NOTES ON SPECTRAL SEQUENCES

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These notes are intended for several students who are taking my class. There is nothing original here, all material is well-known and may be found in different books and papers on algebraic topology. My main purpose here is to provide for my students something which may be readable in our case: I do try to take in account what was covered earlier in the 6-th hundred course on algebraic topology. The final goal of these notes is very modest: we would like to compute a rank of homotopy groups of spheres. Perhaps, the further applications, like computation of the Steenrod algebra and the first few homotopy groups of spheres should be done in a following reading course. The emphasis of this course should be the Adams spectral sequence and some applications. I strongly recommend not to stop at the end of our course. You are already in the position to get to some interesting topics of contemporary homotopy theory.

1. Filtrations and spectral sequences

Let X be a space and there is a sequence (finite) of subspaces $\{X_i\}$ such that

$$\emptyset = X_{-1} \subset X_0 \subset X_1 \subset \cdots \subset X_{k-1} \subset X_k = X. \tag{1}$$

Let $C_q(X)$ be a group of singular or cellular chains of X. We have the filtration:

$$0 \subset C_q(X_0) \subset C_q(X_1) \subset \cdots \subset C_q(X_{k-1}) \subset C_q(X_k) = C_q(X).$$

We identify each group $C_q(X_i)$ with its image in $C_q(X)$. We say that an element $\alpha \in C_q(X)$ has *filtration* i if $\alpha \in C_q(X_i)$ and $\alpha \notin C_q(X_{i+1})$. In other words, the group $C_q(X_i)$ is a subgroup of elements in $C_q(X)$ with the filtration $\leq i$.

Recall that we have an exact sequence of complexes:

$$0 \longrightarrow C_*(X_{i-1}) \longrightarrow C_*(X_i) \longrightarrow C_*(X_i/X_{i-1}) \longrightarrow 0 \tag{2}$$

Let us denote

$$E_0^{i,q-i} = C_q(X_i/X_{i-1}).$$

The boundary operator of the complex $C_*(X_i/X_{i-1})$

$$\partial : C_q(X_i/X_{i-1}) \longrightarrow C_{q-1}(X_i/X_{i-1})$$

is denoted as

$$d_0^{i,q-i} : E_0^{i,q-i} \longrightarrow E_0^{i,q-i-1}$$

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Clearly $d_0^{i,q-i} d_0^{i,q-i-1} = 0$. We have a complex

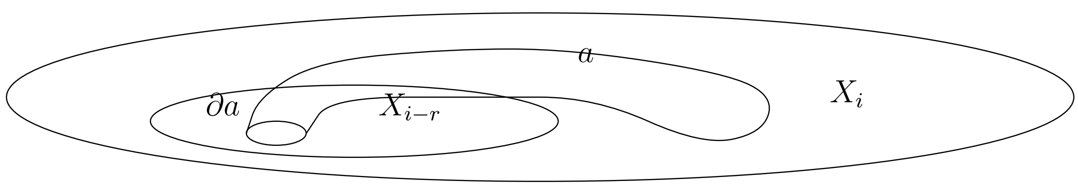
$$\cdots \longrightarrow E_0^{i,q-i+1} \xrightarrow{d_0^{i,q-i+1}} E_0^{i,q-i} \xrightarrow{d_0^{i,q-i}} E_0^{i,q-i-1} \xrightarrow{d_0^{i,q-i-1}} E_0^{i,q-i-2} \longrightarrow \cdots$$

The homology groups of this complex are $H_q(X_i/X_{i-1})$. We denote $E_1^{i,q-i} = H_q(X_i/X_{i-1})$. We are going to define the groups $E_r^{*,*}$ together with the differential

$$d_r^{s,t} : E_r^{s,t} \longrightarrow E_r^{s-r,t+r-1}$$

Definition 1.1. The group $Z_r^{i,q-i} \subset E_0^{i,q-i}$. Let $\alpha \in E_0^{i,q-i} = C_q(X_i/X_{i-1})$. We find a representative $a \in C_q(X_i)$ so that a projects to α:

$$\begin{tikzcd}
 & & a \arrow[r] & \alpha & \\
0 \arrow[r] & C_q(X_{i-1}) \arrow[r] & C_q(X_i) \arrow[r] \arrow[d, "\partial"'] & C_q(X_i/X_{i-1}) \arrow[r] & 0 \\
 & C_{q-1}(X_{i-r}) \arrow[r] & C_{q-1}(X_i) & &
\end{tikzcd}$$



An element $\alpha \in Z_r^{i,q-i}$ if there exists such a representative $a \in C_q(X_i)$ so that its boundary has a filtration $(i-r)$, i.e. $\partial a \in C_q(X_{i-r})$.

Case $r = 0$: Clearly $Z_0^{i,q-i} = E_0^{i,q-i}$.

Case $r = 1$: There exists $a \in \alpha$ so that $\partial a \in C_{q-1}(X_{i-1}) \subset C_{q-1}(X_i)$, i.e. α is a cycle in $C_q(X_i/X_{i-1})$. We conclude that $Z_1^{i,q-i} = Z_q(X_i/X_{i-1})$.

Clearly the groups $Z_r^{i,q-i}$ are decreasing, and for r big enough there is an isomorphism:

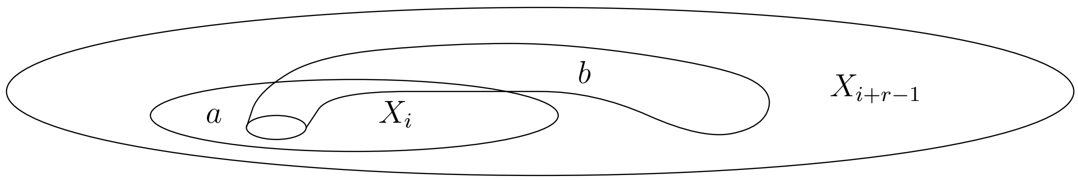
$$Z_r^{i,q-i} = Z_\infty^{i,q-i} \cong Z_q(X_i)/Z_q(X_{i-1}).$$

