

## Math 261: Homework 7 solutions

### Chapter 7

3(ii). Look at  $f(x) = \sin x - x + 1$ . It is continuous. And  $f(0) = 1$ ,  $f(1000) = \sin 1000 - 1000 + 1 < 0$ . So by the IVT there's some  $x$  (between 0 and 1000) with  $f(x) = 0$ .

4(a). Take for example  $(x+1)(x+2)\cdots(x+k)(x^{n-k}+1)$ .

5. If  $f$  is continuous on  $[a, b]$  and  $f(x)$  is always rational, then  $f$  is constant.

Proof. Suppose for a contradiction that  $f$  is not constant. Then, we can find  $x < y \in [a, b]$  such that  $f(x) \neq f(y)$ . Choose an irrational number  $m$  lying between  $f(x)$  and  $f(y)$ . Then, by the intermediate value theorem, there exists  $z \in [x, y]$  with  $f(z) = m$ . Hence,  $f$  takes an irrational value, contradicting the hypotheses.

6. We know  $f(0)$  is either 1 or  $-1$ . Suppose that  $f(0) = 1$  (the other case  $f(0) = -1$  goes in a similar way). I'll show that  $f(x) = \sqrt{1-x^2}$  for all  $x \in [-1, 1]$ . Well suppose not. Then for some  $x \in [-1, 1]$ , we have that  $f(x) = -\sqrt{1-x^2} \neq \sqrt{1-x^2}$ , hence  $x \in (-1, 1)$  and  $f(x) < 0$ . Since  $f(0) = 1$ , the IVT implies there's some  $y$  in between  $x$  and 0, i.e. in  $(-1, 1)$  too, such that  $f(y) = 0$ . But  $f(y) = \pm\sqrt{1-y^2} \neq 0$ , so this is a contradiction.

8. Suppose that  $f(0) > 0$  (the other case  $f(0) < 0$  is similar). I'll show first that  $f(x) > 0$  for all  $x$ . Well, if not then there's some  $x$  with  $f(x) < 0$ , so by IVT since  $f$  is continuous, there's some  $x$  with  $f(x) = 0$ , a contradiction.

Since  $g(0) = \pm f(0)$ , we then either have that  $g(0) > 0$  or  $g(0) < 0$ . In the former case, the argument in the previous paragraph shows that  $g(x) > 0$  for all  $x$ . Hence since  $g(x) = \pm f(x)$  for all  $x$  we actually have that  $g(x) = f(x)$  for all  $x$  (else some  $g(x)$  would be negative).

The case that  $g(0) < 0$  is similar: you get that  $g(x) < 0$  for all  $x$  hence that  $g(x) = -f(x)$  for all  $x$ .

12. (a) We need to show that  $f(x) = 1 - x$  for some  $x \in [0, 1]$  (since  $y = 1 - x$  is the equation of the dashed line). Certainly,  $f(0) \leq 1$  and  $f(1) \geq 0$  by the assumptions. Now set  $g(x) = f(x) - 1 + x$ . Then,  $g(0) \leq 1 - 1 + 0 = 0$  and  $g(1) \geq 0 - 1 + 1 = 0$ . So,  $g(0) \leq 0 \leq g(1)$ . So by the intermediate value theorem, there exists  $x \in [0, 1]$  with  $g(x) = 0$ . But then  $f(x) - 1 + x = 0$  so  $f(x) = 1 - x$  and we're done.

(b) Let  $h(x) = f(x) - g(x)$ .

Case one.  $g(0) = 0, g(1) = 1$ . Then,  $h(0) = f(0) \geq 0$  and  $h(1) = f(1) - 1 \leq 0$ . So  $h(0) \geq 0 \geq h(1)$ , so by the intermediate value theorem, there exists  $x \in [0, 1]$  such that  $h(x) = 0$ . But then,  $f(x) - g(x) = 0$  so  $f(x) = g(x)$  and we're done.

Case two.  $g(0) = 1, g(1) = 0$ . Then,  $h(0) = f(0) - 1 \leq 0$  and  $h(1) = f(1) \geq 0$ . So  $h(0) \leq 0 \leq h(1)$ , so by the intermediate value theorem, there exists  $x \in [0, 1]$  such that  $h(x) = 0$ . But then,  $f(x) - g(x) = 0$  so  $f(x) = g(x)$  and we're done.

### Chapter 8

1.

(ii) 1 is the greatest,  $-1$  is the least.

(iv) 0 is the least element, and the least upper bound is  $\sqrt{2}$  which is not in the set.

(vi) Since  $\{x \mid x^2 + x + 1 < 0\} = ((-1 - \sqrt{5})/2, (-1 + \sqrt{5})/2]$ , the greatest lower bound is  $(-1 - \sqrt{5})/2$  and the least upper bound is  $(-1 + \sqrt{5})/2$ ; neither is in the set.

(viii)  $1 - 1/2$  is the greatest element, and the greatest lower bound is  $-1$  which is not in the set.

6. (a) Suppose not, say  $f(a) \neq 0$  for some  $a$ . Let  $\epsilon = |f(a)|$ . By the definition of continuity, there exists  $\delta > 0$  such that  $|x - a| < \delta$  implies  $|f(x) - f(a)| < |f(a)|$ . By the definition of density there exists such an  $x$  belonging to the set  $A$ , i.e. with  $f(x) = 0$ . But then,  $|f(a)| = |f(x) - f(a)| < |f(a)|$  which is a contradiction.

(b) Apply (a) to the continuous function  $h(x) = f(x) - g(x)$ .

(c) Like in (b) you can reduce to the case that  $g(x) = 0$ , i.e. you know  $f(x) \geq 0$  for all  $x \in A$  and want to prove that  $f(x) \geq 0$  for all  $x \in \mathbb{R}$ . Suppose not, say  $f(a) < 0$  for some  $a$ . Arguing as in (a) with  $\epsilon = -f(a)$ , you then get an  $x \in A$  such that  $|f(x) - f(a)| < \epsilon$ . But then  $0 \leq f(x) < f(a) + \epsilon = 0$  which is the contradiction.

The answer to the final statement is “NO”, e.g. take  $A = \mathbb{R} \setminus \{0\}$  and  $f(x) = x^2$ .

7. Let  $c = f(1)$ . We'll show  $f(x) = cx$  in steps.

(1)  $f(x) = cx$  for all  $x \in \mathbb{N}$ . Well,  $f(n) = f(1 + \dots + 1) = f(1) + \dots + f(1) = nc$ .

(2)  $f(0) = 0$ . Well,  $f(0) + f(0) = f(0 + 0) = f(0)$ , so subtracting  $f(0)$  from both sides gives  $f(0) = 0$ .

(3)  $f(x) = cx$  for all  $x \in \mathbb{N}$ . Well, for  $n \in \mathbb{N}$ ,  $f(n - n) = f(n) + f(-n) = f(0) = 0$ . Hence by (1),  $cn + f(-n) = 0$ , so  $f(-n) = -cn$  which is all that was left to prove after (1) and (2).

(4)  $f(x) = cx$  for all  $x \in \mathbb{Q}$ . Well, say  $x = m/n$  for  $m \in \mathbb{Z}, n \in \mathbb{N}$ . Then,  $f(nx) = nf(x) = f(m) = cm$ . So  $f(x) = cm/n$ .

(5)  $f(x) = cx$  for all  $x \in \mathbb{R}$ . Since  $\mathbb{Q}$  is dense and the functions  $f(x)$  and  $cx$  are continuous, this follows from (4) and 6(b).