Wednesday, December 9, 1998

Honors Calculus I (Math 251, CRN 13791), Final Exam

Name:______________________________

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	TOT.

Show all your work! There are 15 problems at 10 points each.

(1) Define the derivative $f'(a)$. Calculate (from the definition) the derivative of $f(x) = 1/x$.

(2) Let

$$f(x) = \begin{cases} x^2 \sin(1/x) & x \neq 0 \\ 0 & x = 0. \end{cases}$$

Find $f'(0)$.

(3) Calculate the derivatives of the functions below. You may use that the derivative of $\sin(x)$ is $\cos(x)$.

(a) $(x^2 + x)^{30}(x^3 - x)^{40}$

(b) $\sin(x^2 + \sin(x^2 + \sin(x)))$.

(c) $\frac{x^4+x^2}{\sin(x)}$

(d) $(x^2 + x^{-2})^3$.

(4) Suppose $f : [0, 1] \to [0, 1]$ is a continuous function defined on the closed interval $[0, 1]$. Prove $f(x) = x$ for some $x \in [0, 1]$.

(5) Prove that if $f'(a)$ exists, then f is continuous at a.

(6) Use the **Chain rule** and the **Product rule** to prove the **Quotient rule**: [If $f'(a), g'(a)$ exists and $g(a) \neq 0$ then

$$(f/g)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{(g(a))^2}].$$

(7) Find a pair of successive integers so that $4x^3 - 3x^4 + 1$ has a zero between them. State the theorem that you are using.

(8) Prove by induction that

$$1 + r + r^2 + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r}.$$

(9) Find the following limits. In case the limits are ∞ or $-\infty$, indicate.

(a)

$$\lim_{x \to 0} \frac{x^2 + x^3}{x}$$

(b)

$$\lim_{x \to 0} \frac{x}{x^2 + x}$$

(c)

$$\lim_{x \to \infty} \frac{x^2 + 3x^3}{5x^3 + x\sin(x) + 2}$$

(d)

$$\lim_{x \to \infty} \sqrt{x^2 + 9x} - \sqrt{x^2 + x}.$$

(10) Find an example of two function f and g, neither of which is continuous on all of $\mathbf{R}$ but such that their composite $f \circ g$ is continuous on all of $\mathbf{R}$.

(11) Give an example of a function continuous on all of $\mathbf{R}$ and differentiable at every point except at integers. A careful graph is sufficient. Give a graph of the derivative of the function you produced.

(12) Give an example of a function that is continuous on (a, b), and bounded above on (a, b) but so that it does not have a maximum value on (a, b). Give the supremum of the values of the function on (a, b).

(13) Suppose that f and g are even functions. Prove that $f \cdot g$ is an even function. Suppose that f and g are odd functions. Prove that $f \cdot g$ is even.

(14) Give a direct proof, using ε and δ that $\lim_{x\to 4}\sqrt{x} = 2$.

(15) Answer true or false for each of the below. Supply a short justification if possible.

(a) If $(f+g)'(a)$ exists, then $f'(a)$ and $g'(a)$ exist.
(b) If f is continuous at a then f is differentiable at a.
(c) If f is even and g is odd, then $f \cdot g$ is odd.
(d) If f is continuous and bounded above, then f has a maximum value.
(e) If a set A is bounded above, then it has a maximum element.
(f) If $f(x)$ is a polynomial, then $f(x) = 0$ for some x.
(g) If $A \subseteq \mathbf{Q}$ has an upper bound, then $\sup(A)$ may not be in $\mathbf{Q}$.
(h) If $f(x)$ is an odd degree polynomial, then $f(x) = 0$ for some x.