We have a chain of inclusions:

$$Z_\infty^{i,q-i} \subset \cdots \subset Z_{r+1}^{i,q-i} \subset Z_r^{i,q-i} \subset Z_{r-1}^{i,q-i} \subset \cdots \subset Z_0^{i,q-i} = E_0^{i,q-i}$$

Definition 1.2. We define the group $B_r^{i,q-i} \subset E_0^{i,q-i}$. Let $\alpha \in E_0^{i,q-i}$. Then $\alpha \in B_r^{i,q-i}$ if and only if there exists a representative $a \in C_q(X_i)$ so that $a = \partial_{q+1} b$, where $b \in C_{q+1}(X_{i+r-1})$, see a picture below.

$$\begin{tikzcd} & & C_{q+1}(X_i) \arrow[r] \arrow[d, "\partial"'] & C_{q+1}(X_{i+r-1}) & \\ 0 \arrow[r] & C_q(X_{i-1}) \arrow[r] & C_q(X_i) \arrow[r] & C_q(X_i/X_{i-1}) \arrow[r] & 0 \\ & & a \arrow[r, mapsto] & \alpha & \end{tikzcd}$$



We describe the group $B_0^{i,q-i}$. If $\alpha \in B_0^{i,q-i}$, then there exists $a \in \alpha \in E_0^{i,q-i}$ such that $a = \partial_{q+1} b$, where $b \in C_{q+1}(X_{i-1})$, i.e. $\alpha = 0$ and $B_0^{i,q-i} = 0$.

We describe the group $B_1^{i,q-i}$. If $\alpha \in B_1^{i,q-i}$, then there exists $a \in \alpha$ so that $a = \partial_{q+1} b$, where $b \in C_{q+1}(X_i)$. By construction of the complex $C_*(X_i/X_{i-1})$, it means that $\alpha \in C_q(X_i/X_{i-1})$ is a boundary. In other words, there is an isomorphism:

$$B_1^{i,q-i} = B_q(X_i/X_{i-1}),$$

where $B_q(X_i/X_{i-1}) = \mathrm{Im}\,(\partial : C_{q+1}(X_i/X_{i-1}) \longrightarrow C_q(X_i/X_{i-1}))$ is a group of boundaries.

It is clear that $B_r^{i,q-i} \subset B_{r+1}^{i,q-i}$, and for big enough r there is an isomorphism:

$$B_\infty^{i,q-i} = B_r^{i,q-i} = B_q(X) \cap C_q(X_i)/B_q(X) \cap C_q(X_{i-1}).$$

We have a chain of inclusions:

$$0 = B_0^{i,q-i} \subset B_1^{i,q-i} \subset \cdots \subset B_r^{i,q-i} \subset B_{r+1}^{i,q-i} \subset \cdots \subset B_\infty^{i,q-i}.$$

Clearly there is an inclusion of subgroups $B_r^{i,q-i} \subset Z_r^{i,q-i}$. We define the group

$$E_r^{i,q-i} = Z_r^{i,q-i}/B_r^{i,q-i}, \quad r = 0, 1, \ldots$$

We have that $E_0^{i,q-i} = Z_0^{i,q-i}/B_0^{i,q-i} = E_0^{i,q-i}/0 = E_0^{i,q-i}$ (which is a good news since new definition of this group gives the old one). The group $E_1^{i,q-i}$ is our old friend:

$$E_1^{i,q-i} = Z_1^{i,q-i}/B_1^{i,q-i} = Z_q(X_i/X_{i-1})/B_q(X_i/X_{i-1}) = H_q(X_i/X_{i-1}).$$

Also we note that there exists a (big enough) r so that

$$E_r^{i,q-i} = E_{r+1}^{i,q-i} = \cdots = E_\infty^{i,q-i}.$$

We return to the group $E_\infty^{i,q-i}$ a bit later. Now one more definition.

Definition 1.3. Differential $d_r^{i,q-i} : E_r^{i,q-i} \longrightarrow E_r^{i-r,q+r-i-1}$. Let $\alpha \in E_r^{i,q-i}$. We choose a representative $\alpha' \in Z_r^{i,q-i}$ of α. By definition $\alpha' \in Z_r^{i,q-i} \subset E_0^{i,q-i} = C_q(X_i/X_{i-1})$, and there exists a representative $a \in \alpha'$, $a \in C_q(X_i)$ so that $\partial a = b \in C_{q-1}(X_{i-r})$. The element b determines a class

$$\beta' \in C_{q-1}(X_{i-r}/X_{i-r-1}) = E_0^{i-r,q+r-i-1}.$$

Clearly the element β' lives in the subgroup $Z_r^{i-r,q+r-i-1}$, so β' determines an element

$$\beta \in E_r^{i-r,q+r-i-1} = Z_r^{i-r,q+r-i-1}/B_r^{i-r,q+r-i-1}.$$

We define $d_r^{i,q-i}(\alpha) = \beta$.

Remark: You have to check by yourself that the differential $d_r^{i,q-i}$ is well-defined, i.e it does not depend on the choices we made. It is also important to check that $d_r^{i,q-i}$ is a group homomorphism. Please take few minutes to chase a couple of diagrams!

Remark: I would like to remind that a triple of spaces $X \subset Y \subset Z$ gives an exact sequence of chain complexes:

$$0 \longrightarrow C_*(Y,X) \longrightarrow C_*(Z,X) \longrightarrow C_*(Z,Y) \longrightarrow 0,$$

and there is a long exact sequence of homology groups:

$$\cdots \longrightarrow H_q(Y,X) \longrightarrow H_q(Z,X) \longrightarrow H_q(Z,Y) \xrightarrow{\partial} H_{q-1}(Y,X) \longrightarrow \cdots \tag{3}$$

In our case of a triple $X_{i-2} \subset X_{i-1} \subset X_i$ we have the long exact sequence:

$$\cdots \longrightarrow H_q(X_{i-1}/X_{i-2}) \longrightarrow H_q(X_i/X_{i-1}) \longrightarrow H_q(X_i/X_{i-1}) \xrightarrow{\partial} H_{q-1}(X_{i-1}/X_{i-2}) \longrightarrow \cdots.$$

