Math 232 Rapid review

① Recurrence relations ← §10.1, §10.2

To solve $a_{n+2} + b a_{n+1} + c a_n = 0$: (b, c constants) ← homogeneous, linear, second order

Characteristic equation $x^2 + bx + c = 0$. Say that has roots r and s.

Then $a_n = \begin{cases} A r^n + B s^n & \text{if } r \neq s \\ A r^n + B n s^n & \text{if } r = s \end{cases}$

Finally use knowledge of a_0, a_1 to find constants A and B

eg Fibonacci $a_0 = 0$ $a_1 = 1$ $a_{n+1} = a_n + a_{n-1}$: $a_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$

② Graphs ← §11.1 – §11.5

Definitions of graphs, digraphs, multigraphs. Adjacency matrices. $|E|$ #edges $|V|$ #vertices

$K(G)$ = #connected components $\overline{G}$ = complement of G

Basic identity : $2|E| = \sum_{v \in V} \deg(v)$ $\deg(v)$ = #edges ending on v $\deg(\ ) = 2$

Walks, trails, paths, circuits, cycles.

G (connected) has Euler trail if and only if every vertex has even degree

Examples of K_n, $K_{m,n}$, Q_n, Peterson. Notion of bipartite graph

The definition of graph isomorphism – examples that are, examples that are not

Planar graphs. Statement of Kuratowski's theorem (subgraph homeomorphic to K_5 or $K_{3,3}$)

Euler's theorem for planar graphs: $v - e + r = 2$. Proofs that K_5 and $K_{3,3}$ are not planar.

Platonic solids, their graphs and dual graphs.

Examples of Hamilton cycles, eg dodecahedron.

③ Trees ← §12.1, §12.2

Tree = connected graph, no cycles $\iff$ connected, $|V| = |E| + 1$ $\iff$ no cycles, $|V| = |E| + 1$

In a tree, there's a unique path between any two vertices

Any tree has at least two leaves/pendant vertices (Proof?)

A graph is connected $\iff$ it has a spanning tree. ← spanning subgraph that is a tree.

Rooted tree = directed tree with unique root ← $\begin{cases} id = 0 & \text{"in degree"} \\ \text{all other vertices have } id = 1 \end{cases}$

Ordered tree = rooted tree where branches are ordered from left to right

m-ary tree
binary tree } ordered tree where vertices have $od = 0$ or $od = m$
"leaves" "internal vertices"

Lexicographic ordering — Polish notation

Labelled trees — there are n^{n-2} of these with n vertices
Proofs using (a) multinomial coefficients (b) Prüfer codes

④ Generating functions ← §10.4, §10.5

Power series, binomial theorem $(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots$ ↖ any $n \in \mathbb{R}$!!!

$$\binom{n}{r} = \frac{n(n-1)\cdots(n-r+1)}{r!}$$

Special cases:

$$\left.\begin{aligned} \frac{1}{1-x} &= 1 + x + x^2 + \cdots + x^n + \cdots \\ \frac{1}{(1-x)^2} &= 1 + 2x + 3x^2 + \cdots + (n+1)x^n + \cdots \\ \frac{1}{(1-x)^3} &= 1 + 3x + 6x^2 + \cdots + \tfrac{1}{2}(n+1)(n+2)x^n + \cdots \end{aligned}\right\} \text{differentiate!}$$

Examples of solving (possibly inhomogeneous) recurrence relations via generating functions

Binary trees — there are $\frac{1}{n+1}\binom{2n}{n}$ binary rooted trees with n vertices

↑ b_n = nth Catalan number $\begin{cases} b_0 = 1 \\ b_{n+1} = b_0 b_n + b_1 b_{n-1} + \cdots + b_n b_0 \end{cases}$

⑤ Algorithms §5.7, §12.3, §13.1, §13.2

Big-Oh notation

Bubble sort and merge sort algorithms, their complexity.

Reconstruction of labelled tree from its Prüfer code

Dijkstra's shortest path algorithm, practical examples
Prim's minimal spanning tree algorithm } proofs and complexity.