Fall term, 1066

Discrete Mathematics I PRACTISE Second Midterm

Name: ______________________

1	2	3	4	5	6	7	8	TOT.

Answer ALL questions. Each question is worth TWO points. Show all your work and show your working – even if you give the correct answer you will not get full marks without it.

1. (a) Give an example of sets A, B and C such that

$$(A \setminus B) \cap C = (A \setminus C) \cap (B \setminus C).$$

$A = \emptyset$, $B = \emptyset$, $C = \emptyset$ does the job!

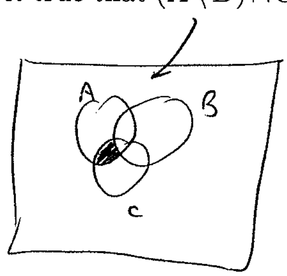
(b) Is it true that $(A \setminus B) \cap C = (A \setminus C) \cap (B \setminus C)$ for arbitrary sets A, B, C?

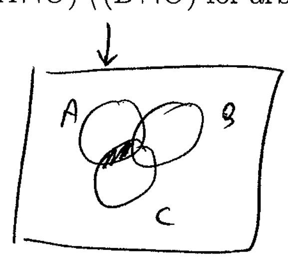
No, eg $A = \{1\}$, $B = \emptyset$, $C = \{1\}$

$(A \setminus B) \cap C = \{1\}$

$(A \setminus C) \cap (B \setminus C) = \emptyset$

(c) Is it true that $(A \setminus B) \cap C = (A \cap C) \setminus (B \cap C)$ for arbitrary sets A, B, C?





yep its true, by considering the Venn diagram

2. How many numbers between 1 and 1000 are divisible by 3, 5 or 7? Explain your method.

Inclusion-exclusion : $\left\lfloor \frac{1000}{3} \right\rfloor + \left\lfloor \frac{1000}{5} \right\rfloor + \left\lfloor \frac{1000}{7} \right\rfloor - \left\lfloor \frac{1000}{15} \right\rfloor - \left\lfloor \frac{1000}{21} \right\rfloor - \left\lfloor \frac{1000}{35} \right\rfloor + \left\lfloor \frac{1000}{105} \right\rfloor$

$= 333 + 200 + 142 - 66 - 47 - 28 + 9$

$= \underline{\underline{619}}$

3. Let $a = 53, b = 42$. Find $gcd(a, b)$. Also find integers s and t such that $sa + tb = gcd(a, b)$.

$53 = 42 + 11$

$42 = 3 \cdot 11 + 9$

$11 = 9 + 2$

$9 = 4 \cdot 2 + 1$

$1 = 9 - 4 \cdot 2 = 9 - 4 \cdot (11 - 9)$

$= 5 \cdot 9 - 4 \cdot 11 = 5 \cdot (\cancel{42}\ 42 - 3 \cdot 11) - 4 \cdot 11$

$= 5 \cdot 42 - 19 \cdot 11 = 5 \cdot 42 - 19 \cdot (53 - 42)$

$= \underset{t}{24} \cdot \underset{b}{42} - \underset{s}{19} \cdot \underset{a}{53}$

$gcd = \underline{\underline{1}}$

$\underline{\underline{s = -19}}$ $\underline{\underline{t = 24}}$

4. In this question you may use the formula $\sum_{i=1}^{n} i = \frac{1}{2}n(n+1)$ if you need to.

(a) Work out the expression $(2i+3)$ for $i = 1, 2, 3$ and 4. Hence calculate $\sum_{i=1}^{4}(2i+3)$.

5, 7, 9, 11 $\quad$ 32

(b) Find a general formula for $\sum_{i=1}^{n}(2i+3)$. (It might be wise to check your answer by making sure your formula agrees with your answer to (a) for $n = 4$!)

$$\sum_{i=1}^{n}(2i+3) = 2\sum_{i=1}^{n} i + \sum_{i=1}^{n} 3 = 2\cdot\tfrac{1}{2}n(n+1) + 3n$$

$$= \underline{\underline{n^2 + 4n}}$$

(c) Give a proof of your formula in (b) by mathematical induction.