We note here that $E_1^{i,q-i} = H_q(X_i/X_{i-1})$ and $E_1^{i-1,q-i} = H_{q-1}(X_{i-1}/X_{i-2})$. I would like to ask you to prove that the above boundary homomorphism

$$H_q(X_i/X_{i-1}) \xrightarrow{\partial} H_{q-1}(X_{i-1}/X_{i-2})$$

coincides with the differential

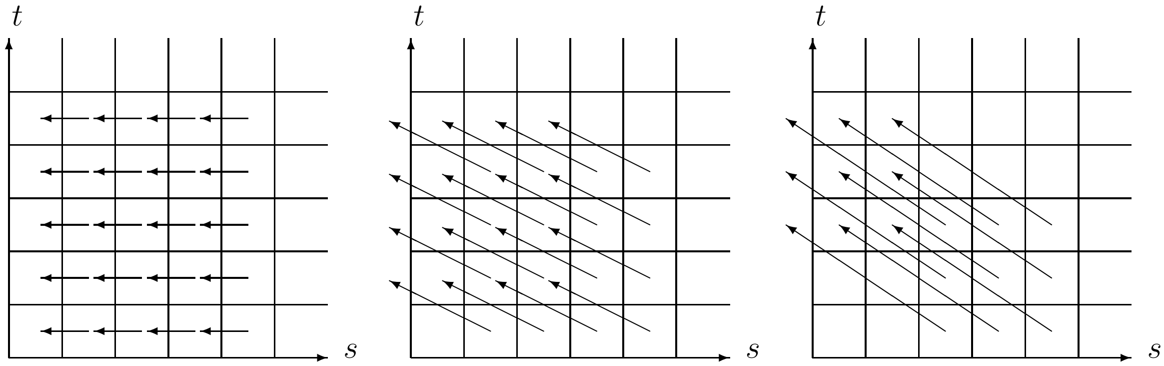
$$d_1^{i,q-i} : E_1^{i,q-i} \longrightarrow E_1^{i-1,q-i}.$$

Again, take few minutes to chase elements in one diagram!

Few words about notations. It is important to think about the r-th term

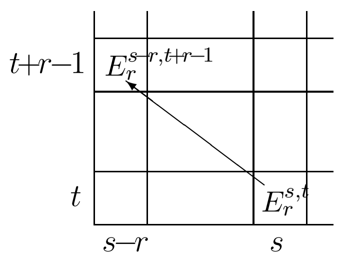
$$E_r^{*,*} = \bigoplus_{s,t} E_r^{s,t}$$

as a bigraded abelian group. Then the differential $d_r^{i,q-i}$ induces a homomorphism of bigraded groups $d_r : E_r^{*,*} \longrightarrow E_r^{*,*}$ of degree $(-r, r-1)$. This algebraic object $(E_r^{*,*}, d_r)$ is called a **homological spectral sequence associated with the filtration (1).** There is a nice way to picture a spectral sequence $(E_r^{*,*}, d_r)$. We can imagine charts of the first three terms:



$E_1^{*,*}$ $E_2^{*,*}$ $E_3^{*,*}$

In general the differential $d_r : E_r^{s,t} \longrightarrow E_r^{s-r,t+r-1}$ may be thought as a "Knight move":



There is one particular property of the differential d_r that we did not prove yet. It is your turn!

Exercise. Prove that $d_r^2 = 0$.

Now we are ready to prove the following technical result.

Theorem 1.4. *There is an isomorphism $E_{r+1}^{i,q-i} \cong Ker\ d_r^{i,q-i}/Im\ d_r^{i-r,q-i+r-1}$. In other words, a homology group of E_r (with respect to the differential d_r) is E_{r+1}.*

Proof. Here we use notations given above to define $d_r^{i,q-i}$. Assume that $d_r^{i,q-i}(\alpha) = 0$. Then the element $\beta' \in C_{q-1}(X_{i-r}/X_{i-r-1})$ belongs to the group $B_r^{i-r,q-i+r-1}$, i.e. there exists a representative $c \in \beta'$ ($c \in C_{q-1}(X_{i-r})$) so that $c = \partial\tau$, where $\tau \in C_q(X_{i-1})$. Recall that α' is a representative of α in the group

$$Z_r^{i,q-i} \subset E_0^{i,q-i} = C_q(X_i/X_{i-1})$$

and a is a representative of α' in the group $C_q(X_i)$. Also recall that $C_q(X_{i-1}) \subset C_q(X_i)$. Clearly the element $a - \tau \in C_q(X_i)$ projects in the same element α'. Since $d_r^{i,q-i}(\alpha)$ does not depend on all choices we made, we could choose $a - \tau$ instead of a in the first place. Then $\partial(a-\tau) \in C_{q-1}(X_{i-r-1})$. It means that the element $\alpha' \in Z_{r+1}^{i,q-i}$ (by definition) and defines some element $\tilde{\alpha}$ in $E_{r+1}^{i,q-i} = Z_{r+1}^{i,q-i}/B_{r+1}^{i,q-i}$. We have constructed a homomorphism:

$$T : \text{Ker } d_r^{i,q-i} \longrightarrow E_{r+1}^{i,q-i} \quad \text{by the formula} \qquad T : \alpha \mapsto \tilde{\alpha}.$$

To complete the proof it remains:

(1) to check that a homomorphism T is well-defined;

(2) to prove that T is epimorphism;

(3) to prove that $\mathrm{Ker}\; T = \mathrm{Im}\; d_r^{i-r,q-i+r-1}$.

I happily leave these statements to you to prove as an exercise. $\square$

Now we return to the "stable" group $E_\infty^{i,q-i}$. Recall that $E_\infty^{i,q-i} = Z_\infty^{i,q-i}/B_\infty^{i,q-i}$.

Let ${}_{(i)}H_q(X) = \mathrm{Im}\; (H_q(X_i) \longrightarrow H_q(X))$, where the homomorphism $H_q(X_i) \longrightarrow H_q(X)$ is induced by the inclusion $X_i \longrightarrow X$. We obtain the following filtration of the group $H_q(X)$:

$$0 = {}_{(-1)}H_q(X) \subset {}_{(0)}H_q(X) \subset {}_{(1)}H_q(X) \subset \cdots \subset {}_{(k)}H_q(X) = H_q(X).$$

Theorem 1.5. *There is an isomorphism* $E_\infty^{i,q-i} \cong {}_{(i)}H_q(X)/{}_{(i-1)}H_q(X)$.

