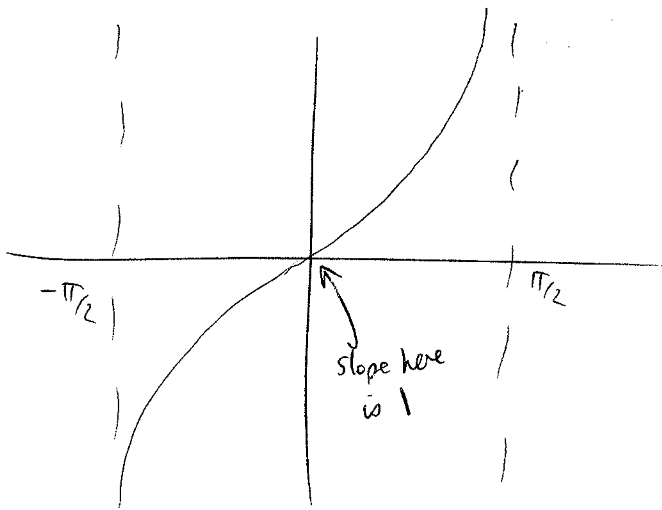
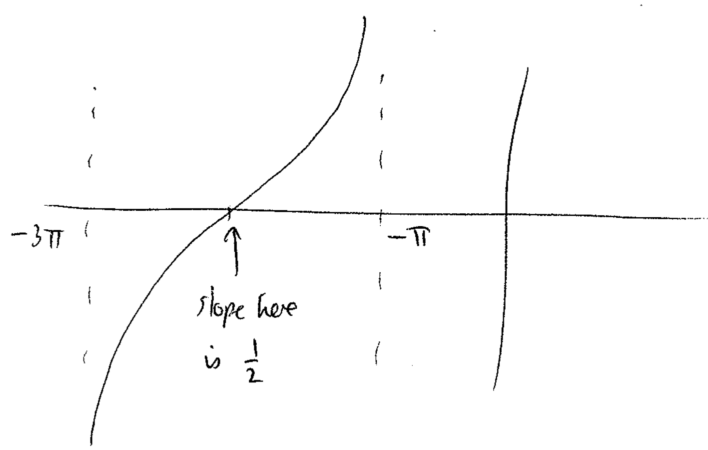
1. (a) Sketch the graph of $f(x) = \tan x$ on the domain $(-\pi/2, \pi/2)$.



(b) For $f(x)$ as in (a), use the transformation rules to sketch the graph of $g(x) = f(\frac{x}{2} + \pi)$.



(c) For $f(x)$ as in (a) and $g(x)$ as in (b), is it true that $g'(x) = f'(\frac{x}{2} + \pi)$?

No. By chain rule, $g'(x) = f'(\frac{x}{2} + \pi) \times \frac{1}{2}$.

2. (a) Complete the following statement of the chain rule: if f and g are differentiable functions then

$$(f \circ g)'(x) = \quad f'(g(x))\,g'(x)$$

(b) The hyperbolic functions $\sinh x$ and $\cosh x$ are defined by

$$\sinh x = \frac{1}{2}(e^x - e^{-x}), \qquad \cosh x = \frac{1}{2}(e^x + e^{-x}).$$

Using these definitions, prove that $\cosh^2 x - \sinh^2 x = 1$.

$$\cosh^2 x = \frac{1}{4}(e^x + e^{-x})^2 = \frac{1}{4}\left((e^x)^2 + 2e^x e^{-x} + (e^{-x})^2\right) = \frac{1}{4}\left(e^{2x} + 2 + e^{-2x}\right)$$

$$\sinh^2 x = \frac{1}{4}(e^x - e^{-x})^2 = \frac{1}{4}\left((e^x)^2 - 2e^x e^{-x} + (e^{-x})^2\right) = \frac{1}{4}\left(e^{2x} - 2 + e^{-2x}\right) \quad -$$

$$\cosh^2 x - \sinh^2 x = \frac{1}{4}(2 - -2) = 1$$

(c) Use the chain rule and the definitions in (b) to show that $\frac{d}{dx}(\sinh x) = \cosh x$ and $\frac{d}{dx}(\cosh x) = \sinh x$.

$$\frac{d}{dx}\left(\frac{1}{2}(e^x - e^{-x})\right) = \frac{1}{2}(e^x + e^{-x}) = \cosh x \quad ✓$$

$$\frac{d}{dx}\left(\frac{1}{2}(e^x + e^{-x})\right) = \frac{1}{2}(e^x - e^{-x}) = \sinh x \quad ✓$$

(d) Let $\tanh x = \frac{\sinh x}{\cosh x}$. Use the quotient rule to prove that $\frac{d}{dx}(\tanh x) = \frac{1}{\cosh^2 x}$.

$$\frac{d}{dx}(\tanh x) \overset{(c)}{=} \frac{\cosh x \cdot \cosh x - \sinh x \cdot \sinh x}{\cosh^2 x} \overset{(b)}{=} \frac{1}{\cosh^2 x}$$

3. Differentiate the following functions, simplifying your answers as far as you judge is reasonable.

(a) $\sin^2 x$.

