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Winter term, 2009

Discrete Mathematics II Practise Final

Name: _____

1	2	3	4	5	6	7	8	TOT.

Answer ALL questions. Each question is worth FIVE points. Show all your work and show your working – even if you give the correct answer you will not get full marks without it.

1. Draw the following graphs:

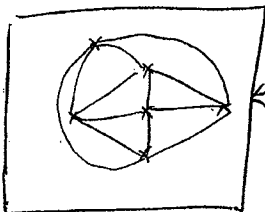
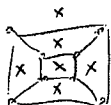
(a) K_5 .



(b) $K_{2,3}$.



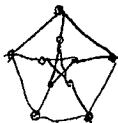
(c) The dual graph to the planar graph Q_3 . Why is this graph also known as the octahedron?



the cube

Its the graph with vertices of octahedron projected onto the plane ...

(d) The Peterson graph.



(e) A forest with two connected components and ten edges.



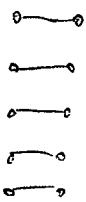
2. Find the maximal length of a circuit in the following graphs:

(a) $K_{4,4}$.



every vertex has degree 4, even, so $K_{4,4}$ has an Euler circuit - that's the longest circuit, length $\boxed{16}$

(b) $K_{5,5}$.



If we remove the edges shown we get a graph in which every vx has degree 4, even, hence an Euler circuit. That's the best we can do - a circuit of length $\boxed{20}$

ie remove 5 edges from 25

(c) K_7 .

Every vx has degree 6, even, so it has Euler circuit, length $\frac{1}{2} \cdot 7 \cdot 6 = \boxed{21}$

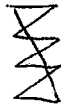
(d) K_8 .



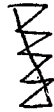
Remove 4 edges to produce regular graph of degree 6 with 8 vertices, which has Euler circuit length $\frac{1}{2} \cdot 8 \cdot 6 - 4 = \boxed{24}$

Now find the maximal length of a cycle in the same graphs.

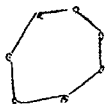
$K_{4,4}$: length 8
(Hamilton) cycle



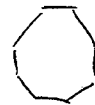
$K_{5,5}$: length 10
(Hamilton)



K_7 : length 7
(Hamilton)



K_8 : length 8
(Hamilton)



3. (a) Write down the general formula for the number of labelled trees with n vertices.

n^{n-2} — fact proved in class

(b) How many spanning trees does the complete graph K_n have? Explain.

n^{n-2} — It's the same problem as in (a) — label the vertices of K_n by $1, 2, \dots, n$, then a spanning tree is the same thing as a labelled tree as in (a).

(c) How many labelled trees with 10 vertices are there in which the vertices numbered $1, 2, \dots, 10$ have degrees $1, 1, 2, 2, 1, 1, 3, 3, 2, 2$, respectively?

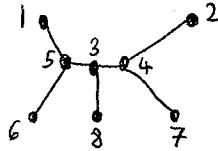
We showed in class for d_1, \dots, d_n with $\sum_{i=1}^n d_i = 2n-2$ that the number of labelled trees with these degrees is $\frac{n!}{2^{n-2} (d_1-1)! \dots (d_{n-1}-1)!}$ (multinomial)

So here it's $\frac{10!}{2 \cdot 2} = \boxed{10,080}$

(d) How many labelled trees with 10 vertices are there in which the vertices numbered $1, 2, \dots, 10$ have degrees $1, 1, 2, 2, 3, 3, 3, 3, 2, 2$, respectively?

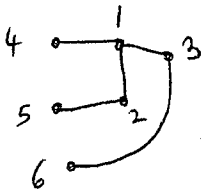
$\boxed{0}$ 9 edges $\therefore 18 = \text{sum of degrees}$ — not the case here!

(e) Find the Prüfer code of the following labelled tree:



5, 4, 5, 3, 4, 3

(f) Which labelled tree has Prüfer code $1, 2, 1, 3$? 6 vertices, leave $\{4, 5, 6\}$



4. (a) Suppose that T is a tree with 12 edges. How many vertices does T have?