Proof. Recall that:

$$\begin{aligned}
Z_\infty^{i,q-i} &= Z_q(X_i)/Z_q(X_{i-1}),\\
B_\infty^{i,q-i} &= B_q(X)\cap C_q(X_i)/B_q(X)\cap C_q(X_{i-1}),\\
{}_{(i)}H_q(X) &= Z_q(X_i)/B_q(X)\cap C_q(X_i),\\
{}_{(i-1)}H_q(X) &= Z_q(X_{i-1})/B_q(X)\cap C_q(X_{i-1}).
\end{aligned}$$

To prove the isomorphism consider the commutative diagram:

$$\begin{tikzcd}
B_q(X)\cap C_q(X_{i-1}) \arrow[r] \arrow[d] & B_q(X)\cap C_q(X_i) \arrow[r] \arrow[d] & B_\infty^{i,q-i} \arrow[d] \\
Z_q(X_{i-1}) \arrow[r] \arrow[d] & Z_q(X_i) \arrow[r] \arrow[d] & Z_\infty^{i,q-i} \arrow[d] \\
{}_{(i-1)}H_q(X) \arrow[r] & {}_{(i)}H_q(X) \arrow[r] & G
\end{tikzcd}$$

where the vertical and horizontal lines are short exact sequences. Clearly
$G = E_\infty^{i,q-i} = {}_{(i)}H_q(X)/{}_{(i-1)}H_q(X)$ $\square$

Remark. Clearly it is important that the bottom row in the above diagram is a short exact sequence.

We summarize the construction:

Theorem 1.6. (Leray Theorem) *Let X be a space filtered by its subspaces:*

$$\emptyset = X_{-1} \subset X_0 \subset X_1 \subset \cdots \subset X_{k-1} \subset X_k = X.$$

Then there exist groups $E_r^{s,t}$, defined for all $r \geq 0$ and all s, t (where $E_r^{s,t} = 0$ if $s < 0$ or $t < 0$), and homomorphisms

$$d_r^{s,t} : E_r^{s,t} \longrightarrow E_r^{s-r,t+r-1}$$

(where $d_r^{s-r,t+r-1} \circ d_r^{s,t} = 0$), such that

(i) $E_{r+1}^{s,t} = Ker\ d_r^{s,t} / Im\ d_r^{s+r,t-r+1}$;

(ii) $E_0^{s,t} = C_{s+t}(X_p/X_{p-1})$;

(iii) $E_\infty^{s,t} = \dfrac{{}_{(s)}H_{s+t}(X)}{{}_{(s-1)}H_{s+t}(X)}$.

Remark. Let A be an abelian group, and $0 \subset A_0 \subset A_1 \subset \cdots \subset A_k = A$ be its filtration by subgroups. We denote $\bar{A} = \bigoplus_{i=0}^{k} A_i/A_{i-1}$. This is a group "associated with A with respect to a given filtration". We note several evident properties:

(1) If the group $\bar{A}$ is finitely generated, then A is finitely generated.
(2) If the group $\bar{A}$ is finite, then A is finite, and $|A| = |\bar{A}|$.
(3) If all the groups A_i/A_{i-1} are free abelian, then $A \cong \bar{A}$.
(4) If all the groups A_i/A_{i-1} are vector spaces over a field k, then $A \cong \bar{A}$.

Again, I happily leave to you to prove these statements.

2. Leray-Serre spectral sequence for a fiber bundle

Let $\pi : E \longrightarrow B$ be a Serre fiber bundle with a fiber F, where B is a finite connected CW-complex.

Warning: I will give all constructions and proofs in the case of *locally-trivial fiber bundles*, however **all results** hold for a Serre fiber bundle. The finiteness condition on B may be dropped without any loss of generality.

Our goal here is to find homology and cohomology groups of the total space E provided that we know homology and cohomology groups of B and F.

Let $B^{(i)}$ be the i-th skeleton of B. We have a filtration of B by its skeletons:

$$\emptyset = B^{(-1)} \subset B^{(0)} \subset \cdots \subset B^{(k)} = B. \tag{4}$$

Exercise. Analyze a spectral sequence associated with the filtration (4). In particular, prove that $E_\infty = E_2$.

Now we construct a filtration of the total space E as follows. Let $E_i = \pi^{-1}(B^{(i)})$, so we have:

$$\emptyset = E_{-1} \subset E_0 \subset \cdots \subset E_k = E. \tag{5}$$

We consider a spectral sequence associated with the filtration (5). It turns out that the E_2-term of this spectral sequence may be computed in terms of homology groups of the base space and the fiber. To see this clearly we have to analyze a geometry of a fiber bundle over a CW-complex in more detail.

Recall that $E_0^{p,q} = C_{p+q}(E_p, E_{p-1})$. Then we have that $E_1^{p,q} = H_{p+q}(E_p, E_{p-1})$. The following statement is very important for this spectral sequence.

Proposition 2.1. *There is an isomorphism $H_{p+q}(E_p, E_{p-1}) \cong C_p(B; H_q(F))$ (here we mean a cellular chain group).*

Proof. First, it is important to understand a structure of the space E_p/E_{p-1}. We choose p-cells of B: $\sigma_1^p, \dots, \sigma_m^p$, and the characteristic and attaching maps for each one:

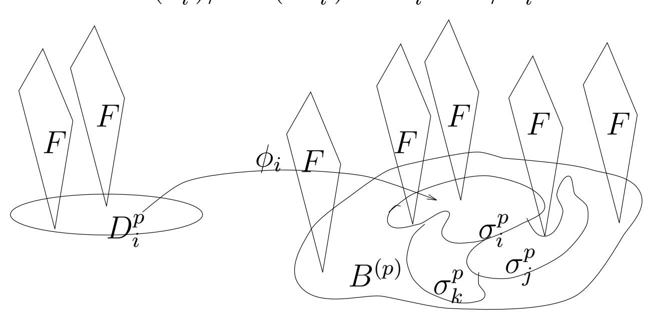
$$\begin{array}{ccc} D_i^p & \xrightarrow{\phi_i} & B^{(p)} \\ \uparrow & & \uparrow \\ S_i^{p-1} & \xrightarrow{\psi_i} & B^{(p-1)} \end{array}$$

