Theorem The number of compositions of r is 2^{r-1}

Example Compositions of 4:

$$\left.\begin{array}{lll} 4 & & \\ 3+1 & 1+3 & \\ 2+1+1 & 1+2+1 & 1+1+2 \\ 2+2 & & \\ 1+1+1+1 & & \end{array}\right\} 8 = 2^3$$

Proof There are lots of steps!

$\boxed{\text{Step 1}}$ The number of ways to put r balls in n boxes is $\binom{n+r-1}{r}$

Proof: $\left.\begin{array}{c} x\,x \mid x \mid x\,x \mid x \\ n-1 \text{ barriers} \\ r \text{ balls} \end{array}\right\}$ $\left.\begin{array}{c} \#\text{ of words with } n+r-1 \text{ letters,} \\ r \text{ of which are } x, \\ n-1 \text{ of which are } \mid \end{array}\right\}$ $\binom{n+r-1}{r}$ $\square$

$\boxed{\text{Step 2}}$ The number of integer solutions to the equation

$$\left.\begin{array}{l} x_1+\cdots+x_n=r \\ x_1,\ldots,x_n \geq 0 \end{array}\right\} \text{ is } \binom{n+r-1}{r}$$

Proof: It's the same problem as in step 1: x_i is the number of balls in the ith box $\square$

$\boxed{\text{Step 3}}$ The number of integer solutions to the equation ($r \geq n$ of course!)

$$\left.\begin{array}{l} x_1+\cdots+x_n=r \\ x_1,\ldots,x_n > 0 \end{array}\right\} \text{ is } \binom{r-1}{r-n}$$

Proof: Let $x_i' = x_i - 1$. Then $x_1',\ldots,x_n'$ solve the equation $\left.\begin{array}{l} x_1'+\cdots+x_n' = r-n \\ x_1',\ldots,x_n' \geq 0 \end{array}\right\}$

By Step 2 there are $\binom{n+r-n-1}{r-n} = \binom{r-1}{r-n}$ solutions, its the same as now! $\square$

$\boxed{\text{Step 4}}$ The number of compositions of r with n summands is $\binom{r-1}{r-n}$

Proof: Its the same problem as Step 3 $\boxtimes$

$\boxed{\text{Step 5}}$ The number of compositions of r is 2^{r-1}

Proof: Its $\sum_{n=1}^{r}$ (the number of compositions of r with n summands)

$$\stackrel{\text{Step 4}}{=} \sum_{n=1}^{r} \binom{r-1}{r-n}$$

$$= \binom{r-1}{r-1} + \binom{r-1}{r-2} + \cdots + \binom{r-1}{1} + \binom{r-1}{0}$$

= the sum of all the numbers on $(r-1)$th row of Pascals triangle

$= (1+1)^{r-1}$ by binomial theorem

$= 2^{r-1}$

$\boxtimes$