Fall 2006

Discrete Mathematics I Midterm

Name: ____________________

1	2	3	4	5	TOT.

Answer ALL questions. Each question is worth THREE points. Show all your work and show your working – even if you give the correct answer you will not get full marks without it.

1. (a) How many integer solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 = 100$$

with $x_1, x_2, x_3, x_4 \geq 0$?

$\binom{103}{3}$ ← think of as putting 100 balls in four boxes, three barriers

100 0's, 3 1's $\frac{103!}{100!\,3!}$

(b) How many integer solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 = 100$$

with $x_1, x_2, x_3, x_4 > 0$?

Subtract one from each x ... get $\left.\begin{array}{l} y_1+y_2+y_3+y_4 = 96 \\ y_1, y_2, y_3, y_4 \geq 0 \end{array}\right\}$ which is a problem like (a)

So it's $\binom{99}{3}$

(c) How many integer solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 \leq 100$$

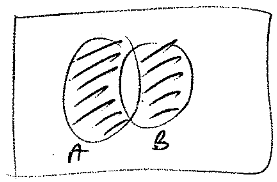
with $x_1, x_2, x_3, x_4 \geq 0$?

Let $x_5 = 100 - (x_1+x_2+x_3+x_4)$, so $\left.\begin{array}{l} x_1+x_2+x_3+x_4+x_5 = 100 \\ x_1, x_2, x_3, x_4, x_5 \geq 0 \end{array}\right\}$

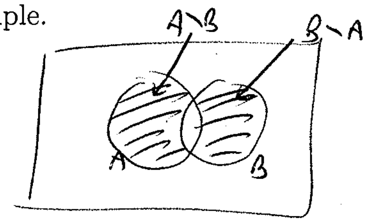
$\binom{104}{4}$ solutions.

2. In this question, A, B and C are arbitrary sets. You should remember the definitions of the sets $A \setminus B$ ("set minus") and $A \Delta B$ ("symmetric difference")...

(a) $A\Delta B = (A \setminus B) \cup (B \setminus A)$. True or False? Justify your answer by giving a proof or a counterexample.



$A \Delta B$



$(A \setminus B) \cup (B \setminus A)$

(T) by Venn diagram

(b) $(A\Delta B) \setminus C = A\Delta(B \setminus C)$. True or False? Justify your answer by giving a proof or a counterexample.

(F) eg $A = B = C = \{1\}$

$(A \Delta B) \setminus C = \emptyset$

$A \Delta (B \setminus C) = \{1\}$

(c) $(A \setminus B)\Delta C = A \setminus (B\Delta C)$. True or False? Justify your answer by giving a proof or a counterexample.

(F) eg $A = B = \{1\}$ $\quad C = \{1, 2\}$

$(A \setminus B) \Delta C = \{1, 2\}$

$A \setminus (B \Delta C) = \{1\}$

3. Answer the following True or False, justifying your answers carefully.

(a) $(p \wedge (\neg q)) \to (p \to q)$ is a tautology. Fa**l**se

No! take $p=1, q=0$

then $p \wedge (\neg q) = 1$ $\quad p \to q = 0$

$\therefore (p \wedge (\neg q)) \to (p \to q) = 0$ $\quad \therefore$ It's not a tautology.

(b) If x is a real number with the property that $x + y$ is irrational for every irrational number y, then x is rational.

True Suppose for a contradiction that x is irrational. Take $y = -x$. Then y is irrational, so by the given property $x+y$ is irrational. But $x+y=0$ which is rational ※

(c) If n is an integer such that n^2 is divisible by 4 then n is divisible by 4.

False eg $n=2$.

(d) $\sqrt{5}$ is rational.

~~Suppose for a contradiction that $\sqrt{5}$ is~~ False!

If it was, say $\sqrt{5} = \frac{a}{b}$, fraction in lowest terms. Then $5b^2 = a^2$ $\therefore a^2$ is divisible by 5 $\therefore a$ is divisible by 5, say $a = 5k$ $\therefore 5b^2 = 25k^2$ $\therefore b^2 = 5k^2$ $\therefore b^2$ is divisible by 5 $\therefore b$ is divisible by 5

This contradicts the assumption that $\frac{a}{b}$ was in its lowest terms.

(e) If x is rational and y is irrational than $x \cdot y$ is irrational.

False eg $x=0, y=\sqrt{2}, x \cdot y = 0$.

4. (a) How many arrangements are there of the word ABRACADABRA? How many of those involve no pair of consecutive A's?

$\frac{11!}{5!2!2!}$

No consecutive A's:

5 A's here

↓ B ↓ R ↓ C ↓ D ↓ B ↓ R ↓

$\frac{6!}{2!2!} \binom{7}{5}$

(b) Find simple expressions for each of the following:

- $\sum_{i=1}^{10}(2i-1)$. 100

- $\sum_{i=0}^{n}\binom{n}{i}2^i$. 3^n

- $\sum_{i=1}^{n}\binom{n}{i}2^i$. $3^n - 1$

5. True or False? If True give a proof, if False give a counterexample. (You might not need to draw out the full truth table to do this!)

(a) If $(q \wedge r) \Rightarrow p$ and $q \Rightarrow \neg r$ then p is true.

If q is false then $q \wedge r$ is false so p could be false

$\therefore$ This is <u>false</u> eg $\boxed{q=0,\ r=1,\ p=0.}$ is a counterexample.

$(q \wedge r) \rightarrow p$ is true, $q \rightarrow \neg r$ is true, p is false

(b) If $q \vee \neg r$ and $\neg(r \rightarrow q) \rightarrow \neg p$ are both true then p is true.

$\underbrace{\neg(r \rightarrow q) \rightarrow \neg p}_{p \rightarrow (r \Rightarrow q)}$

This is <u>false</u>, eg $\boxed{p=0,\ q=1,\ r=1}$ is a counterexample

$q \vee \neg r$ is true

$\neg(r \rightarrow q) \rightarrow \neg p$ is true

p is false

(c) If $p \Rightarrow (q \vee r)$, $q \Rightarrow s$ and $r \Rightarrow \neg p$ then $p \Rightarrow s$.

If p is false then $p \Rightarrow s$

If p is true then $q \vee r$

if q then s ✓

if r then $\neg p$ ✗

$\therefore$ $p \Rightarrow s$

This is <u>true</u>