Considering σ_i^p as geometric objects we let $\sigma_i^p = \phi_i(D_i^p)$ and $\partial\sigma_i^p = \psi_i(S_i^{p-1})$. We note that the factor-space E_p/E_{p-1} is decomposed as

$$E_p/E_{p-1} = \bigvee_{i=1}^{m} \left(\pi^{-1}(\sigma_i^p)/\pi^{-1}(\partial\sigma_i^p)\right),$$

and since a pull-back of the fiber bundle over D_i^p is trivial, we have a homeomorphism

$$\pi^{-1}(\sigma_i^p)/\pi^{-1}(\partial\sigma_i^p) \cong D_i^p \times F/S_i^{p-1} \times F.$$



Now we have:

$$C_p(B; H_q(F)) \cong H_p(B^{(p)}/B^{(p-1)}; H_q(F))$$

$$\cong \bigoplus_{i=1}^{m} H_p(D_i^p/S_i^{p-1}; H_q(F)) \cong \bigoplus_{i=1}^{m} H_q(F)(\sigma_i^p)$$

where $H_q(F)(\sigma_i^p)$ is a copy of the group $H_q(F)$ with a generator σ_i^p. (To be more precise we should say that $H_q(F)(\sigma_i^p)$ is a "free module over $H_q(F)$ with a generator σ_i^p".) On the other hand, we have the following isomorphisms:

$$\begin{aligned} H_{p+q}(E_p/E_{p-1}) &\cong H_{p+q}\left(\bigvee_{i=1}^m (\pi^{-1}(\sigma_i^p)/\pi^{-1}(\partial\sigma_i^p))\right) \\ &\cong \bigoplus_{i=1}^m H_{p+q}\left(\pi^{-1}(\sigma_i^p)/\pi^{-1}(\partial\sigma_i^p)\right) \\ &\cong \bigoplus_{i=1}^m H_{p+q}(D_i^p \times F/S_i^{p-1}\times F) \\ &\cong \bigoplus_{i=1}^m H_q(F)(\sigma_i^p). \end{aligned}$$

The last isomorphism here may be seen as follows. We choose the three-cell decomposition of D_i^p: e^0 (a point), e^{p-1} (the sphere S^{p-1}) and the cell e^p (the disk D_i^p itself). The CW-complex $D_i^p \times F$ has three types of cells: $e^0\times\omega$, $e^{p-1}\times\omega$, and $e^p\times\omega$ where ω is a cell of F. While we compute the group $H_{p+q}(D_i^p\times F/S_i^{p-1}\times F)$ we can ignore the first two types since we factor them out anyway. The remaining cells are in one-to-one correspondence with the cells of F with a dimensional shift (by p). □

Remark. There is a rather delicate point here. Indeed, the isomorphism $E_1^{p,q}\cong C_p(B;H_q(F))$ is **not** canonical. Let $\alpha\in C_p(B;H_q(F))$, $\alpha=\sum_i\lambda_i\sigma_i^p$, where $\lambda_i\in H_q(F)$. Now we would like to see the image of α in the group

$$E_1^{p,q}\cong\bigoplus_{i=1}^m H_{p+q}\left(\pi^{-1}(\sigma_i^p)/\pi^{-1}(\partial\sigma_i^p)\right).$$

To do this we have to choose (for each i) a homeomorphism

$$\pi^{-1}(\sigma_i^p)/\pi^{-1}(\partial\sigma_i^p)\cong D_i^p\times F/S_i^{p-1}\times F,$$

which is **not** unique (even up to homotopy). However we may choose a homeomorphism of the space $\{0\}\times F\subset D_i^p\times F$ with the fiber $F_{x_0}=\pi^{-1}(x_0)$, where x_0 is the image of 0 under the characteristic map $\phi_i : D_i^p\longrightarrow\sigma_i^p$. This gives a particular choice of the isomorphism $E_1^{p,q}\cong C_p(B;H_q(F))$.

The next question is: is it possible to choose the homeomorphisms $F\cong F_x$ in a canonical way for all points $x\in B$? We recall that a path connecting x_1 and x_2 gives a homeomorphism $F_{x_1}\cong F_{x_2}$ (up to homotopy), moreover two homotopic paths give homotopic homeomorphisms. It means that a homotopy class of this homeomorphism does not depend on a choice of path provided that the base B is simply-connected. In this case we may choose a single point $x_0\in B$ and then define homeomorphisms $F_{x_1}\cong F_{x_0}\cong F$ by choosing any path connecting x_0 and x_1. This remains true in the nonsimply-connected case provided that a fiber bundle $E\longrightarrow B$ is "simple", i.e. any path connecting x_1 and

x_2 gives a unique (up to homotopy) homeomorphism $F_{x_1} \cong F_{x_2}$. There is something very interesting is going on when a fiber bundle is not "simple"; however we will stay away from this in our course. $\square$

Now we have the first differential

$$d_1^{p,q} : E_1^{p,q} = C_p(B; H_q(F)) \longrightarrow E_1^{p-1,q} = C_{p-1}(B; H_q(F))$$

Exercise. Prove that the differential $d_1^{p,q}$ coincides with the boundary operator

$$\partial : C_p(B; H_q(F)) \longrightarrow C_{p-1}(B; H_q(F)).$$

Once you are done with this exercise, then the following isomorphism is immediate:

$$E_2^{p,q} \cong H_p(B; H_q(F)). \tag{6}$$

The isomorphism (6) is extremely important for computations and theoretical arguments. We shall return to the construction and properties of this spectral sequence (Leray-Serre spectral sequence). Now we take a look on the groups $E_2^{p,q}$. We have that

$$E_2^{p,0} = H_p(B; H_0(F)) = H_p(B; \mathbf{Z})$$

(in the case of a connected fiber F), and $E_2^{0,q} = H_0(B; H_q(F)) \cong H_q(F)$. In other words, the homology groups of the base are in the zero row, and the homology groups of the fiber are in the zero column. We also note that if the groups $H_*(F)$ or $H_*(F)$ are torsion free, then $E_2^{p,q} = H_p(B) \otimes H_q(F)$. It is also true if we work with coefficients in a field.

