Wednesday, December 9, 1998

Honors Calculus I (Math 251, CRN 13791), Final Exam

Name:____________________________

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	TOT.

Show all your work! There are 15 problems at 10 points each.

(1) Define the derivative $f'(a)$. Calculate (from the definition) the derivative of $f(x) = 1/x$.

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}.$$

If $f(x) = 1/x$, then

$$f'(a) = \lim_{h\to 0} \frac{\frac{1}{a+h} - \frac{1}{a}}{h} = \lim_{h\to 0} \frac{\frac{a-(a+h)}{a(a+h)}}{h}$$
$$= \lim_{h\to 0} \frac{\frac{-h}{a(a+h)}}{h} = \lim_{h\to 0} \frac{-1}{a(a+h)} = -\frac{1}{a^2}.$$

(2) Let

$$f(x) = \begin{cases} x^2 \sin(1/x) & x \neq 0 \\ 0 & x = 0. \end{cases}$$

Find $f'(0)$.

We need

$$\lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{f(h)}{h} = \lim_{h\to} \frac{h^2 \sin(1/h)}{h} = \lim_{h\to 0} h \sin(1/h) = 0.$$

Thus $f'(0) = 0$.

(3) Calculate the derivatives of the functions below. You may use that the derivative of $\sin(x)$ is $\cos(x)$.

(a) $(x^2 + x)^{30}(x^3 - x)^{40}$

(b) $\sin(x^2 + \sin(x^2 + \sin(x)))$.

(c) $\frac{x^4+x^2}{\sin(x)}$

(d) $(x^2 + x^{-2})^3$.

(a) *Using the product rule and the chain rule, the derivative is*
$30(x^2 + x)^{29}(2x + 1)(x^3 - x)^{40} + (x^2 + x)^{30} 40(x^3 - x)^{39}(3x^2 - 1)$.

(b) *Using the chain rule repeatedly*
$\cos(x^2 + \sin(x^2 + \sin(x)))(2x + \cos(x^2 + \sin(x))(2x + \cos(x)))$.

(c)

$$\frac{(4x^3 + 2x)\sin(x) - (x^4 + x^2)\cos(x)}{\sin^2(x)}.$$

(d) $3(x^2 + x^{-2})^2(2x - 2x^{-3})$.

(4) Suppose $f : [0, 1] \to [0, 1]$ is a continuous function defined on the closed interval $[0, 1]$. Prove $f(x) = x$ for some $x \in [0, 1]$.

We consider the function $g(x) = f(x) - x$. We wish to show that $g(x) = 0$ for some $x \in [0, 1]$.

Note that if $f(0) = 0$ we're done. Otherwise, $g(0) > 0$. Note also that if $f(1) = 1$ we're done. Otherwise $g(1) < 0$. Then by one of the theorems of chapter 7, $g(x) = 0$ for some $x \in (0, 1)$.

(5) Prove that if $f'(a)$ exists, then f is continuous at a.

Our hypothesis implies that

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = f'(a)$$

exists. It follows that the limit

$$\lim_{x \to a} (f(x) - f(a)) = \lim_{x \to a} (x - a) \frac{f(x) - f(a)}{x - a}$$

exists, and is $0 \cdot f'(a) = 0$. *So* f *is continuous at* a.

(6) Use the **Chain rule** and the **Product rule** to prove the **Quotient rule**: [If $f'(a), g'(a)$ exists and $g(a) \neq 0$ then

$$(f/g)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{(g(a))^2}].$$

We let $h(x) = 1/x$. *Then* $h'(x) = \frac{-1}{x^2}$. *By the chain rule, the derivative of* $1/g(x) = h(g(x))$ *is* $(-1/(g(x))^2)g'(x)$.

We want the derivative of $f(x)/g(x) = f(x)h(x)$, *which is (by the product rule)*

$$f'(x)h(x) + f(x)\frac{-1}{(g(x))^2}g'(x) = \frac{f'(x)}{g(x)} + \frac{-f(x)g'(x)}{g^2(x)}$$
$$= \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}.$$

(7) Find a pair of successive integers so that $4x^3 - 3x^4 + 1$ has a zero between them. State the theorem that you are using.

We use the theorem that if f is continuous and $f(a) > 0$, $f(b) < 0$ then there is an $x \in [a, b]$ so that $f(x) = 0$.

In our case $f(1) = 2 > 0$ and $f(2) = -15 < 0$. So there is a 0 between 1 and 2.

(8) Prove by induction that

$$1 + r + r^2 + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r}.$$

As our base case, we check $n = 1$. The left hand side is $1 + r$. The right hand side is $\frac{1-r^2}{1-r} = 1 + r$. So our base case is true.

Suppose that the proposition is true for k, that is that

$$1 + r + r^2 + \cdots + r^k = \frac{1 - r^{k+1}}{1 - r}.$$

Then adding r^{k+1}, we get

$$1 + r + r^2 + \cdots + r^k + r^{k+1} = \frac{1 - r^{k+1}}{1 - r} + r^{k+1} =$$

$$\frac{1 - r^{k+1}}{1 - r} + \frac{r^{k+1}(1 - r)}{1 - r} = \frac{1 - r^{k+1}}{1 - r} + \frac{r^{k+1} - r^{k+2}}{1 - r} = \frac{1 - r^{k+2}}{1 - r}.$$

So our proposition is true for $k + 1$. Then by induction it is true for all n.

