Math 261: Homework 4 solutions

Sorry there are lots of graphs in these solutions which I can't do in PDF... you need the hard copy.

Ch. 4

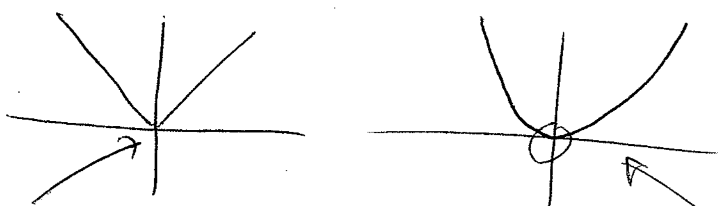
2. (a) Let $x \in [0, b]$. Set $t = x/b$, so that $0 \leq t \leq 1$. Then, $x = tb$. The number t indicates how far between 0 and b the number x is: if $t = 0$ then $x = 0$, if $t = 0.5$ then x is exactly half way between 0 and b, and if $x = 1$ then $x = b$.

(b) Let $x \in [a, b]$. Then, $x - a \in [0, b - a]$. Set $t = \frac{x-a}{b-a}$ so that $x - a = t(b - a)$ as in (a) and $0 \leq t \leq 1$. Then, $x = a + t(b - a) = (1 - t)a + tb$. The midpoint of the interval is $(1 - \frac{1}{2})a + \frac{1}{2}b$; the point 1/3 of the way from a to b is $(1 - 1/3)a + 1/3b$.

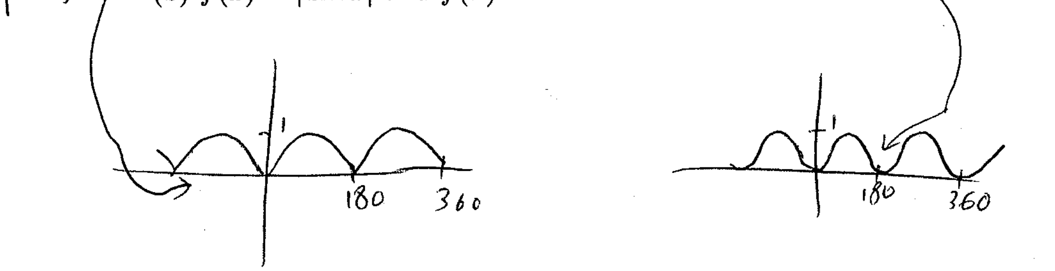
(c) Let $0 \leq t \leq 1$. Then, as $(b - a) > 0$, $0 \leq t(b - a) \leq b - a$, so $a \leq a + t(b - a) \leq b$. Hence, $a \leq (1 - t)a + tb \leq b$. You should think of t as a *parameter* saying how far the number $(1 - t)a + tb$ is across the interval $[a, b]$, as t goes between 0 and 1.

(d) Just observe that $(1 - t)a + tb = a$ if and only if $t = 0$, and $(1 - t)a + tb = b$ if and only if $t = 1$. Since by (b) and (c) we know that $[a, b] = \{(1 - t)a + tb \mid 0 \leq t \leq 1\}$ it follows that $(a, b) = \{(1 - t)a + tb \mid 0 < t < 1\}$.

13. (a) $f(x) = |x|$ and $f(x) = x^2$.

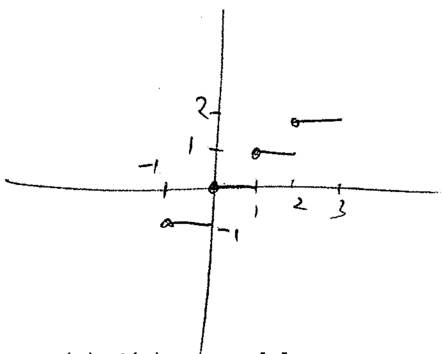


(b) $f(x) = |\sin x|$ and $f(x) = \sin^2 x$.

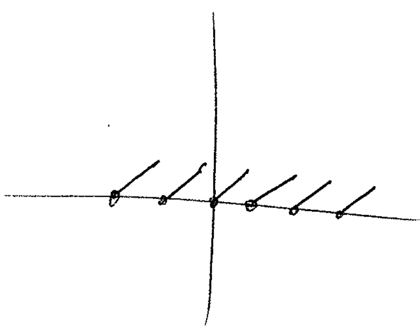


The important difference in (b) is that the graph $f(x) = |\sin x|$ has "points" at the bottom of each loop, whereas $f(x) = \sin^2 x$ is a nice "smooth" curve at the bottoms.

