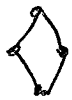
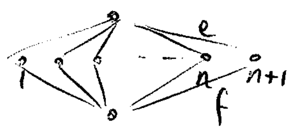
HW7 Selected solutions

Supp. ex. to Ch. 12 : # 12

$t_n = \#$ spanning trees of $K_{2,n}$

 $t_1 = 1$  $t_2 = 4$

In general, 

$t_{n+1} = 2 \cdot t_n + 2^n$

spanning trees involving one of e or f only

spanning trees involving both e and f (must erase one of each edge pair 

$1 \leq i \leq n$)

So to find t_n, must solve recurrence

$$t_1 = 1 \qquad t_{n+1} = 2t_n + 2^n. \qquad \text{Note } t_0 = 0$$

Let $f(x) = \sum_{n \geq 0} t_n x^n = x \sum_{n \geq 0} t_{n+1} x^n = x \sum_{n \geq 0} (2t_n + 2^n) x^n$

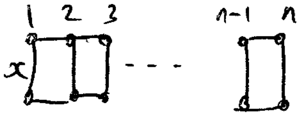
$$= 2x \sum_{n \geq 0} t_n x^n + x \sum_{n \geq 0} (2x)^n = 2x f(x) + \frac{x}{1-2x}$$

$$\therefore f(x)[1-2x] = \frac{x}{1-2x}$$

$$\therefore f(x) = \frac{x}{(1-2x)^2} = x + 2 \cdot 2x^2 + 3 \cdot 4x^3 + 4 \cdot 8x^4 + \cdots + n \cdot 2^{n-1} x^n + \cdots$$

$$\therefore \boxed{t_n = n \cdot 2^{n-1}}$$

Supp ex. to Ch. 12 : #13

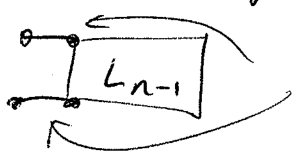


L_n "Ladder"

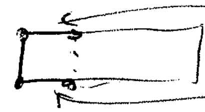
a_n = # spanning trees in L_n b_n = # spanning trees with edge x present.

$$a_n = a_{n-1} + b_n$$

$\uparrow$ a spanning tree without edge x is same as spanning tree for L_{n-1} with



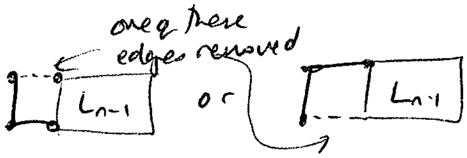
two edges added.



And these have neither edge removed.

$$b_n = 2a_{n-1} + b_{n-1}$$

$\uparrow$ These are the ones with



So must solve: $\left.\begin{array}{l} a_n = a_{n-1} + b_n \\ b_n = 2a_{n-1} + b_{n-1} \end{array}\right\}$ $b_n = a_n - a_{n-1}$ $\therefore b_{n-1} = a_{n-1} - a_{n-2}$

Well $a_n = a_{n-1} + (2a_{n-1} + b_{n-1}) = 3a_{n-1} + b_{n-1} = 3a_{n-1} + (a_{n-1} - a_{n-2})$

$$\left.\begin{array}{l} a_n = 4a_{n-1} - a_{n-2} \\ a_0 = 0 \quad a_1 = 1 \quad a_2 = 4 \end{array}\right\}$$

For this we use method of §10.2. Characteristic equation is $x^2 - 4x + 1 = 0$

roots $2 \pm \sqrt{3}$

$\therefore$ General solution is $a_n = A(2+\sqrt{3})^n + B(2-\sqrt{3})^n$

$n=0$ gives $A + B = 0$ $\therefore a_n = A\left[(2+\sqrt{3})^n - (2-\sqrt{3})^n\right]$

$n=1$ gives $A\left[2+\sqrt{3}-2+\sqrt{3}\right] = 1$ $\therefore A = \frac{1}{2\sqrt{3}}$

$$\therefore a_n = \frac{1}{2\sqrt{3}}\left[(2+\sqrt{3})^n - (2-\sqrt{3})^n\right]$$