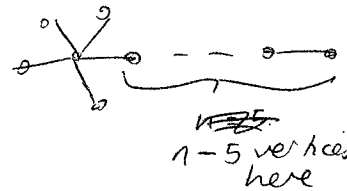
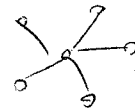
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(b) For which n does there exist a tree T with n edges and 5 pendant vertices? Explain.

Not possible with ≤ 5 vertices

Possible for $n = 6$

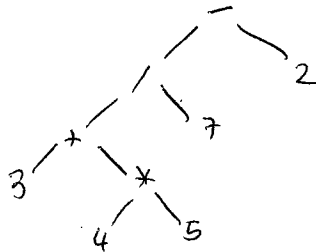
Possible for any $n \geq 6$



(c) Draw a binary tree representing the expression

$$(3 + (4 * 5)) / 7 - 2.$$

Hence rewrite this expression in Polish notation.



- / + * 4 5 7 2

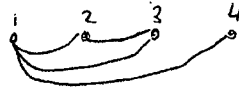
(d) I am organizing a knockout soccer tournament for 3000 teams. What is the fewest number of matches that need to be played to decide the winner? How many rounds of matches are needed?

Next power of 2 down is $2048 = 2^{11}$ which takes
 11 rounds of matches, plus 952 more matches (1 round)
 to cut from 3000 to 2048 \rightarrow 12 rounds
2999 matches

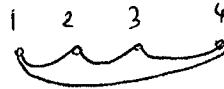


5. (a) Draw the loop-free undirected graphs with the following adjacency matrices:

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$



$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$



Are these two graphs isomorphic? Explain.

No - the first has a vertex of degree 3, the second does not.

(b) Work out how many different spanning trees each of the graphs you drew in (a) has.

The first: 3 spanning trees

The second: 4 spanning trees

(c) Use Euler's formula to prove that the graph K_5 is not a planar graph.

Suppose K_5 is planar. Then $v - e + r = 2$ $v = 5, e = \frac{1}{2} \cdot 5 \cdot 4 = 10$
 $\# \text{vs} \quad \# \text{edges} \quad \# \text{regions} \quad \therefore r = 7$

But $\sum_{A=\text{a region}} \text{deg}(A) = 2e = 20$ each region has degree at least 3 $\triangle A$
 7 regions

$$\therefore \sum_{A=\text{a region}} \text{deg}(A) \geq 3 \cdot 7 = 21 \quad \text{X}$$

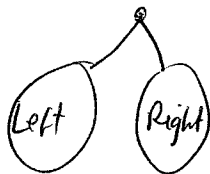
6. (a) There are five binary rooted trees with 3 vertices. Draw them all:



(b) Let t_n be the number of binary rooted trees with n vertices (in particular $t_0 = 1$ according to convention). Explain why

$$t_{n+1} = t_0 t_n + t_1 t_{n-1} + \dots + t_n t_0.$$

Take a binary tree with $n+1$ vertices



Say there are k vertices in its left subtree, so $0 \leq k \leq n$. That subtree is one of t_k different trees.

There are $(n-k)$ vertices in its right subtree, so one of t_{n-k} trees

$$\text{So } t_{n+1} = \sum_{k=0}^n t_k t_{n-k}$$

(c) Write down the sequence $t_0, t_1, t_2, t_3, t_4, t_5$. What are these numbers called?

1, 1, 2, 5, 14, 42 Catalan numbers

$$t_n = \frac{1}{n+1} \binom{2n}{n}$$

7. (a) Use the binomial theorem to write down the first five terms in the expansion of $(1 + 2x)^{\frac{1}{2}}$ as a power series:

$$\begin{aligned}
 & 1 + \binom{\frac{1}{2}}{1} \cdot 2x + \binom{\frac{1}{2}}{2} (2x)^2 + \binom{\frac{1}{2}}{3} (2x)^3 + \binom{\frac{1}{2}}{4} (2x)^4 \\
 &= \left(1 + x + \frac{\frac{1}{2} \cdot (-\frac{1}{2})}{2} \cdot 4x^2 + \frac{\frac{1}{2} \cdot (-\frac{1}{2}) \cdot (-\frac{3}{2})}{6} \cdot 8x^3 + \frac{\frac{1}{2} \cdot (-\frac{1}{2}) \cdot (-\frac{3}{2}) \cdot (-\frac{5}{2})}{24} \cdot 16x^4 \right) \\
 &= \left(1 + x - \frac{1}{2}x^2 + \frac{1}{2}x^3 - \frac{5}{8}x^4 \right)
 \end{aligned}$$

