Math 261: Homework 2 solutions

Ch.1 22 We are told that $y_0 \neq 0$ and

$$|y - y_0| < \min \left( \frac{|y_0|}{2}, \frac{\varepsilon|y_0|^2}{2} \right).$$

In other words we know both that $|y - y_0| < \frac{|y_0|}{2}$ and that $|y - y_0| < \frac{\varepsilon|y_0|^2}{2}$. Let me first show that

$$|y| > \frac{|y_0|}{2},$$

which gives in particular that $y \neq 0$ since $y_0 \neq 0$. Well, $|y| = |y_0 - y + y| \leq |y_0 - y| + |y|$ by the triangle inequality. By what we’re told $|y_0 - y| + |y| < \frac{|y_0|}{2} + |y|$. So we’ve shown that $|y_0| < \frac{|y_0|}{2} + |y|$. Rearranging this inequality gives that $|y| > \frac{|y_0|}{2}$.

Now we can prove the main thing. We already know

$$|y - y_0| < \frac{\varepsilon|y_0|^2}{2}.$$

I just showed that $\frac{|y_0|}{2} < |y|$. So we get

$$|y - y_0| < \frac{\varepsilon|y_0|^2}{2} < \varepsilon |y_0||y|.$$

Dividing both sides by $|y_0||y|$ gives

$$\left| \frac{y - y_0}{yy_0} \right| < \varepsilon.$$

But

$$\left| \frac{y - y_0}{yy_0} \right| = \frac{1}{y} - \frac{1}{y_0} < \varepsilon$$

so its just what we were after!

Ch.2 1 I’ll just do (ii) assuming part (i). So we want to prove

$$1^3 + \cdots + n^3 = \sum_{i=1}^{n} i^3 = (1 + \cdots + n)^2.$$

First, let me rewrite the right hand side: we know $1 + \cdots + n = \frac{1}{2}n(n + 1)$. So we need to prove

$$1^3 + \cdots + n^3 = \frac{1}{4}n^2(n + 1)^2.$$
Base case. Check the formula is true for $n = 1$. Yes, both left and right hand sides are 1.

Induction step. Assume the formula is true for $n = k$, i.e.

$$1^3 + \cdots + k^3 = \frac 1 4 k^2 (k + 1)^2.$$ 

Add $(k + 1)^3$ to both sides. Get

$$1^3 + \cdots + k^3 + (k + 1)^3 = \frac 1 4 k^2 (k + 1)^2 + (k + 1)^3$$

$$= \frac 1 4 (k + 1)^2 (k^2 + 4k + 4)$$

$$= \frac 1 4 (k + 1)^2 (k + 2)^2$$

which is exactly what we’re trying to prove with $n = k + 1$.

Done by P.M.I.

Ch.2 2 (i) We have that

$$\sum_{i=1}^{n} (2i - 1) = \sum_{i=1}^{2n} i - \sum_{i=1}^{n} 2i = \sum_{i=1}^{2n} i - 2 \sum_{i=1}^{n} i$$

This says: sum of first $n$ odd numbers is sum of first $2n$ numbers minus sum of first $n$ even numbers. Think about it with some examples $n = 3, 4, 5, \ldots$. So

$$\sum_{i=1}^{n} (2i - 1) = \frac 1 2 2n(2n + 1) - 2 \frac 1 2 n(n + 1)$$

by stuff we already know. The right hand side simplifies to $n^2$, which is the answer.

(ii) This time

$$\sum_{i=1}^{n} (2i - 1)^2 = \sum_{i=1}^{2n} i^2 - \sum_{i=1}^{n} (2i)^2 = \sum_{i=1}^{2n} i^2 - 4 \sum_{i=1}^{n} i^2$$

Plugging in what we know,

$$\sum_{i=1}^{n} (2i - 1)^2 = \frac 1 6 2n(2n + 1)(4n + 1) - \frac 4 6 n(n + 1)(2n + 1).$$

The right hand side simplifies to

$$\frac 1 3 n(2n + 1)(4n + 1 - 2n - 2) = \frac 1 3 n(2n + 1)(2n - 1).$$
Ch. 2 5. Done in class.