$E_2^{0,q}$						
$\vdots$						
$E_2^{0,2}$						
$E_2^{0,1}$						
$E_2^{0,0}$	$E_2^{1,0}$	$E_2^{2,0}$	$E_2^{3,0}$	$\cdots$	$E_2^{p,0}$	$E_2^{p+1,0}$

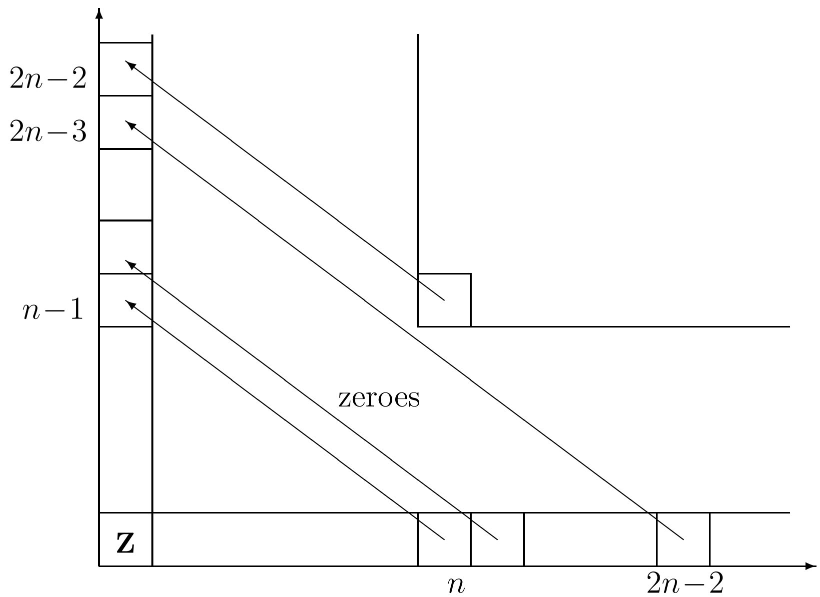
$E_2^{*,*}$

3. First applications

Let X be a space with a base point x_0, and $P(X) \xrightarrow{\pi} X$ be a map from the space of paths starting at x_0 into X. This is a Serre fiber bundle with a fiber ΩX, the loops on X. The exact sequence in homotopy groups immediately implies that $\pi_i X \cong \pi_{i-1}\Omega X$ for all $i \geq 1$. A relationship between homology groups of X and ΩX is more complicated.

Theorem 3.1. *Let X be a $(n-1)$-connected CW-complex. Then there is an isomorphism $H_i(X) \cong H_{i-1}(\Omega X)$ if $i \leq 2n-2$.*

Proof. First we recall that since X is $(n-1)$-connected $H_j(X) = 0$ if $1 \leq j \leq n-1$, and $H_0(X) \cong \mathbf{Z}$, and $H_n(X) \cong \pi_n X$. Consider the spectral sequence for the fiber bundle $P(X) \xrightarrow{\pi} X$. The E_2-term looks as follows:



The zero row consists of the groups $H_j(X)$, consequently $E_2^{0,0} = \mathbf{Z}$, and $E_2^{p,0} = 0$ for $1 \leq p \leq n-1$. It implies that the columns $E_2^{p,*}$ are all zero, as it is shown on the picture. The total space $P(X)$ is contractable, so the homology groups of $P(X)$ are the same as of a point. We observe that the cells $E_2^{0,q} = 0$ for $1 \leq q \leq n-2$. Indeed, if a group $E_2^{0,q} = 0$ would be nontrivial, then there is no nontrivial differential d_r which have the group $E_2^{0,q}$ as a target (since all cells which could support the differential are zeroes). In more detail one should give an inductive argument: the group $H_1(\Omega X)$ should be zero: otherwise there is no differential that would hit the cell $E_2^{0,1}$, (all possible candidates to support a differential with this target are zeroes). Then it implies that the second row is zero since $E_2^{p,q} = H_p(X; H_q(\Omega X))$. Then a similar argument shows that the groups $E_2^{p,q} = 0$ for $1 \leq q \leq n-2$.

The cell $E_2^{n,0} \cong H_n(X)$ is nontrivial, however it must disappear in the E_∞-term, and the only possible nontrivial differential which the group $E_2^{n,0}$ may support, is

$$d_n^{n,0} : E_n^{n,0} \cong E_2^{n,0} \longrightarrow E_2^{0,n-1} \cong E_n^{0,n-1}.$$

It implies that the differential $d_n^{n,0}$ is an isomorphism, so $E_2^{0,n-1} = H_{n-1}(\Omega X) \cong H_n(X)$. It is clear that the same argument implies that the differential

$$d_p^{p,0} : E_p^{p,0} \cong E_2^{p,0} \longrightarrow E_2^{0,p-1} \cong E_p^{0,p-1}$$

is an isomorphism provided that $n \leq p \leq 2n-2$. Clearly this argument does not work for the cell $E_2^{2n-1,0}$ because it is possible that the differential

$$d_n^{n,n-1} : E_n^{n,n-1} = E_2^{n,n-1} \longrightarrow E_2^{0,2n-2} = E_n^{0,2n-2}$$

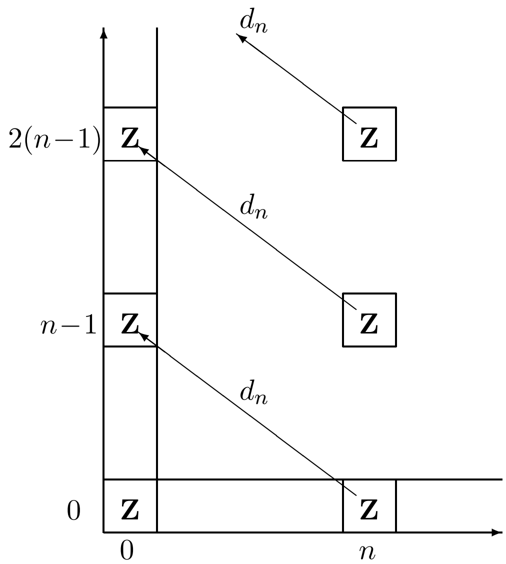
is nontrivial. $\square$

Now we can compute homology groups of ΩS^n.