Base case $n=1$: $5 = 1^2 + 4\cdot 1$ ✓

Induction step Assume true for $n=k$, i.e.

$$\sum_{i=1}^{k}(2i+3) = k^2 + 4k$$

Add $2(k+1)+3$ to both sides:

$$\sum_{i=1}^{k}(2i+3) + 2(k+1) + 3 = k^2 + 4k + 2(k+1) + 3$$

$$\therefore \sum_{i=1}^{k+1}(2i+3) = k^2 + 6k + 5 = (k+1)^2 + 4(k+1) \checkmark$$

Done by PMI

5. I've just written the numbers 1 through 10 on ten separate pieces of paper, folded each of them in half and put them in a hat. Then I draw out first one then a second piece of paper. What is the probability that the number on the second piece is greater or equal to the number on the first piece?

There are 10×9 ways of picking in total

$9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 = \frac{1}{2} \cdot 9 \cdot 10 = 45$ ways

first $= 1$, second ≥ 2 ; first $= 2$, second ≥ 3 ; . . . ; first $= 9$, second $= 10$

go small big

$\therefore$ Prob. $= \frac{45}{90} = \frac{1}{2}$ Should have been obvious!

6. Consider the sequence $a_0, a_1, a_2, \ldots$ defined recursively be

$$a_0 = 1, a_1 = 2, a_2 = 3,$$
$$a_n = a_{n-1} + a_{n-2} + a_{n-3} \text{ for } n \geq 3.$$

Use mathematical induction to prove that $a_n \leq 3^n$ for all $n \geq 0$.

Base case : $n = 0, 1$ and 2 :

$a_0 = 1 \leq 3^0$ ✓
$a_1 = 2 \leq 3^1$ ✓
$a_2 = 3 \leq 3^2$ ✓

Induction step Assume true for $n = k-1$, $n = k-2$ and $n = k-3$ $(k \geq 3)$ some fixed

Consider $n = k$:

$a_k = a_{k-1} + a_{k-2} + a_{k-3} \underset{\text{using hypothesis}}{\leq} 3^{k-1} + 3^{k-2} + 3^{k-3}$

$\leq 3^{k-1} + 3^{k-1} + 3^{k-1} = 3 \cdot 3^{k-1} = 3^k$ ✓

Done by PMI

7. On the Island of Knights and Liars, there are two villages. All the residents of one of the villages are liars and all the residents of the other are knights. Liars always lie and knights always tell the truth.
On a recent visit to this island, you met a group of three locals. The first of them said that the other two are from the same village. The second also said that the other two are from the same village. What did the third respond when you ask him if the other two are from the same village?
GIVE A COMPLETE PROOF THAT YOUR ANSWER IS CORRECT!

Local 1	Local 2	Local 3	local 1's response	local 2's response	Local 3's response
K	K	K	Yes	Yes	Yes
K	K	L	No	No	No
K	L	K	No	No	No
K	L	L	Yes	Yes	Yes
L	K	K	No	No	No
L	K	L	Yes	Yes	Yes
L	L	K	Yes	Yes	Yes
L	L	L	No	No	No

local 1 and local 2 both responded Yes
from the table you can see local 3 must respond yes too!
(In fact all three always give the same answer)

8. (a) Let a and b be integers with $b \neq 0$. What does it meet precisely to say that b divides a?

$a = bk$ for some integer k

(b) Suppose that $d|a$ and $d|b$. Prove that d divides any linear combination $sa + tb$ of a and b.

$d|a \therefore a = dk$ some integer k

$d|b \therefore b = dl$ some integer l

$sa + tb = sdk + tdl = d(\underbrace{sk + tl}_{\text{an integer}})$

$\therefore d \mid sa + tb$

(c) Given non-zero integers a, b and q, prove that $gcd(a, b) = gcd(b, a - qb)$.

If $d|a$ and $d|b$ then $d|b$ and $d|a-qb$

If $d|b$ and $d|a-qb$ then $d|a = (a-qb)+qb$ and $d|b$

$\therefore$ The common divisors of a and b are the same as the common divisors of b and $a - qb$

$\therefore gcd(a,b) = gcd(b, a-qb)$