Ch. 2 6. I’m just going to do (iv). We need to sum things looking like
\[ \frac{2n + 1}{n^2(n + 1)^2}. \]

Going to try to use the method of differences, so can I come up with a
difference that equals this? Guess:
\[ \frac{1}{n^2} - \frac{1}{(n + 1)^2}. \]

Work it out, its
\[ \frac{(n + 1)^2 - n^2}{n^2(n + 1)^2} = \frac{2n + 1}{n^2(n + 1)^2}. \]

Lucky guess! So:
\[
\sum_{i=1}^{n} \frac{2i + 1}{i^2(i + 1)^2} = \sum_{i=1}^{n} \left( \frac{1}{i^2} - \frac{1}{(i + 1)^2} \right) = 1 - \frac{1}{(n + 1)^2}.
\]

You can simplify this to
\[ \frac{n^2 + 2n}{(n + 1)^2} \]
if you want.

Ch. 2 12 (a) First part: yes. Proof: use contradiction. Suppose \( c = a + b \)
is rational. Then, as \( a \) is rational and rational minus rational is rational,
you get that \( b = c - a \) is rational, contradicting the assumption on \( b \).

Second part: not necessarily. For example, \( a = \sqrt{2}, b = -\sqrt{2} \), when
\( a + b = 0 \) which is rational!

(b) Not necessarily, e.g. \( a = 0, b = \sqrt{2} \).

(c) The fourth root of 2 is such a number.

(d) Yes. Try \( \sqrt{2} \) and \( -\sqrt{2} \).

Ch. 2 26

I’ll just explain how to describe an \textit{inductive} algorithm for moving \( n \)
rings from spindle 1 to spindle 3 in \( 2^n - 1 \) moves. This is half the question:
proving this is the minimal number of moves is a little bit harder.

Clearly, we can move one ring in just 1 move. Now suppose that we
already have an algorithm for moving \((n - 1)\) rings in \(2^{n-1} - 1\) moves. Here
is an algorithm to move \( n \) rings in \(2^n - 1\) moves. First: move the top \((n - 1)\)
rings from spindle 1 to spindle 2 using the algorithm given. Second: move
the largest ring from spindle 1 to spindle 3. Third: move the \((n - 1)\) rings
from spindle 2 to spindle 3 using the algorithm given. Total number of moves = \(2^{n-1} - 1 + 1 + 2^{n-1} - 1 = 2^n - 1\).

Note this question is an example of a definition by induction. We defined the algorithm by first defining it for \(n = 1\), then defining the algorithm for \(n\) in terms of the inductively defined algorithm involving \(n - 1\).

Ch.3 1 (i) \(\frac{1+x}{2+x}\).
(ii) \(\frac{x}{1+x}\).
(iii) \(\frac{1}{1+cx}\).
(iv) \(\frac{1}{1+x+y}\).
(v) \(\frac{1}{1+x} + \frac{1}{1+y} = \frac{2+x+y}{(1+x)(1+y)}\).
(vi) Need \(\frac{1}{1+cx} = \frac{1}{1+x}\), so \(1 + x = 1 + cx\) which is always true if \(x = 0\) regardless of what \(c\) is. So the answer is: ALL \(c\).

(vii) Okay, so regardless of what \(c\) is, \(x = 0\) is one solution. To get another non-zero solution, we need that \((c-1)x = 0\), so for \(x \neq 0\) we must have that \(c-1 = 0\) so \(c = 1\). So the answer is: \(c = 1\) only.

Ch.3 3
(i) \([-1, 1]\).
(ii) \([-1, 1]\).
(iii) All \(x \in \mathbb{R}\) except \(x = 1, 2\).
(iv) \(\{\pm 1\}\).
(v) Need \(x \leq 1, x \geq 2\). So the domain is \(\emptyset\), the empty set.

Ch.3 4
(i) \((2^y)^2 = 2^{2y} = 4^y\).
(ii) \(\sin^2 y\).
(iii) \(4 \sin x + \sin(2t)\).
(iv) \(\sin(t^3)\).