Solutions to practise midterm II

1. Let p, q and r be propositions.

(a) Calculate the truth table for the compound proposition $((p \vee q) \wedge r) \rightarrow ((p \wedge q) \vee r)$.

p	q	r	$p \vee q$	$(p \vee q) \wedge r$	$p \wedge q$	$(p \wedge q) \vee r$	$(p \vee q) \wedge r \rightarrow (p \wedge q) \vee r$
0	0	0	0	0	0	0	1
0	0	1	0	0	0	1	1
0	1	0	1	0	0	0	1
0	1	1	1	1	0	1	1
1	0	0	1	0	0	0	1
1	0	1	1	1	0	1	1
1	1	0	1	0	1	1	1
1	1	1	1	1	1	1	1

(b) Is $((p \vee q) \wedge r) \rightarrow ((p \wedge q) \vee r)$ a tautology?

Yes

(c) $((p \vee q) \wedge r) \Leftrightarrow ((p \wedge q) \vee r)$. True or False?

False (the columns $(p \vee q) \wedge r$ and $(p \wedge q) \vee r$ in the truth table aren't the same)

8

2. Let n be an integer. Prove carefully that n is a multiple of 3 if and only if $n^2 - 3n + 2$ is not divisible by 3.

($\Rightarrow$) Suppose n is a multiple of 3.

So $n = 3k$ for some $k \in \mathbb{Z}$

$\therefore n^2 - 3n + 2 = (3k)^2 - 3(3k) + 2$

$= 9k^2 - 9k + 2$

It's not a multiple of 3!

$\therefore n^2 - 3n + 2$ is not divisible by 3.

($\Leftarrow$) Let's check the contrapositive: if n is not a multiple of 3 then $n^2 - 3n + 2$ is divisible by 3.

Case one $n = 3k+1$ for some $k \in \mathbb{Z}$

$n^2 - 3n + 2 = (3k+1)^2 - 3(3k+1) + 2 = 9k^2 + 6k + 1 - 9k - 3 + 2$

$= 9k^2 - 3k$ ✓ a multiple of 3.

Case two $n = 3k+2$ for some $k \in \mathbb{Z}$

$n^2 - 3n + 2 = (3k+2)^2 - 3(3k+2) + 2 = 9k^2 + 12k + 4 - 9k - 6 + 2$

$= 9k^2 + 3k$ ✓ a multiple of 3

3. True or False? If True give a proof, if False give a counterexample.

(a) If n is an integer such that n^2 is divisible by 4 then n is divisible by 4.

False, eg $n = 2 \quad n^2 = 4 \quad \#$

(b) If n is an integer such that n^2 is divisible by 5 then n is divisible by 5.

True. Prove the contrapositive: if $5 \nmid n$ then $5 \nmid n^2$.

Cases: $n=5k+1 \quad n^2 = 25k^2+10k+1$ ✓ not div. by 5

$n=5k+2 \quad n^2 = 25k^2+20k+4$ ✓ "

$n=5k+3 \quad n^2 = 25k^2+30k+9$ ✓ "

$n=5k+4 \quad n^2 = 25k^2+40k+16$ ✓ "

(c) $\sqrt{5}$ is rational.

False. For if $\sqrt{5} = \frac{m}{n}$ is a fraction with $GCD(m,n)=1$

Then: $5n^2=m^2 \therefore 5|m^2 \therefore 5|m \therefore m=5k$ for some $k \in \mathbb{Z}$

$\therefore 5n^2=25k^2 \therefore n^2=5k^2 \therefore 5|n^2 \therefore 5|n$

$\therefore GCD(m,n) \geq 5 \quad \#$

(d) If x is rational and y is irrational than $x \cdot y$ is irrational.

False, eg $x=0,\ y=\sqrt{2},\ x \cdot y = 0$

4. Let $f : S \to T$ be a function. Write down precise definitions of the following:

(a) f is 1–1.

If $f(x) = f(x')$ then $x = x'$

(Different inputs map to different outputs is ok too)

(b) f is onto.

For every $y \in T$ there exists $x \in S$ such that $f(x) = y$.

("Image = Codomain" is ok too providing you define the image : $\{f(x) \mid x \in S\}$).

Now consider the following functions. For each of them work out whether they are 1–1, onto or both. Explain your answer.

(c) $f : \mathbb{R} \to \mathbb{R}, x \mapsto 10^x$.

1–1 , not onto — all outputs are >0 so eg -1 is not in image

has a left inverse eg : $x \mapsto \begin{cases} \log_{10} x & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$

(d) $f : \mathbb{P} \times \mathbb{P} \to \mathbb{P}, (x, y) \mapsto GCD(x, y)$.

not 1–1 (eg $GCD(2,3) = GCD(3,2)$)

onto as $n = GCD(n,n)$ for each $n \in \mathbb{P}$

(e) $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}, (x, y) \mapsto (x + y, x - y)$.

This is 1–1 , for if $f(x,y) = f(x',y')$ then $\left.\begin{array}{l} x+y = x'+y' \\ x-y = x'-y' \end{array}\right\}$ add : $2x = 2x'$ subtract : $2y = 2y'$

$\therefore x = x', y = y'$

$\therefore (x,y) = (x',y')$

it is not onto, eg $(0,1)$ is not in the image

because if $\left.\begin{array}{l} x+y=0 \\ x-y=1 \end{array}\right\}$ then adding gives $2x = 1$

$\therefore x = \frac{1}{2}$ which is not allowed $(x,y) \in \mathbb{Z} \times \mathbb{Z}$

5. True or False? If True give a proof, if False give a counterexample. (You might not need to draw out the full truth table to do this!)

(a) If $(q \wedge r) \to p$ and $q \to \neg r$ then p.

Means: $((q \wedge r) \to p) \wedge (q \to \neg r) \Rightarrow p$

This is false – to see that enough to show one row of the truth table that is not true. Take $p = q = r = 0$ ($q \wedge r = 0$ so $(q \wedge r) \to p = 1$ and $q \to \neg r = 1$ but $p = 0$.)

(b) If $q \vee \neg r$ and $\neg(r \to q) \to \neg p$ then p.

False. Take $p = q = r = 0$

$q \vee \neg r = 1$

$\neg(r \to q) = 0$ $\qquad \neg(r \to q) \to \neg p = 1$ $\qquad$ But $p = 0$.

(c) (Hard) If $p \to (q \vee r)$, $q \to s$ and $r \to \neg p$ then $p \to s$.

This is true. You could draw the truth table for

$(p \to (q \vee r)) \wedge (q \to s) \wedge (r \to \neg p) \to (p \to s)$. But that take too long.

So reason more directly:

How could $p \to s$ be false? Only if $\boxed{p = 1,\ s = 0}$

But in that case, its not possible all of $p \to (q \vee r)$, $q \to s$ and $r \to \neg p$ are true:

If they were, $s = 0$, $q \to s = 1$ so $\boxed{q = 0}$

$\neg p = 0$, $r \to \neg p = 1$ so $\boxed{r = 0}$

Hence $q \vee r = 0$, so $p \to (q \vee r) = 0$, a contradiction.