Winter 2007

Calculus I Practise Midterm

Name: Jon.

1	2	3	4	5	6	TOT.

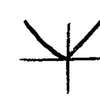
Answer ALL questions. Each question is worth TEN points. Show all your work and try to justify your answers whenever possible – that way I can give some credit even for wrong answers.

1. True or False?

(a) If $f(x) = e^2$ then $f'(x) = 2e$. ← False ($f'(x) = 0$ as $f(x)$ is constant!)

(b) If f is differentiable then $\frac{d}{dx}(\sin(f(x))) = \cos(f(x)) \cdot f'(x)$. ← True (chain rule)

(c) If $\lim_{x \to 6} f(x)$ exists then the limit must be $f(6)$. ← False (only if f is continuous at $x=6$)

(d) If $f(x)$ is continuous at $x = a$ then $f(x)$ is differentiable at $x = a$. ← False (differentiable implies continuous, not the other way round, eg )

(e) If $f''(6)$ exists then the function $f'(x)$ is continuous at $x = 6$.

↑ True, $f'(x)$ is differentiable at $x=6$, hence continuous.

2. (a) Write down the *quotient rule* as precisely as you can.

If f and g are differentiable, then

$$\left(\frac{f}{g}\right)'(x) = \frac{g(x)\cdot f'(x) - f(x)\cdot g'(x)}{(g(x))^2}$$

(b) Differentiate the following functions. Simplify your answer as far as you think is reasonable. Show your working to get partial credit!

(i) $\frac{4-x}{3+x}$.

$$\frac{-3-x-4+x}{(3+x)^2} = -\frac{7}{(3+x)^2}$$

(ii) $x(\cos^2 x + 1)$.

$$\cos^2 x + 1 + x\left(2\cos x \cdot (-\sin x)\right)$$

$$= \cos^2 x + 1 - 2x\cos x \sin x$$

(iii) $\tan(\sqrt{xe^x})$.

$$\sec^2(\sqrt{xe^x}) \cdot \frac{1}{2\sqrt{xe^x}} \cdot (e^x + xe^x)$$

$$= \frac{\sec^2(\sqrt{xe^x}) \cdot (e^x + xe^x)}{2\sqrt{xe^x}}.$$

(iv) $\sin(\ln x)$.

$$\frac{\cos(\ln x)\cdot}{x}$$

3. At which point of the curve $y = x^2 - 3x + 7$ is the tangent line parallel to the line $\underbrace{x + y = 1}_{\text{slope } -1}$?

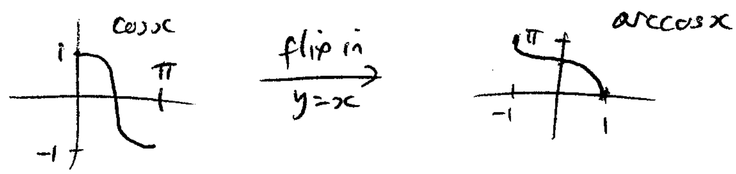
$$\frac{dy}{dx} = 2x - 3 = -1$$

$x = 1, y = 5$

$(1, 5)$

4. (a) What is the definition of the function $\arccos x$? What is its domain?

Its the inverse function to $\cos x$ on the domain $[0, \pi]$.



Domain $= [-1, 1]$

(b) If $y = \arccos x$, use implicit differentiation to calculate $\frac{dy}{dx}$ in terms of x.

$\cos y = x$

$-\sin y \dfrac{dy}{dx} = \cancel{x}\, 1$

$\dfrac{dy}{dx} = -\dfrac{\cancel{x}\,1}{\sin y} = -\dfrac{1}{\sqrt{1-\cos^2 y}} = -\dfrac{1}{\sqrt{1-x^2}}$

5. (a) Precisely what does it mean to say that a function $f(x)$ is *differentiable* at $x = a$?

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \quad \text{exists}$$

(b) Suppose f and g are differentiable functions such that $f(9) = 12, f'(9) = -1, f'(5) = 8, f'(6) = 7, g(9) = 6, g'(9) = 5$. Calculate

(i) $(f \circ g)'(9)$.

$$f'(g(9)) \cdot g'(9) = f'(6) \cdot g'(9)$$
$$= 7 \cdot 5 = 35$$

(ii) $(f \cdot g)'(9)$.

$$f'(9)g(9) + f(9)g'(9) = -6 + 12 \cdot 5 = 54$$

(iii) $\left(\frac{d}{dx} \ln f(x)\right)\big|_{x=9}$.

$$\left.\frac{f'(x)}{f(x)}\right|_{x=9} = \frac{f'(9)}{f(9)} = \frac{-1}{12}$$

"evaluated at $x=9$"

(in seconds)

6. A particle oscillates so that its displacement $f(t)$ from the origin at time t is given by the equation $f(t) = e^{-3t}\sin t$ (in meters).
(a) How fast is the particle moving at time $t = 0$? Include the units!

$$f'(t) = e^{-3t}\cos t - 3e^{-3t}\sin t$$

$$f'(0) = 1 \quad m/sec.$$

(b) What is the acceleration of the particle at time $t = 0$? Include the units!

$$f''(t) = -e^{-3t}\sin t - 3e^{-3t}\cos t - 3e^{-3t}\cos t + 9e^{-3t}\sin t$$

$$f''(0) = -6 \quad m/sec^2.$$

(c) Calculate $\lim_{t\to\infty} f(t)$. What does this tell you about the long term behavior of the particle?

$e^{-3t} \underbrace{\sin t}_{\text{between } -1 \text{ and } 1}$

$e^{-3t} \to 0$ as $t \to \infty$

$\therefore \lim_{t\to\infty} f(t) = 0$ (policeman lemma)

the particle eventually comes to rest at the origin.