Math 261: Homework 2 solutions

Ch.1 22 We are told that $y_0 \neq 0$ and

$$|y - y_0| < \min\left(\frac{|y_0|}{2}, \frac{\varepsilon|y_0|^2}{2}\right).$$

In other words we know both that $|y - y_0| < \frac{|y_0|}{2}$ and that $|y - y_0| < \frac{\varepsilon|y_0|^2}{2}$. Let me first show that

$$|y| > \frac{|y_0|}{2},$$

which gives in particular that $y \neq 0$ since $y_0 \neq 0$. Well, $|y_0| = |y_0 - y + y| \leq |y_0 - y| + |y|$ by the triangle inequality. By what we're told $|y_0 - y| + |y| < \frac{|y_0|}{2} + |y|$. So we've shown that $|y_0| < \frac{|y_0|}{2} + |y|$. Rearranging this inequality gives that $|y| > \frac{|y_0|}{2}$.

Now we can prove the main thing. We already know

$$|y - y_0| < \frac{\varepsilon|y_0|^2}{2}.$$

I just showed that $\frac{|y_0|}{2} < |y|$. So we get

$$|y - y_0| < \frac{\varepsilon|y_0|^2}{2} < \varepsilon|y_0||y|.$$

Dividing both sides by $|y_0||y|$ gives

$$\left|\frac{y - y_0}{yy_0}\right| < \varepsilon.$$

But

$$\left|\frac{y - y_0}{yy_0}\right| = \left|\frac{1}{y} - \frac{1}{y_0}\right| < \varepsilon$$

so its just what we were after!

Ch.2 1 I'll just do (ii) assuming part (i). So we want to prove

$$1^3 + \cdots + n^3 = \sum_{i=1}^{n} i^3 = (1 + \cdots + n)^2.$$

First, let me rewrite the right hand side: we know $1 + \cdots + n = \frac{1}{2}n(n+1)$. So we need to prove

$$1^3 + \cdots + n^3 = \frac{1}{4}n^2(n+1)^2.$$

Base case. Check the formula is true for $n = 1$. Yes, both left and right hand sides are 1.

Induction step. Assume the formula is true for $n = k$, i.e.

$$1^3 + \cdots + k^3 = \frac{1}{4}k^2(k+1)^2.$$

Add $(k+1)^3$ to both sides. Get

$$\begin{aligned} 1^3 + \cdots + k^3 + (k+1)^3 &= \frac{1}{4}k^2(k+1)^2 + (k+1)^3 \\ &= \frac{1}{4}(k+1)^2(k^2 + 4k + 4) \\ &= \frac{1}{4}(k+1)^2(k+2)^2 \end{aligned}$$

which is exactly what we're trying to prove with $n = k + 1$.

Done by P.M.I.

Ch.2 2 (i) We have that

$$\sum_{i=1}^{n}(2i-1) = \sum_{i=1}^{2n} i - \sum_{i=1}^{n} 2i = \sum_{i=1}^{2n} i - 2\sum_{i=1}^{n} i$$

This says: sum of first n odd numbers is sum of first $2n$ numbers minus sum of first n even numbers. Think about it with some examples $n = 3, 4, 5, \ldots$. So

$$\sum_{i=1}^{n}(2i-1) = \frac{1}{2}2n(2n+1) - 2\frac{1}{2}n(n+1)$$

by stuff we already know. The right hand side simplifies to n^2, which is the answer.

(ii) This time

$$\sum_{i=1}^{n}(2i-1)^2 = \sum_{i=1}^{2n} i^2 - \sum_{i=1}^{n}(2i)^2 = \sum_{i=1}^{2n} i^2 - 4\sum_{i=1}^{n} i^2$$

Plugging in what we know,

$$\sum_{i=1}^{n}(2i-1)^2 = \frac{1}{6}2n(2n+1)(4n+1) - \frac{4}{6}n(n+1)(2n+1).$$

The right hand side simplifies to

$$\frac{1}{3}n(2n+1)\left(4n+1-2n-2\right) = \frac{1}{3}n(2n+1)(2n-1).$$

Ch.2 5. Done in class.