(9) Find the following limits. In case the limits are ∞ or $-\infty$, indicate.

(a)

$$\lim_{x\to 0} \frac{x^2 + x^3}{x}$$

Dividing by x, we get 0.

(b)

$$\lim_{x\to 0} \frac{x}{x^2 + x}$$

Dividign by x, we get 1.

(c)

$$\lim_{x\to\infty} \frac{x^2 + 3x^3}{5x^3 + x\sin(x) + 2}$$

Dividing by x^3, we get $3/5$.

(d)

$$\lim_{x\to\infty} \sqrt{x^2 + 9x} - \sqrt{x^2 + x}.$$

We use the usual trick of multiplying by 1 in the form $\frac{\sqrt{x^2+9x}+\sqrt{x^2+x}}{\sqrt{x^2+9x}+\sqrt{x^2+x}}$. Then we take the limit

$$\lim_{x\to\infty} \frac{(x^2 + 9x) - (x^2 + x)}{\sqrt{x^2 + 9x} + \sqrt{x^2 + x}} = \lim_{x\to\infty} \frac{8x}{\sqrt{x^2 + 9x} + \sqrt{x^2 + x}}$$

$$= \lim_{x\to\infty} \frac{8}{\sqrt{1 + 9/x} + \sqrt{1 + 1/x}} = 4.$$

(10) Find an example of two function f and g, neither of which is continuous on all of $\mathbf{R}$ but such that their composite $f \circ g$ is continuous on all of $\mathbf{R}$.

There are a number of ways to do this. Take

$$f(x) = \begin{cases} 1 & x < 0 \\ 2 & x \geq 0 \end{cases}$$

and $g(x)$ to be the same function. Then $f \circ g$ is the constant function 2. f and g are not continuous at 0, but the constant function $f \circ g$ is continuous everywhere.

(11) Give an example of a function continuous on all of $\mathbf{R}$ and differentiable at every point except at integers. A careful graph is sufficient. Give a graph of the derivative of the function you produced.

One such function is

$$f(x) = \left\{ \; |x-(2n+1)| \quad 2n \leq x \leq 2n+2. \right.$$

Then the derivative is given by

$$f'(x) = \begin{cases} 1 & 2n < x < 2n+1 \\ -1 & 2n+1 < x < 2n+2. \end{cases}$$

(12) Give an example of a function that is continuous on (a,b), and bounded above on (a,b) but so that it does not have a maximum value on (a,b). Give the supremum of the values of the function on (a,b).

One example is

$$(0,1) = (a,b), f(x) = x.$$

Then the supremum of the values is 1, *but* $f(x)$ *is never* 1 *on* $(0,1)$.

(13) Suppose that f and g are even functions. Prove that $f \cdot g$ is an even function. Suppose that f and g are odd functions. Prove that $f \cdot g$ is even.

In the first case, $f(-x)g(-x) = f(x)g(x)$. *In the second case,* $f(-x)g(-x) = (-1)f(x)(-1)g(x) = f(x)g(x)$.

(14) Give a direct proof, using ε and δ that $\lim_{x\to 4}\sqrt{x}=2$.

If $0<|x-4|<\delta$, *then we have* $4-\delta<x<4+\delta$. *If we restrict* $\delta\le 1$, *then we get*

$$4-4\delta+\delta^2<4-\delta<x<4+\delta<4+4\delta+\delta^2.$$

Taking square roots gives

$$2-\delta<\sqrt{x}<2+\delta \text{ so } |\sqrt{x}-2|<\delta.$$

So if $\varepsilon>0$, *and we take* $\delta=\min(\varepsilon,1)$ *then* $0<|x-4|<\delta$ *implies* $|\sqrt{x}-2|<\varepsilon$.

(15) Answer true or false for each of the below. Supply a short justification if possible.

(a) If $(f+g)'(a)$ exists, then $f'(a)$ and $g'(a)$ exist.
(b) If f is continuous at a then f is differentiable at a.
(c) If f is even and g is odd, then $f\cdot g$ is odd.
(d) If f is continuous and bounded above, then f has a maximum value.
(e) If a set A is bounded above, then it has a maximum element.
(f) If $f(x)$ is a polynomial, then $f(x)=0$ for some x.
(g) If $A\subseteq \mathbf{Q}$ has an upper bound, then $\sup(A)$ may not be in $\mathbf{Q}$.
(h) If $f(x)$ is an odd degree polynomial, then $f(x)=0$ for some x.

(a) *False. For example,* $a=0$, $f(x)=|x|$ *and* $g(x)=-|x|$.
(b) *False.* $a=0$, $f(x)=|x|$.
(c) *True.* $f(-x)g(-x)=(-x)f(x)g(x)$.
(d) *False. For example* $f(x)=1-1/x$ *and* $x\in[1,\infty)$.
(e) *False.* $(0,1)$.
(f) *False.* $1+x^2$.
(g) *False. For example* $\{x|x^2<2, x\in\mathbf{Q}\}$.
(h) *True. This was a theorem in chapter 7.*