x = 11$ or $x = -5$.

(ii) $|x - 3| < 8$. Case one: $x \geq 3$. Then, $|x - 3| = x - 3 < 8$ so $x < 11$. Case two: $x < 3$. Then, $|x - 3| = 3 - x < 8$ so $-5 < x$. So the answer is: $-5 < x < 11$.

(iii) $|x + 4| < 2$. Case one: $x \geq -4$. Then $|x + 4| = x + 4 < 2$ so $x < -2$. So in this case, we have that $-4 \leq x < -2$. Case two: $x < -4$. Then, $|x + 4| = -x - 4 < 2$ so $-6 < x$. So in this case we have that $-6 < x < -4$. Combining, we have shown overall that $-6 < x < -2$.

18.

(a) Take $x = \frac{-b+\sqrt{b^2-4c}}{2}$. Then,

$$\begin{aligned} x^2+bx+c &= \frac{(-b+\sqrt{b^2-4c})^2}{4} + \frac{-b^2+b\sqrt{b^2-4c}}{2} + c \\ &= \frac{b^2-2b\sqrt{b^2-4c}+b^2-4c}{4} + \frac{-2b^2+2b\sqrt{b^2-4c}}{4} + \frac{4c}{4} \\ &= \frac{b^2-2b\sqrt{b^2-4c}+b^2-4c-2b^2+2b\sqrt{b^2-4c}+4c}{4} = 0. \end{aligned}$$

The other one is similar.

(b) Suppose that $b^2-4c<0$, so $4c-b^2>0$. Then,

$$\begin{aligned} x^2+bx+c &= (x+\frac{b}{2})^2 + c - \frac{b^2}{4} \\ &= (x+\frac{b}{2})^2 + \frac{1}{4}(4c-b^2) > (x+\frac{b}{2})^2 \geq 0. \end{aligned}$$

This shows that $x^2+bx+c>0$ for all x.

20. Suppose $|x-x_0|<\frac{\epsilon}{2}$ and $|y-y_0|<\frac{\epsilon}{2}$. Then, by the theorem proved in class,

$$|(x+y)-(x_0+y_0)| = |(x-x_0)+(y-y_0)| \leq |x-x_0|+|y-y_0| < \frac{\epsilon}{2}+\frac{\epsilon}{2} = \epsilon.$$

For the second one,

$$|(x-y)-(x_0-y_0)| = |(x-x_0)-(y_0-y)| \leq |x-x_0|+|y_0-y| = |x-x_0|+|y-y_0| < \epsilon.$$