

Discrete Mathematics I Practise Final

Our final is at 10.15 on Friday of finals week, in our regular classroom. It will have eight questions, each worth five points.

Here's a rough list of topics it will cover:

- Divisors, multiples, floor and ceiling. Simple proofs involving these definitions.
- Primes, GCD's, LCM's, the "super" Euclidean algorithm, the fundamental theorem of arithmetic (learn the precise statement!).
- Sets, \cup , \cap , \oplus , \times , ..., Venn diagrams.
- Truth tables and logical connectives, logical equivalence.
- Functions, especially the definitions 1-1 and onto.
- Simple proofs involving contrapositives or contradictions, e.g. about rational and irrational numbers.
- Partial orders and digraphs. Equivalence relations and graphs, especially the theorem about an equivalence relation partitioning the set into disjoint subsets.
- MOD and DIV, $\equiv \pmod{p}$, modular arithmetic, solving equations like $2x \equiv 7 \pmod{23}$.
- Loop invariants, simple computer algorithms involving recurrence relations, solving linear recurrence relations.
- Sequences and \sum notation for sums, proof by mathematical induction.

In the text book, this is roughly sections 1.1–1.7, 2.1–2.4, 3.1–3.5, 4.1–4.2, 4.4–4.5, 4.7. Now is the time to sit down and systematically review all that material. Write down for yourself all the main definitions! Copy out for yourself at least one example of each thing!

1. (a) What is the definition of a prime number p ?

(b) Use the Euclidean algorithm to find integers s and t such that

$$17s + 101t = 1.$$

Show your working!

(c) Use your answer to (b) to solve the congruence equation $17x \equiv 66 \pmod{101}$.

2. Let G be the digraph with

$$\begin{aligned}V(G) &= \{v_1, v_2, v_3, v_4\}, \\E(G) &= \{e_1, e_2, e_3, e_4, e_5, e_6\}\end{aligned}$$

and directed edges defined by the function

$$\begin{aligned}\gamma : E(G) &\rightarrow V(G) \times V(G) \\e_1 &\mapsto (v_3, v_1), \\e_2 &\mapsto (v_3, v_2), \\e_3 &\mapsto (v_4, v_3), \\e_4 &\mapsto (v_4, v_1), \\e_5 &\mapsto (v_4, v_2), \\e_6 &\mapsto (v_2, v_4).\end{aligned}$$

(a) Draw a picture representing the digraph G .

(b) Write down the adjacency matrix of G .

(c) What is the length of the shortest path that passes through every vertex?

3. (a) What does it mean to say an integer m *divides* an integer n ? Prove directly from your definition that if m divides n and m divides p then m divides $n + p$.

(b) Draw the digraph of the partial order $|$ on $\{1, 2, 3, 4, 5, 6, 7, 8\}$ defined by $m|n$ if m divides n .

(c) Define a relation \trianglelefteq on \mathbb{Z} by $m \trianglelefteq n$ if $m^2 \leq n^2$. Is \trianglelefteq a partial order?

4. (a) Work out the truth table for the compound propositions $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$.

(b) Using your answer to (a), explain why $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$.

(c) Explain as clearly as you can how to translate the statement you just proved into the following theorem about sets: $A \cup (B \cap C) = (A \cap B) \cup (A \cap C)$.

5. (a) Prove that $369^{123456789} - 13579$ is divisible by 10.

(b) If x is rational and y is irrational prove that $x + y$ is irrational.

6. Consider the function $f : \mathbb{N} \rightarrow \mathbb{N}$ defined by the formula $f(n) = \lfloor \sqrt{n} \rfloor$.

(a) Compute $f^{-1}(1)$.

(b) Is this function 1-1, onto, or a 1-1 correspondence?

7. Prove by induction on $n = 1, 2, \dots$ that

$$\sum_{i=1}^n i2^i = (n-1)2^{n+1} + 2.$$

8. Suppose $(S(n))_{n \geq 0}$ is the sequence defined by the following pseudo-computer program:

```
S(n)
Input : an integer  $n \geq 0$ 
begin
  if  $n = 0$  then return 1
  if  $n = 1$  then return 4
  return  $4S(n - 1) - 4S(n - 2)$ 
end
```

(a) Compute $S(2)$

(b) Write down the recurrence relation defining $S(n)$ in usual mathematical notation.

(c) Solve the recurrence relation to find an explicit formula for $S(n)$.