Solutions to my question 1

(a) $a_{n+1} = a_n + 3^n \qquad a_0 = 1$

$$f(x) = \sum_{n\geq 0} a_n x^n = 1 + \sum_{n\geq 1} a_n x^n = 1 + x\sum_{n\geq 0} a_{n+1} x^n$$

$$= 1 + x\sum_{n\geq 0}(a_n + 3^n)x^n = 1 + xf(x) + \frac{x}{1-3x}$$

$$\therefore (1-x)f(x) = 1 + \frac{x}{(1-3x)}$$

$$\therefore \quad f(x) = \frac{1}{1-x} + \frac{x}{(1-x)(1-3x)}$$

$$\therefore a_n = 1 + \tfrac{1}{2}\cdot 3^n - \tfrac{1}{2} = \underline{\underline{\tfrac{1}{2} + \tfrac{1}{2}\cdot 3^n}}$$

<u>Partial fractions</u>

$$\frac{x}{(1-x)(1-3x)} = \frac{A}{1-x} + \frac{B}{1-3x}$$

$$A(1-3x) + B(1-x) = x$$

$$\therefore \left.\begin{array}{l} A + B = 0 \\ 3A + B = -1 \end{array}\right\} \quad A = -\tfrac{1}{2},\ B = \tfrac{1}{2}$$

$$\frac{x}{(1-x)(1-3x)} = \frac{1}{2}\cdot\frac{1}{1-3x} - \frac{1}{2}\cdot\frac{1}{1-x}$$

(b) $a_{n+1} = 2a_n + 2^n \qquad a_0 = 0$

$$f(x) = \sum_{n\geq 1} a_n x^n = x\sum_{n\geq 0} a_{n+1}x^n$$

$$= x\sum_{n\geq 0}(2a_n + 2^n)x^n$$

$$= 2xf(x) + \frac{x}{1-2x}$$

$$\therefore (1-2x)f(x) = \frac{x}{1-2x}$$

$$\therefore \quad f(x) = \frac{x}{(1-2x)^2} = \frac{1}{2}\cdot\frac{2x}{(1-2x)^2}$$

$$\frac{x}{(1-x)^2} = x + 2x^2 + \dots + nx^n + \dots$$

$$\therefore \frac{2x}{(1-2x)^2} = \dots + n(2x)^n + \dots$$

$$\therefore \quad a_n = \tfrac{1}{2}\cdot 2^n\cdot n = \underline{\underline{n\cdot 2^{n-1}}}$$

(c) $a_{n+1} = a_n + n^2 \qquad a_0 = 1 \qquad$ Let $f(x) = \sum_{n \geq 0} a_n x^n$

$$f(x) = 1 + x \sum_{n \geq 0} (a_n + n^2) x^n \qquad \text{(as usual)}$$
$$= 1 + x f(x) + x \sum_{n \geq 0} n^2 x^n$$

$$\therefore (1-x) f(x) = 1 + \frac{2x}{(1-x)^3} - \frac{3x}{(1-x)^2} + \frac{x}{1-x}$$

$$n^2 = 2 \cdot \frac{(n+1)(n+2)}{2} - 3(n+1) + 1$$

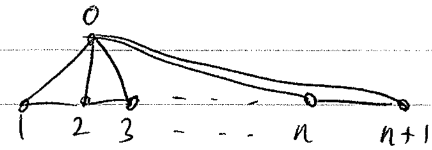
$$\therefore \sum_{n \geq 0} n^2 x^n = \frac{2}{(1-x)^3} - \frac{3}{(1-x)^2} + \frac{1}{1-x}$$

$$\therefore f(x) = \frac{1}{1-x} + \frac{x}{(1-x)^2} + \frac{2x}{(1-x)^4} - \frac{3x}{(1-x)^3}$$

$$\therefore a_n = 1 + n + \frac{2}{6} n(n+1)(n+2) - \frac{3}{2} n(n+1)$$

$$\therefore a_n = \frac{1}{6}(n+1)\left[6 + 2n(n+2) - 9n\right] = \frac{1}{6}(n+1)(2n^2 - 5n + 6)$$

Solutions to Supp. Ex. to Ch. 12 #20

20 

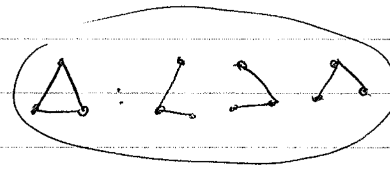
t_{n+1} = # spanning trees

$t_0 = 1$

(a) Find a recurrence for t_n.

$t_1 = 1$

The following is okay for $n \geq 1$:

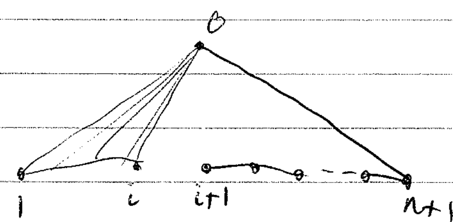
$t_2 = 3$ 

t_{n+1} = # spanning trees without edge $0-(n+1)$

+ # spanning trees with edge $0-(n+1)$

= t_n + $\underbrace{\#\text{ spanning trees with edge } 0-(n+1)}$

Consider such with edges $i-(i+1)$ removed but $(i+1)-(i+2), \dots, n-(n+1)$ all present $(0 \leq i \leq n)$



all edges $0-(i+1), \dots, 0-n$ must be gone too!

∴ There are t_i such.

Correct even for $i=0$!

$$\Rightarrow \boxed{\underset{(n\geq 1)}{t_{n+1} = t_n + \sum_{i=0}^{n} t_i} \qquad t_0 = 1 = t_1}$$

(b) $$t_{n+1} - t_n = \left(t_n + \sum_{i=0}^{n} t_i\right) - \left(t_{n-1} + \sum_{i=0}^{n-1} t_i\right)$$ okay for $n \geq 2$ but not for $n=1$!

$$= t_n + t_n - t_{n-1}$$

$$\therefore \boxed{t_{n+1} = 3t_n - t_{n-1}} \qquad t_0 = 1 \qquad \boxed{t_1 = 1 \quad t_2 = 3 \quad t_3 = 8}$$

good here

(c) Solve! Char. poly: $x^2 - 3x + 1 = 0$ $\quad x = \frac{3}{2} \pm \sqrt{5}$

$$\therefore t_n = A\left(\frac{3}{2} + \frac{\sqrt{5}}{2}\right)^n + B\left(\frac{3}{2} - \frac{\sqrt{5}}{2}\right)^n \quad \text{for } n \geq 1$$

$n=1$: $1 = \frac{3}{2}(A+B) + \frac{\sqrt{5}}{2}(A-B)$ $\quad \therefore A = \frac{1}{\sqrt{5}}$

$t_0 := 0 \rightarrow n=0$: $0 = A + B$ $\quad \therefore B = -A$

follows recurrence!

Compare with ex. 10.10:

$$F_{2n} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{2n} - \left(\frac{1-\sqrt{5}}{2}\right)^{2n}\right]$$

$$\boxed{\therefore t_n = \frac{1}{\sqrt{5}}\left(\frac{3+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{3-\sqrt{5}}{2}\right)^n} \quad (n \geq 1)$$

$$\left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{3+\sqrt{5}}{2} \qquad \left(\frac{1-\sqrt{5}}{2}\right)^2 = \frac{3-\sqrt{5}}{2}$$

So $t_n = F_{2n}$!