(b) Use the method of generating functions to solve the recurrence relation

$$a_{n+1} = 3a_n + 3^n(n+1), \quad a_0 = 1.$$

You might find it helpful to use the identities:

$$\begin{aligned}
 \frac{1}{(1-x)^2} &= 1 + 2x + 3x^2 + \dots + (n+1)x^n + \dots, \\
 \frac{1}{(1-x)^3} &= 1 + 3x + 6x^2 + \dots + \frac{1}{2}(n+1)(n+2)x^n + \dots.
 \end{aligned}$$

$$\begin{aligned}
 \frac{1}{1-3x} &= \dots + (3x)^n + \dots \\
 \frac{3x}{(1-3x)^2} &= \dots + \frac{1}{2}n(n+1)(3x)^n + \dots
 \end{aligned}$$

$$\begin{aligned}
 \text{Let } f(x) &= \sum_{n \geq 0} a_n x^n = 1 + x \sum_{n \geq 0} a_{n+1} x^n = 1 + x \sum_{n \geq 0} (3a_n + 3^n(n+1)) x^n \\
 &= 1 + 3xf(x) + x \sum_{n \geq 0} (n+1)(3x)^n
 \end{aligned}$$

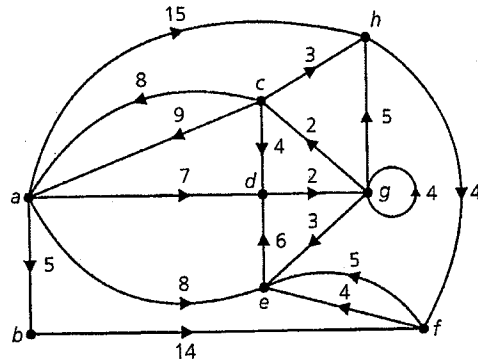
$$\therefore (1-3x)f(x) = 1 + \frac{x}{(1-3x)^2}$$

$$\therefore f(x) = \frac{1}{1-3x} + \frac{1}{3} \frac{3x}{(1-3x)^2}$$

$$\begin{aligned}
 \therefore a_n &= 3^n + \frac{1}{3} \cdot \frac{1}{2} n(n+1) 3^n \\
 &= \boxed{3^n + \frac{n(n+1)}{2} \cdot 3^{n-1}}
 \end{aligned}$$

check: $n=0 \checkmark$ $n=1$ $3+1=4 \checkmark$
 $n=2$ $9+9=18 \checkmark$

8. Apply Dijkstra's algorithm to the following weighted directed multigraph to find the shortest distance from vertex a to the other seven vertices of the graph. Show all the steps of the algorithm so I can see what you are doing!



$$S = \{a\}$$

$$\bar{S} = \{b, c, d, e, f, g, h\}$$

$$\bar{S} = \{c, d, e, f, g, h\}$$

$$\bar{S} = \{c, e, f, g, h\}$$

$$\bar{S} = \{c, f, g, h\}$$

$$\bar{S} = \{c, f, h\}$$

$$\bar{S} = \{f, h\}$$

$$\bar{S} = \{f\}$$

$a-b$ is shortest, length 5

$a-d$ is shortest, length 7

$a-e$ is shortest, length 8

$a-d-g$ is shortest, length 9

$a-d-g-c$ is shortest, length 11

$a-d-g-h$ is shortest, length 14

$a-d-g-h-f$ is shortest, length 18

$$S = \{a, b\}$$

$$S = \{a, b, d\}$$

$$S = \{a, b, d, e\}$$

$$S = \{a, b, d, e, g\}$$

$$S = \{a, b, c, d, e, g\}$$

$$S = \{a, b, c, d, e, g, h\}$$

Results

- a-b length 5
- a-d-g-c length 11
- a-d length 7
- a-e length 8
- a-d-g-h-f length 18
- a-d-g length 9
- a-d-g-h length 14

Shortest distances
and a shortest path.