Math 261: Homework 7 solutions

Chapter 7

3(ii). Look at \( f(x) = \sin x - x + 1 \). It is continuous. And \( f(0) = 1 \), \( f(1000) = \sin 1000 - 1000 + 1 < 0 \). So by the IVT there’s some \( x \) (between 0 and 1000) with \( f(x) = 0 \).

4(a). Take for example \((x + 1)(x + 2) \cdots (x + k)(x^{n-k} + 1)\).

5. If \( f \) is continuous on \([a, b]\) and \( f(x) \) is always rational, then \( f \) is constant.
   Proof. Suppose for a contradiction that \( f \) is not constant. Then, we can find \( x < y \in [a, b] \) such that \( f(x) \neq f(y) \). Choose an irrational number \( m \) lying between \( f(x) \) and \( f(y) \). Then, by the intermediate value theorem, there exists \( z \in [x, y] \) with \( f(z) = m \). Hence, \( f \) takes an irrational value, contradicting the hypotheses.

6. We know \( f(0) \) is either 1 or \(-1\). Suppose that \( f(0) = 1 \) (the other case \( f(0) = -1 \) goes in a similar way). I’ll show that \( f(x) = \sqrt{1 - x^2} \) for all \( x \in [-1, 1] \). Well suppose not. Then for some \( x \in [-1, 1] \), we have that \( f(x) = \sqrt{1 - x^2} \neq \sqrt{1 - x^2} \), hence \( x \in (-1, 1) \) and \( f(x) < 0 \). Since \( f(0) = 1 \), the IVT implies there’s some \( y \) in between \( x \) and 0, i.e. in \((-1, 1)\) too, such that \( f(y) = 0 \). But \( f(y) = \pm \sqrt{1 - y^2} \neq 0 \), so this is a contradiction.

8. Suppose that \( f(0) > 0 \) (the other case \( f(0) < 0 \) is similar). I’ll show first that \( f(x) > 0 \) for all \( x \). Well, if not then there’s some \( x \) with \( f(x) < 0 \), so by IVT since \( f \) is continuous, there’s some \( x \) with \( f(x) = 0 \), a contradiction.

Since \( g(0) = \pm f(0) \), we then either have that \( g(0) > 0 \) or \( g(0) < 0 \). In the former case, the argument in the previous paragraph shows that \( g(x) > 0 \) for all \( x \). Hence since \( g(x) = \pm f(x) \) for all \( x \) we actually have that \( g(x) = f(x) \) for all \( x \) (else some \( g(x) \) would be negative).

The case that \( g(0) < 0 \) is similar: you get that \( g(x) < 0 \) for all \( x \) hence that \( g(x) = -f(x) \) for all \( x \).

12. (a) We need to show that \( f(x) = 1 - x \) for some \( x \in [0, 1] \) (since \( y = 1 - x \) is the equation of the dashed line). Certainly, \( f(0) \leq 1 \) and \( f(1) \geq 0 \) by the assumptions. Now set \( g(x) = f(x) - 1 + x \). Then, \( g(0) \leq 1 - 1 + 0 = 0 \) and \( g(1) \geq 0 - 1 + 1 = 0 \). So, \( g(0) \leq 0 \leq g(1) \). So by the intermediate value theorem, there exists \( x \in [0, 1] \) with \( g(x) = 0 \). But then \( f(x) - 1 + x = 0 \) so \( f(x) = 1 - x \) and we’re done.

(b) Let \( h(x) = f(x) - g(x) \).
   Case one. \( g(0) = 0, g(1) = 1 \). Then, \( h(0) = f(0) \geq 0 \) and \( h(1) = f(1) - 1 \leq 0 \). So \( h(0) \geq 0 \geq h(1) \), so by the intermediate value theorem, there exists \( x \in [0, 1] \) such that \( h(x) = 0 \). But then, \( f(x) - g(x) = 0 \) so \( f(x) = g(x) \) and we’re done.

Case two. \( g(0) = 1, g(1) = 0 \). Then, \( h(0) = f(0) - 1 \leq 0 \) and \( h(1) = f(1) \geq 0 \). So \( h(0) \leq 0 \leq h(1) \), so by the intermediate value theorem, there exists \( x \in [0, 1] \) such that \( h(x) = 0 \). But then, \( f(x) - g(x) = 0 \) so \( f(x) = g(x) \) and we’re done.

Chapter 8
1. (ii) 1 is the greatest, \(-1\) is the least.
(iv) 0 is the least element, and the least upper bound is \( \sqrt{2} \) which is not in the set.

(vi) Since \( \{ x \mid x^2 + x + 1 < 0 \} = ((-1 - \sqrt{5})/2, (-1 + \sqrt{5})/2) \), the greatest lower bound is \((-1 - \sqrt{5})/2\) and the least upper bound is \((-1 + \sqrt{5})/2\); neither is in the set.

(viii) \( 1 - 1/2 \) is the greatest element, and the greatest lower bound is \(-1\) which is not in the set.

6. (a) Suppose not, say \( f(a) \neq 0 \) for some \( a \). Let \( \epsilon = |f(a)| \). By the definition of continuity, there exists \( \delta > 0 \) such that \( |x - a| < \delta \) implies \( |f(x) - f(a)| < |f(a)| \). By the definition of density there exists such an \( x \) belonging to the set \( A \), i.e. with \( f(x) = 0 \). But then, \( |f(a)| = |f(x) - f(a)| < |f(a)| \) which is a contradiction.

(b) Apply (a) to the continuous function \( h(x) = f(x) - g(x) \).

(c) Like in (b) you can reduce to the case that \( g(x) = 0 \), i.e. you know \( f(x) \geq 0 \) for all \( x \in A \) and want to prove that \( f(x) \geq 0 \) for all \( x \in \mathbb{R} \). Suppose not, say \( f(a) < 0 \) for some \( a \). Arguing as in (a) with \( \epsilon = -f(a) \), you then get an \( x \in A \) such that \( |f(x) - f(a)| < \epsilon \). But then \( 0 \leq f(x) < f(a) + \epsilon = 0 \) which is the contradiction.

The answer to the final statement is “NO”, e.g. take \( A = \mathbb{R} \setminus \{0\} \) and \( f(x) = x^2 \).

7. Let \( c = f(1) \). We’ll show \( f(x) = cx \) in steps.

(1) \( f(x) = cx \) for all \( x \in \mathbb{N} \). Well, \( f(n) = f(1 + \cdots + 1) = f(1) + \cdots + f(1) = nc \).

(2) \( f(0) = 0 \). Well, \( f(0) + f(0) = f(0 + 0) = f(0) \), so subtracting \( f(0) \) from both sides gives \( f(0) = 0 \).

(3) \( f(x) = cx \) for all \( x \in \mathbb{N} \). Well, for \( n \in \mathbb{N} \), \( f(n) - n = f(n + f(-n)) = f(0) = 0 \). Hence by (1), \( cn + f(-n) = 0 \), so \( f(-n) = -cn \) which is all that was left to prove after (1) and (2).

(4) \( f(x) = cx \) for all \( x \in \mathbb{Q} \). Well, say \( x = m/n \) for \( m \in \mathbb{Z}, n \in \mathbb{N} \). Then, \( f(nx) = nf(x) = f(m) = cm \). So \( f(x) = cm/n \).

(5) \( f(x) = cx \) for all \( x \in \mathbb{R} \). Since \( \mathbb{Q} \) is dense and the functions \( f(x) \) and \( cx \) are continuous, this follows from (4) and 6(b).