Fall 1999

Discrete Mathematics I Final

Name:______________________________

1	2	3	4	5	6	7	8	TOT.

Answer ALL questions. Each question is worth FIVE points. Justify all your answers carefully and show your work!

1. (a) What does it mean to say an integer m *divides* an integer n? Prove carefully using your definition that if m divides n then m divides pn for any integer p.

(b) State the fundamental theorem of arithmetic as carefully as you can.

(c) Suppose $n \geq 1$ and $p_1 < p_2 < \cdots < p_n$ are distinct primes. How many positive divisors does the number $p_1 p_2 \cdots p_n$ (their product) have? Justify your answer carefully.

2. (a) Use the Euclidean algorithm to find integers s and t such that

$$31s + 97t = 1.$$

Show your working!

(b) Use mathematical induction to prove that $57|(7^{n+2}+8^{2n+1})$ for all $n \geq 0$.

3. (a) What does it mean to say that a function $f : A \to B$ is 1–1?

Onto?

Bijective?

Invertible?

(b) Suppose that $f : A \to B$ has a right inverse $g : B \to A$. Prove that f is onto.

(c) How many 1–1 functions are there from $\{1, 2, 3, 4, 5\}$ to $\{6, 7, 8, 9\}$? How many onto functions are there from $\{1, 2, 3, 4, 5\}$ to $\{6, 7, 8, 9\}$?

4. Calculate the truth tables of the following compound propositions. Which of them is a tautology?

(a) $[(p \to q) \vee (q \to r)] \to (p \to r)$.

(b) $[(p \to q) \wedge (q \to r)] \to (p \to r)$.

5. (a) Let $g : \mathbb{Z} \to \mathbb{N}$ be the function defined by $g(n) = |n|$. For each $n \in \mathbb{N}$, describe the set $g^{-1}(\{n\})$ of all pre-images of n. Is g 1–1 and/or onto?

(b) Answer the same questions as in (a) for the function $g : \mathbb{Z} \to \mathbb{N}$ defined by $g(n) = 1 + (-1)^n$.

6. (a) Write the sum $1 + 3 + 5 + \cdots + (2n - 1)$ of the first n odd numbers using the $\sum$ notation.

(b) Prove by mathematical induction that $1 + 3 + 5 + \cdots + (2n - 1) = n^2$.

7. How many integer solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 = 100$$

subject to the constraints $x_1 \geq 1, x_2 \geq 2, x_3 \geq 3$ and $x_4 \geq 4$? Explain.

8. (a) Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(xy) = xf(y) + f(x)y$ for every $x, y \in \mathbb{R}$. Prove that $f(x^n) = nx^{n-1}f(x)$.

(b) Solve the following recurrence relations:
(i) $2a_n - 3a_{n-1} = 0$ for $n \geq 1$, subject to the boundary condition $a_4 = 81$.

(ii) $a_n = 5a_{n-1} + 6a_{n-2}$ for $n \geq 2$, subject to the boundary conditions $a_0 = 1$ and $a_1 = 3$.