Lemma 3.2. *Let* $n \geq 2$*. Then*

$$H_j(\Omega S^n) \cong \begin{cases} \mathbf{Z} & \text{if } j = k(n-1), k = 0, 1, 2, \dots \\ 0 & \text{else} \end{cases}$$

Proof. We examine the spectral sequence of the fibration $P(S^n) \xrightarrow{\Omega S^n} S^n$. It is clear that the E_2-term looks as follows:



There are some nontrivial groups only in $E_2^{0,*}$ and $E_2^{n,*}$. The argument given in the proof of the Theorem 3.1 shows that $E_2^{0,j} = 0$ and $E_2^{n,j} = 0$ for $1 \leq j \leq n-1$. The differential $d_n : E_n^{n,0} \longrightarrow E_n^{0,n-1}$ is an isomorphism. Now we repeat the argument to show that $E_2^{0,j} = 0$ and $E_2^{n,j} = 0$ for $n+1 \leq j \leq 2n-3$. Clearly $d_n : E_n^{n,n-1} \longrightarrow E_n^{0,2(n-1)}$ is an isomorphism. An induction completes the argument. □

Exercise. Investigate all fiber bundles over S^2 with a fiber S^1. Compute homology groups of the corresponding total spaces.

Exercise. Investigate the spectral sequence of the Hopf fiber bundle $S^{2n+1} \longrightarrow \mathbf{CP}^n$.

These examples conclude the first applications. Our next goal is to switch to the cohomological spectral sequence.

4. Cohomological Leray-Serre spectral sequence

I would like to start here by quoting Robert Switzer who says in his book: *"There can be scarcely have been a student of mathematics who had to deal with spectral sequences and was not repelled or at least very confused by them in his (her) first encounter. One needs much practice with spectral sequences before all those indices stop swiming before ones eyes and begin to take on some sensible meaning."* I hope this may help us to deal with the confusion and difficulties we have now looking at spectral sequences. We have to make one more effort to switch to the cohomological version of the spectral sequences. Then we will be able to proceed with actual calculations.

Let X be a CW-comlpex. I would like to take a risk and state right away the main result concerning the cohomological Leray-Serre spectral sequence.

Theorem 4.1. (Leray Theorem) *Let X be a space filtered by its subspaces:*

$$\emptyset = X_{-1} \subset X_0 \subset X_1 \subset \cdots \subset X_{k-1} \subset X_k = X.$$

Then there exist groups $E_r^{s,t}$, defined for all $r \geq 0$ and all s,t (where $E_r^{s,t} = 0$ if $s < 0$ or $t < 0$), and homomorphisms $d_r^{s,t} : E_r^{s,t} \longrightarrow E_r^{s+r,t-r+1}$ (where $d_r^{s+r,t-r+1} \circ d_r^{s,t} = 0$), such that

(i) $E_{r+1}^{s,t} = Ker\ d_r^{s,t} / Im\ d_r^{s-r,t+r-1}$;

(ii) $E_0^{s,t} = C^{s+t}(X_p/X_{p-1})$;

(iii) $E_\infty^{s,t} = \dfrac{{}_{(s)}H^{s+t}(X)}{{}_{(s-1)}H_{s+t}(X)}$ *where*

$$_{(s)}H^{s+t}(X) = Ker\ \left(H^{s+t}(X) \longrightarrow H^{s+t}(X_s)\right)$$

is a homomorphism induced by the inclusion $X_s \subset X$.

Comments for someone who would like to prove Theorem 4.1. It is very good idea to go through the construction of the spectral sequence one more time "dualizing" all groups and homomorphisms. It is interesting that instead of the filtration $0 \subset C_q(X_0) \subset C_q(X_1) \subset \cdots \subset C_q(X_k) = C_q(X)$ in homological case we have the filtration: ${}_{(0)}C^q(X) \subset {}_{(1)}C^q(X) \subset \cdots \subset_{(k)} C^q(X_k) = C^q(X)$, where ${}_{(i)}C^q(X) = \text{Ker } (C^q(X) \longrightarrow C^q(X_i))$. The group $E_0^{i,q-i} = {}_{(i)}C^q(X)/{}_{(i-1)}C^q(X)$ should be idntified with the group $C^q(X_i/X_{i-1})$, and the boundary operator ∂ shoud be replaced by the coboundary operator δ. Good luck! □

Theorem 4.1 is almost identical to Theorem 1.6, however the cohomological spectral sequence has an additional structure which is crucial in many ways, as for computations, as for theoretical arguments. The spectral sequence which is of most interest for us is the Leray-Serre spectral sequence for a fiber bundle $E \longrightarrow B$ with a fiber F. (We still keep all our assumptions on a fiber bundle.) Recall that the Leray-Serre spectral sequence is induced by the filtration $\emptyset \subset E_{-1} \subset E_0 \subset \cdots \subset E_k = E$, where $E_i = \pi^{-1}(B^{(i)})$, and $B^{(i)}$ is the i-th skeleton of B. The E_2-term of the cohomological Leray-Serre spectral sequence may be identified with

$$E_2^{p,q} = H^p(B; H_q(F)).$$

We note that $E_2^{*,*}$ is a bigraded ring as a cohomology ring of B with coefficients in the graded ring $H^*(F)$:

$$E_2^{p,q} \otimes E_2^{p',q'} \longrightarrow E_2^{p+p',q+q'}.$$

Here is the main result concerning the cohomological Leray-Serre spectral sequence.