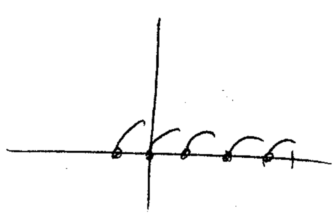
17.(i) $f(x) = [x]$



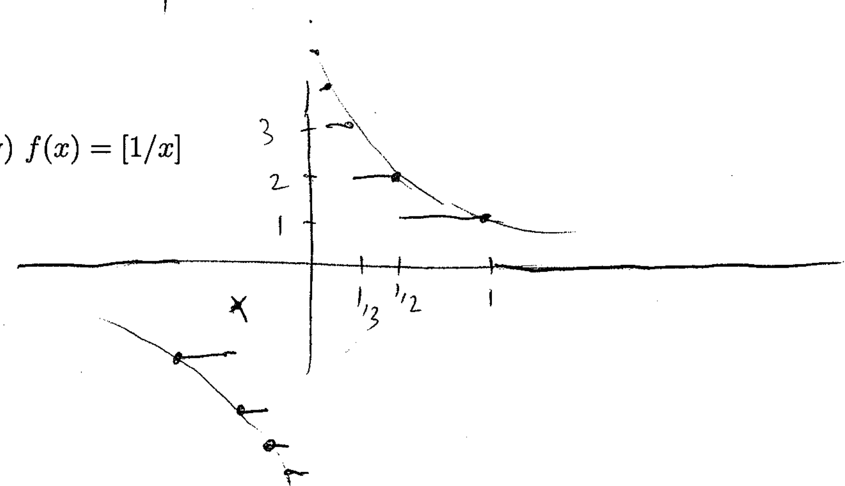
(ii) $f(x) = x - [x]$



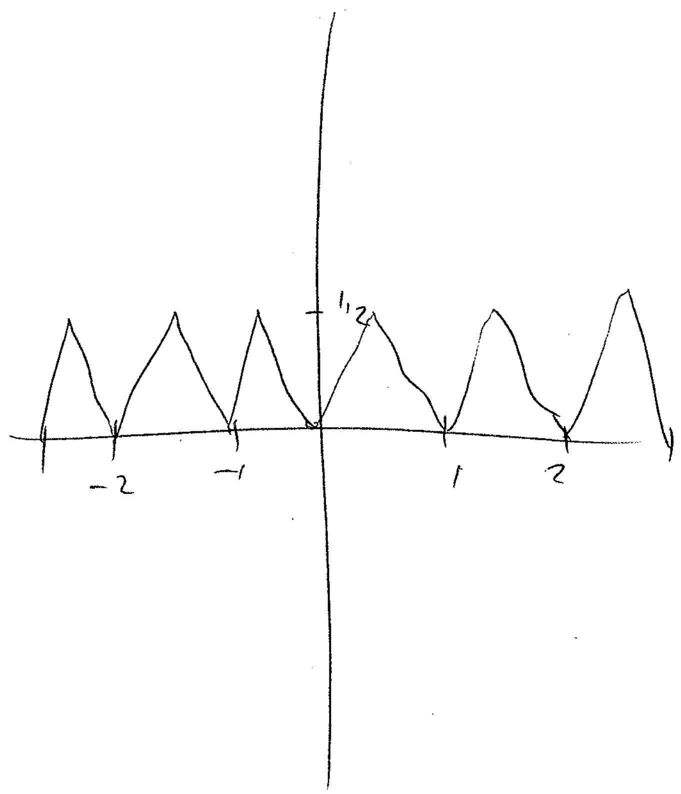
(iii) $f(x) = \sqrt{x - [x]}$



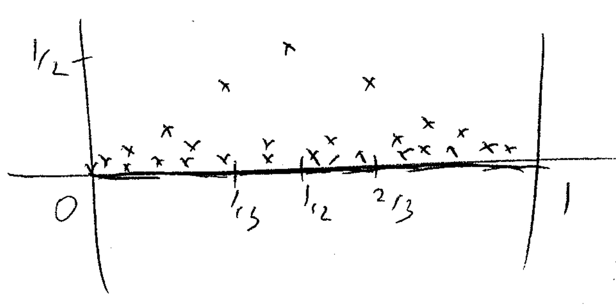
(v) $f(x) = [1/x]$



18.(i) See page 500 in the text book!



20. Frankly, if you sprinkle points randomly on the paper, you'll get a pretty good approximation to this one! This is a famous function with unusually pathological behaviour (as you probably found if you tried to plot it!)



Ch. 3

21. (a) False. Take $f(x) = x^2, g(x) = h(x) = 2$. Then $(g+h)(x) = 4$ so $(f \circ (g+h))(x) = 16$, while $((f \circ g) + (f \circ h))(x) = 8$.

(b) True. Left hand side is $((g+h) \circ f)(x) = g(f(x)) + h(f(x))$. Right hand side is $g(f(x)) + h(f(x))$. Yes they're equal.

(c) True. Left hand side on x is $\frac{1}{f(g(x))}$. Right side on x is $\frac{1}{f}(g(x)) = \frac{1}{f(g(x))}$.

(d) False. Take $f(x) = x+1$, $g(x) = x$. Then $\frac{1}{f \circ g}(x) = \frac{1}{x+1}$. And $f \circ \frac{1}{g}(x) = \frac{1}{x} + 1$.

23. (a) Prove the contrapositive statement instead (equivalent): if $g(x) = g(y)$ then $x = y$. Proof: Apply f to both sides, get $f(g(x)) = f(g(y))$. Hence $I(x) = I(y)$. Hence $x = y$.

(b) $b = f(g(b)) = f(a)$ where $a = g(b)$.

25. Take $f(x) = 10^x$. It has a left inverse, namely, $\log_{10}$. But it doesn't have a right inverse: -1 is not equal to $f(a)$ for any a which contradicts 23(b).

Ch. 5

1. (i) $\lim_{x \to 1} \frac{x^2-1}{x+1} = \lim_{x \to 1}(x-1) = 0$.

(ii) $\lim_{x \to 2} \frac{x^3-8}{x-2} = \lim_{x \to 2} \frac{(x-2)(x^2+2x+4)}{x-2} = \lim_{x \to 2}(x^2+2x+4) = 12$.

(iii) $\lim_{x \to 3} \frac{x^3-8}{x-2} = \frac{19}{1} = 19$.

(iv) $\lim_{x \to y} \frac{x^n-y^n}{x-y} = \lim_{x \to y}(x^{n-1}+x^{n-2}y+\cdots+x^2y^{n-3}+xy^{n-2}+y^{n-1}) = ny^{n-1}$.