

Math 261: Midterm practise questions

1. (i) Say precisely what it means for the limit of $f(x)$ as x approaches a to be l (that is, $\lim_{x \rightarrow a} f(x) = l$).

(ii) Prove, using the definition you provided above, that if

$$f(x) = \begin{cases} -3x + 6 & x \neq 3 \\ 0 & x = 3 \end{cases}$$

then

$$\lim_{x \rightarrow 3} f(x) = -3$$

(iii) Prove, using the definition you provided above, that if $f(x) = x^2$ then

$$\lim_{x \rightarrow 5} f(x) = 25.$$

Solution.

(i) For all $\epsilon > 0$, there exists $\delta > 0$ such that $0 < |x - a| < \delta$ implies $|f(x) - l| < \epsilon$.

(ii) Take $\epsilon > 0$. Set $\delta = \epsilon/3$. Then, let $0 < |x - 3| < \delta$. In other words, $3 - \epsilon/3 < x < 3 + \epsilon/3$ and $x \neq 3$. Then, $-3(3 + \epsilon/3) + 6 < -3x + 6 < -3(3 - \epsilon/3) + 6$ so $-3 - \epsilon < -3x + 6 < -3 + \epsilon$. This says that $|f(x) - (-3)| < \epsilon$ which is what we wanted.

(iii) Take $\epsilon > 0$. Set $\delta = \min(\epsilon/11, 1)$. Then, if $0 < |x - 5| < \delta$, we have that $f(x) - 25 = x^2 - 25 = (x - 5)(x + 5) = (x - 5)(x - 5 + 10)$. So,

$$|f(x) - 25| \leq |x - 5|(|x - 5| + 10) < \delta(\delta + 10) \leq \frac{\epsilon}{11}(1 + 10) = \epsilon.$$

This is what we needed.

2. (i) Define what it means to say the function $f(x)$ is continuous at a .

(ii) Prove (using ϵ and δ) that the function

$$f(x) = \begin{cases} x & x \in \mathbf{Q} \\ 0 & \text{else} \end{cases}$$

is continuous at 0.

Solution.

(i) It means $\lim_{x \rightarrow a} f(x) = f(a)$.

(ii) Take $\epsilon > 0$. Set $\delta = \epsilon$. Then, for $|x| < \delta$, we have that $|f(x)| = |x| < \delta$ for $x \in \mathbf{Q}$ or $|f(x)| = 0$ otherwise. Either way, $|f(x)| < \epsilon$. This proves that $\lim_{x \rightarrow 0} f(x) = 0$. But $f(0) = 0$ by definition. Hence, f is continuous at 0.

3. Answer the following true or false. If you can, give a very short justification for your answer.

- (i) $f(x + y) = f(x) + f(y)$.
- (ii) $[(g + h) \circ f](x) = g(f(x)) + h(f(x))$.
- (iii) $|x + y| \leq |x| + |y|$.
- (iv) $|x - y| \leq |x| - |y|$.
- (v) $|x| - |y| \leq |x - y|$.
- (vi) If $f(a) = f(b)$ then $a = b$.

Solution.

- (i) False. Consider $f(x) = x^2$. Then, $f(4) = 16 \neq f(2) + f(2)$!
- (ii) True. By the definitions, $[(g + h) \circ f](x) = (g + h)(f(x)) = g(f(x)) + h(f(x))$.
- (iii) True. We proved this in class.
- (iv) False. Take $x = 2, y = 3$. Then LHS = 1, RHS = -1.
- (v) True. Set $x' = x + y, y' = y$ in (iii), so $x = x' - y'$. Then, (iii) reads $|x'| \leq |x' - y'| + |y'|$. Rearranging gives $|x'| - |y'| \leq |x' - y'|$.
- (vi) False. Take $f(x) = x^2$. Here, $f(1) = f(-1)$, but $1 \neq -1$.

4. Find the following limits.

(i)

$$\lim_{x \rightarrow \infty} \frac{2x^2 - \cos(x)}{x^2 + x + 1}$$

(ii)

$$\lim_{x \rightarrow a} \frac{x^3 - a^3}{x - a}$$

(iii)

$$\lim_{x \rightarrow 4} \frac{\sqrt{x} - 2}{x - 4}$$

(iv)

$$\lim_{x \rightarrow \infty} (\sqrt{x^2 + 2x} - x).$$

Solution.