Ch.2 6. I'm just going to do (iv). We need to sum things looking like

$$\frac{2n+1}{n^2(n+1)^2}.$$

Going to try to use the method of differences, so can I come up with a difference that equals this? Guess:

$$\frac{1}{n^2} - \frac{1}{(n+1)^2}.$$

Work it out, its

$$\frac{(n+1)^2 - n^2}{n^2(n+1)^2} = \frac{2n+1}{n^2(n+1)^2}.$$

Lucky guess! So:

$$\sum_{i=1}^{n} \frac{2i+1}{i^2(i+1)^2} = \sum_{i=1}^{n} \left(\frac{1}{i^2} - \frac{1}{(i+1)^2} \right) = 1 - \frac{1}{(n+1)^2}.$$

You can simplify this to

$$\frac{n^2+2n}{(n+1)^2}$$

if you want.

Ch.2 12 (a) First part: yes. Proof: use contradiction. Suppose $c = a + b$ is rational. Then, as a is rational and rational minus rational is rational, you get that $b = c - a$ is rational, contradicting the assumption on b.

Second part: not necessarily. For example, $a = \sqrt{2}, b = -\sqrt{2}$, when $a + b = 0$ which is rational!

(b) Not necessarily, e.g. $a = 0, b = \sqrt{2}$.

(c) The fourth root of 2 is such a number.

(d) Yes. Try $\sqrt{2}$ and $-\sqrt{2}$.

Ch.2 26

I'll just explain how to describe an *inductive* algorithm for moving n rings from spindle 1 to spindle 3 in $2^n - 1$ moves. This is half the question: proving this is the minimal number of moves is a little bit harder.

Clearly, we can move one ring in just 1 move. Now suppose that we *already* have an algorithm for moving $(n-1)$ rings in $2^{n-1} - 1$ moves. Here is an algorithm to move n rings in $2^n - 1$ moves. First: move the top $(n-1)$ rings from spindle 1 to spindle 2 using the algorithm given. Second: move the largest ring from spindle 1 to spindle 3. Third: move the $(n-1)$ rings from spindle 2 to spindle 3 using the algorithm given. Total number of moves $= 2^{n-1} - 1 + 1 + 2^{n-1} - 1 = 2_n - 1$.

Note this question is an example of a definition by induction. We defined the algorithm by first defining it for $n = 1$, then defining the algorithm for n *in terms of* the inductively defined algorithm involving $n - 1$.

Ch.3 1 (i) $\frac{1+x}{2+x}$.

(ii) $\frac{x}{1+x}$.

(iii) $\frac{1}{1+cx}$.

(iv) $\frac{1}{1+x+y}$.

(v) $\frac{1}{1+x} + \frac{1}{1+y} = \frac{2+x+y}{(1+x)(1+y)}$.

(vi) Need $\frac{1}{1+cx} = \frac{1}{1+x}$, so $1 + x = 1 + cx$ which is always true if $x = 0$ regardless of what c is. So the answer is: ALL c.

(vii) Okay, so regardless of what c is, $x = 0$ is one solution. To get another non-zero solution, we need that $(c - 1)x = 0$, so for $x \neq 0$ we must have that $c - 1 = 0$ so $c = 1$. So the answer is: $c = 1$ only.

Ch.3 3

(i) $[-1, 1]$.

(ii) $[-1, 1]$.

(iii) All $x \in \mathbb{R}$ except $x = 1, 2$.

(iv) $\{\pm 1\}$.

(v) Need $x \leq 1$, $x \geq 2$. So the domain is $\emptyset$, the empty set.

Ch.3 4

(i) $(2^y)^2 = 2^{2y} = 4^y$.

(ii) $sin^2 y$.

(iii) $4^{\sin x} + \sin(2^t)$.

(iv) $\sin(t^3)$.