$2\sin x \cos x$

(b) 2^x.

$2^x \cdot \ln 2$

(c) $\ln(\arctan x)$.

$$\frac{1}{\arctan x\,(1+x^2)}$$

(d) $\sin x \cdot \cos x$.

$\cos^2 x - \sin^2 x$

(e) $\frac{x^2+1}{(x+1)^2}$.

$$\frac{2x(x+1)^2 - 2(x^2+1)(x+1)}{(x+1)^4}$$

4. Calculate the following limits using any method you like. If the limit is ∞ or $-\infty$, say so. If the limit does not exist, write DNE.

(a) $\lim_{x\to 0} \frac{\sin 2x}{x}$.

$$= \lim_{x\to 0} \frac{2\cos 2x}{1} = 2$$

(b) $\lim_{x\to 2} \frac{x^2-2x+1}{x^3-9x^2+3}$.

$$= \frac{1}{8-36+3} = \frac{1}{-25} = -\frac{1}{25}$$

(c) $\lim_{x\to\infty} \frac{e^{3x}-2-3x}{x^2}$.

$$= \lim_{x\to\infty} \frac{3e^{3x}-3}{2x} = \lim_{x\to\infty} \frac{9e^{3x}}{2} = \infty$$

(d) $\lim_{x\to 0^+} x^2 \ln x$.

$$\lim_{x\to 0^+} \frac{\ln x}{x^{-2}} = \lim_{x\to 0^+} \frac{1}{-2x\cdot x^{-3}} = -\frac{1}{2}\lim_{x\to 0^+} \frac{x^3}{x} = -\frac{1}{2}\lim_{x\to 0^+} x^2 = 0$$

(e) $\lim_{v\to 4^-} \frac{4-v}{|4-v|}$. for $v<4$, $|4-v| = 4-v$

So its $\lim_{v\to 4^-} \frac{4-v}{4-v} = 1$

5. Determine the constant a such that the function $f(x) = x^2 + \frac{a}{x}$ has a local minimum at $x = 2$. Does the function have a ~~relative~~ global maximum anywhere?

$$f'(x) = 2x - \frac{a}{x^2}$$

Want $f'(2) = 0$ for local min.

So $f'(2) = 4 - \frac{a}{4} = 0$

$$\therefore a = 16$$

As $x \to \infty$, $f(x) \to \infty$.

So it cannot have a global maximum anywhere

(it just gets bigger and bigger ...).