(i)

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{2x^2 - \cos(x)}{x^2 + x + 1} &= \lim_{x \rightarrow \infty} \frac{2x^2}{x^2 + x + 1} - \lim_{x \rightarrow \infty} \frac{\cos x}{x^2 + x + 1} \\ &= \lim_{x \rightarrow \infty} \frac{2(x^2 + x + 1) - 2x - 2}{x^2 + x + 1} - 0 = 2 - \lim_{x \rightarrow \infty} \frac{2x + 2}{x^2 + x + 1} = 2 - 0 = 2. \end{aligned}$$

(ii)

$$\lim_{x \rightarrow a} \frac{x^3 - a^3}{x - a} = \lim_{x \rightarrow a} \frac{(x - a)(x^2 + ax + a^2)}{x - a} = \lim_{x \rightarrow a} (x^2 + ax + a^2) = 3a^2.$$

(iii)

$$\lim_{x \rightarrow 4} \frac{\sqrt{x} - 2}{x - 4} = \lim_{x \rightarrow 4} \frac{\sqrt{x} - 2}{(\sqrt{x} - 2)(\sqrt{x} + 2)} = \frac{1}{\lim_{x \rightarrow 4} \sqrt{x} + 2} = \frac{1}{4}.$$

(iv)

$$\begin{aligned} \lim_{x \rightarrow \infty} (\sqrt{x^2 + 2x} - x) &= \lim_{x \rightarrow \infty} \frac{(\sqrt{x^2 + 2x} - x)(\sqrt{x^2 + 2x} + x)}{\sqrt{x^2 + 2x} + x} \\ &= \lim_{x \rightarrow \infty} \frac{x^2 + 2x - x^2}{\sqrt{x^2 + 2x} + x} = \lim_{x \rightarrow \infty} \frac{2}{(\sqrt{x^2 + 2x})/x + 1} \\ &= \lim_{x \rightarrow \infty} \frac{2}{\sqrt{1 + 2/x} + 1} = \frac{2}{1 + 1} = 1. \end{aligned}$$

5. Find the following limits.

(i)

$$\lim_{x \rightarrow \infty} \frac{3x^2 - x \sin(x) + x^2 \sin(x)}{x^2 + 4x + 3}$$

(ii)

$$\lim_{x \rightarrow a} \frac{\frac{1}{x} - \frac{1}{a}}{x - a}.$$

(iii)

$$\lim_{x \rightarrow 4} \frac{x^2 - 16}{x - 4}$$

(iv)

$$\lim_{x \rightarrow \infty} (\sqrt{x^2 + 2x} - \sqrt{x^2 + 3x}).$$

Solution.

(i)

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{3x^2 - x \sin(x) + x^2 \sin(x)}{x^2 + 4x + 3} &= \lim_{x \rightarrow \infty} \frac{(x^2 + 4x + 3)(3 + \sin x) - (4x + 3)(3 + \sin x) - x \sin x}{x^2 + 4x + 3} \\ &= \lim_{x \rightarrow \infty} (3 + \sin x) - \lim_{x \rightarrow \infty} \frac{(4x + 3)(3 + \sin x) - x \sin x}{x^2 + 4x + 3} \\ &= \lim_{x \rightarrow \infty} (3 + \sin x) - 0 \end{aligned}$$

which does not exist.

(ii)

$$\lim_{x \rightarrow a} \frac{\frac{1}{x} - \frac{1}{a}}{x - a} = \lim_{x \rightarrow a} \frac{\frac{a-x}{ax}}{x - a} = - \lim_{x \rightarrow a} \frac{1}{ax} = -\frac{1}{a^2}.$$

(iii)

$$\lim_{x \rightarrow 4} \frac{x^2 - 16}{x - 4} = \lim_{x \rightarrow 4} \frac{(x - 4)(x + 4)}{x - 4} \lim_{x \rightarrow 4} (x + 4) = 8.$$

(iv)

$$\begin{aligned} \lim_{x \rightarrow \infty} (\sqrt{x^2 + 2x} - \sqrt{x^2 + 3x}) &= \lim_{x \rightarrow \infty} \frac{-x}{\sqrt{x^2 + 2x} + \sqrt{x^2 + 3x}} \\ &= - \lim_{x \rightarrow \infty} \frac{1}{\sqrt{1 + 2/x} + \sqrt{1 + 3/x}} = -\frac{1}{2}. \end{aligned}$$

6.

Suppose that $f(x) > 7$ for all x and that $\lim_{x \rightarrow 11} f(x) = \ell$. Prove that $\ell \geq 7$. Is it possible that $\ell = 7$?

Solution.

Suppose for a contradiction that $\ell < 7$. Let $\epsilon = 7 - \ell > 0$. By the definition of limit, there exists $\delta > 0$ such that $0 < |x - 11| < \delta$ implies $|f(x) - \ell| < \epsilon$. But that means that for such an x , $f(x) < \ell + \epsilon = 7$ which contradicts the assumption that $f(x) > 7$ for all x .

For the second part its possible. For example you could have

$$f(x) = \begin{cases} 7 + |x - 11| & x \neq 11, \\ 8 & x = 11. \end{cases}$$