

Fall term, 1066

Discrete Mathematics I PRACTISE Second Midterm

Name: _____

1	2	3	4	5	6	7	8	TOT.

Answer ALL questions. Each question is worth TWO points. Show all your work and show your working – even if you give the correct answer you will not get full marks without it.

1. (a) Give an example of sets A, B and C such that

$$(A \setminus B) \cap C = (A \setminus C) \cap (B \setminus C).$$

(b) Is it true that $(A \setminus B) \cap C = (A \setminus C) \cap (B \setminus C)$ for arbitrary sets A, B, C ?

(c) Is it true that $(A \setminus B) \cap C = (A \cap C) \setminus (B \cap C)$ for arbitrary sets A, B, C ?

2. How many numbers between 1 and 1000 are divisible by 3, 5 or 7? Explain your method.

3. Let $a = 53, b = 42$. Find $\gcd(a, b)$. Also find integers s and t such that $sa + tb = \gcd(a, b)$.

4. In this question you may use the formula $\sum_{i=1}^n i = \frac{1}{2}n(n+1)$ if you need to.

(a) Work out the expression $(2i+3)$ for $i = 1, 2, 3$ and 4 . Hence calculate $\sum_{i=1}^4 (2i+3)$.

(b) Find a general formula for $\sum_{i=1}^n (2i+3)$. (It might be wise to check your answer by making sure your formula agrees with your answer to (a) for $n = 4$!)

(c) Give a proof of your formula in (b) by mathematical induction.

5. I've just written the numbers 1 through 10 on ten separate pieces of paper, folded each of them in half and put them in a hat. Then I draw out first one then a second piece of paper. What is the probability that the number on the second piece is greater or equal to the number on the first piece?

6. Consider the sequence a_0, a_1, a_2, \dots defined recursively by

$$\begin{aligned} a_0 &= 1, a_1 = 2, a_2 = 3, \\ a_n &= a_{n-1} + a_{n-2} + a_{n-3} \text{ for } n \geq 3. \end{aligned}$$

Use mathematical induction to prove that $a_n \leq 3^n$ for all $n \geq 0$.

7. On the Island of Knights and Liars, there are two villages. All the residents of one of the villages are liars and all the residents of the other are knights. Liars always lie and knights always tell the truth.

On a recent visit to this island, you met a group of three locals. The first of them said that the other two are from the same village. The second also said that the other two are from the same village. What did the third respond when you ask him if the other two are from the same village?

GIVE A COMPLETE PROOF THAT YOUR ANSWER IS CORRECT!

8. (a) Let a and b be integers with $b \neq 0$. What does it mean precisely to say that b divides a ?

(b) Suppose that $d|a$ and $d|b$. Prove that d divides any linear combination $sa + tb$ of a and b .

(c) Given non-zero integers a, b and q , prove that $\gcd(a, b) = \gcd(b, a - qb)$.