6. Find local maxima, minima and points of inflection for the function $f(x) = x^2 e^{5x}$. Sketch the graph.

$$f'(x) = 2xe^{5x} + 5x^2 e^{5x} = 0$$

$$x e^{5x}(2+5x) = 0$$

local min   local max

$$\boxed{x = 0 \quad \text{or} \quad x = -\frac{2}{5}}$$

critical point

$$f''(x) = 2e^{5x} + 10xe^{5x} + 10xe^{5x} + 25x^2 e^{5x}$$

$$= e^{5x}(25x^2 + 20x + 2) = 0$$

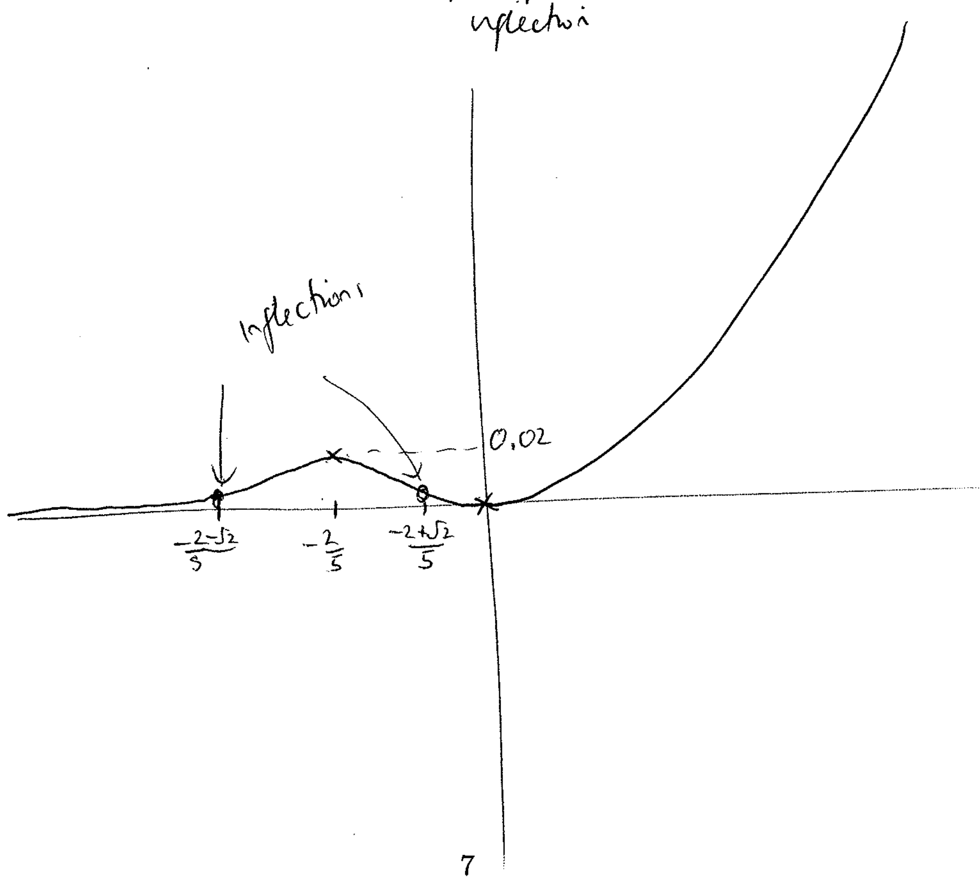
$$f''(0) = 2 > 0$$

$$f''(-\tfrac{2}{5}) = -0.27 < 0$$

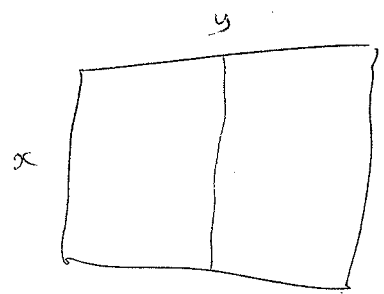
$$x = \frac{-20 \pm \sqrt{200}}{50} = \boxed{-\frac{2}{5} \pm \frac{\sqrt{2}}{5}}$$

-0.68, -0.12

point of inflection



7. A farmer wants to fence an area of 1.5 million square feet in a rectangular field, then divide it in half with a fence parallel to one of the sides of the rectangle. How should he do this so as to minimize the cost of fence?



$xy = 1.5$

$\therefore y = \frac{1.5}{x}$

$F(x) = 3x + 2y = \underline{\underline{3x + \frac{3}{x}}}$

fencing needed

minimize on $(0, \infty)$

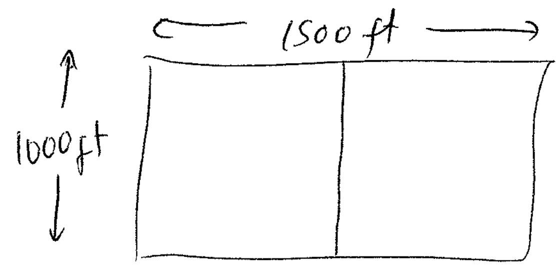
not at endpoints!

$\therefore$ Critical points.

$F'(x) = 3 - \frac{3}{x^2} = 0$

$x^2 = 1$

$x = 1, \quad y = 1.5.$

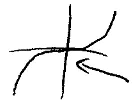


answer.

8. True or False? Justify your answers briefly if you can.

(a) If $f'(c) = 0$ then f has a local maximum or minimum at c.

False eg $f(x) = x^3$



$f'(0) = 0$

not max or min.

(b) If f has a global ~~min~~ max imum at c then $f'(c) = 0$.

False eg

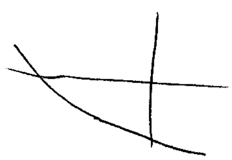
$f'(c)$ DNE here but global max!

(c) If $f'(x) = g'(x)$ for all x then $f(x) = g(x)$.

False eg $f(x) = x^2$ $\neq$ $g(x) = x^2 + 57$ $\Big\rangle$ $f'(x) = 2x = g'(x)$

(d) There exists a function $f(x)$ such that $f(x) < 0$, $f'(x) < 0$ and $f''(x) > 0$ for all x.

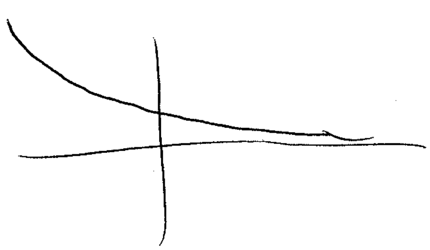
negative decreasing concave up



$\leftarrow$ False – can't do it!

(e) There exists a function $f(x)$ such that $f(x) > 0$, $f'(x) < 0$ and $f''(x) > 0$ for all x.

positive decreasing concave up



True – eg $f(x) = e^{-x}$