Theorem 4.2. *Let $E \xrightarrow{\pi} B$ be a Serre fiber bundle with a fiber F. Let G be an abelian group. Then there exists a spectral sequence $(E_r^{*,*}, d_r^{*,*})$ such that*

(i) $E_2^{p,q} = H^p(B; H^q(B;G))$;

(ii) *for any* r *the* E_r*-term* $E_r^{*,*} = \bigoplus_{p,q} E_r^{p,q}$ *is a bigraded ring, i.e. there are given bilinear products*

$$\mu_r : E_r^{p,q} \otimes E_r^{p',q'} \longrightarrow E_r^{p+p',q+q'};$$

(iii) *the differential* $d_r^{p,q} : E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1}$ *so that* $d_r^{p+r,q+r-1} \circ d_r^{p,q} = 0$, *and*

$$d_r(ab) = d_r(a) \cdot b + (-1)^{p+q} a \cdot d_r(b), \quad \textit{where} \quad a \in E_r^{p,q};$$

(iv) *the product* $\mu_2 : E_2^{p,q} \otimes E_2^{p',q'} \longrightarrow E_2^{p+p',q+q'}$ *coincides with the product induced by the ring structures of* $H^*(B)$ *and* $H^*(F;G)$;

(v) *the product* $\mu_\infty : E_\infty^{p,q} \otimes E_\infty^{p',q'} \longrightarrow E_\infty^{p+p',q+q'}$ *is "adjoint" to the product in the ring* $H^*(E;G)$.

The last statement here should be explained in more detail. Let A be a ring with multiplication $\mu : A \otimes A \longrightarrow A$ and let $0 \subset A_0 \subset A_1 \subset \cdots \subset A_k = A$ be a filtration. This filtration is *multiplicative* if $A_i \otimes A_j \subset A_{i+j}$. In this case a graded abelian group $\bar{A} = \bigoplus_i A_i/A_{i-1}$ becomes a graded ring with multiplication $\bar{\mu} : \bar{A} \otimes \bar{A} \longrightarrow \bar{A}$ induced by μ. In our case we have a filtration

$$0 \subset {}_{(0)}H^*(X) \subset {}_{(0)}H^*(X) \subset \cdots \subset {}_{(k)}H^*(X) = H^*(X)$$

and a product $\mu : H^*(X) \otimes H^*(X) \longrightarrow H^*(X)$ induces a product

$$\bar{\mu} : E_\infty^{*,*} \otimes E_\infty^{*,*} \longrightarrow E_\infty^{*,*}$$

since $E_\infty^{s,t} = \dfrac{{}_{(s)}H^{s+t}(X)}{{}_{(s-1)}H_{s+t}(X)}$. Now the statement (v) claims that the products μ_∞ and $\bar{\mu}$ coincide.

Remark. In general the product $\bar{\mu}$ does not contain all information on the product μ.

To understand better an advantage of the cohomological spectral sequence we compute the cohomology ring $H^*(\Omega S^n; \mathbf{Z})$. To state the result we should define some particular numbers. Let

$$\alpha_{k,\ell} = \begin{cases} \binom{k+\ell}{\ell} & \text{if } n \text{ is odd,} \\ \binom{(k+\ell)/2}{\ell/2} & \text{if } n \text{ is even and } k,\ \ell \text{ are even,} \\ \binom{(k+\ell-1)/2}{\ell/2} & \text{if } n \text{ is even, } k \text{ is odd, and } \ell \text{ is even,} \\ 0 & \text{if } n \text{ is even, } k,\ \ell \text{ are odd.} \end{cases}$$

The numbers $\alpha_{k,\ell}$ may be defined inductively: let $\alpha_{0,1} = \alpha_{1,0} = 1$ and

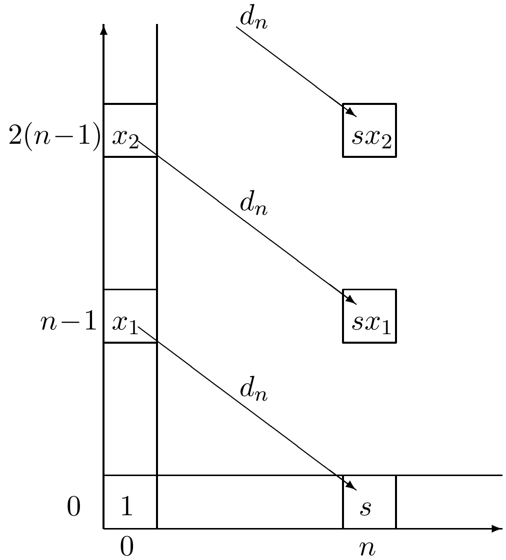
$$\alpha_{k,\ell} = \begin{cases} \alpha_{k-1,\ell} + \alpha_{k,\ell-1} & \text{if } n \text{ is odd,} \\ \alpha_{k-1,\ell} + (-1)^k \alpha_{k,\ell-1} & \text{if } n \text{ is even.} \end{cases} \tag{7}$$

Theorem 4.3. *$H^{k(n-1)}(\Omega S^n;\mathbf{Z}) \cong \mathbf{Z}$ for $k = 0, 1, \ldots,$ $H^q(\Omega S^n;\mathbf{Z}) = 0$ if $q \neq k(n-1)$. Moreover, there exist generators $x_k \in H^{k(n-1)}(\Omega S^n;\mathbf{Z})$ such that $x_k x_\ell = \alpha_{k,\ell} x_{k+\ell}$.*

Proof. We consider the cohomological Leray-Serre spectral sequence for the fiber bundle

$$P(S^n) \xrightarrow{\Omega S^n} S^n.$$

The E_2-term looks as follows:



There are only two columns $E_2^{0,*}$ and $E_2^{n,*}$ with nontrivial groups, and clearly the differential d_n is the only possible nontrivial differential. Even more, d_n should wipe out all nontrivial groups except $E_2^{0,0}$. Clearly it implies that $H^q(\Omega S^n;\mathbf{Z}) \cong \mathbf{Z}$ if $q = k(n-1)$ and $H^q(\Omega S^n;\mathbf{Z}) = 0$ otherwise. Let $s \in H^n(S^n;\mathbf{Z}) \cong E_2^{n,0}$ be a generator. There is a choice (actually a unique one) of generators $x_k \in H^{k(n-1)}(\Omega S^n;\mathbf{Z})$ so that $d_n x = s$ and $d_n(x_k) = s x_{k-1}$. Also there exist integers $\alpha_{k,l}$ such that $x_k x_l = \alpha_{k,l} x_{k+l}$. Now we use multiplicative formula for d_n to get:

$$d_n(x_k x_l) = d_n(\alpha_{k,l} x_{k+l}) = \alpha_{k,l} s x_{k+l-1}$$

On the other hand we have:

$$\begin{aligned} d_n(x_k x_l) &= d_n(x_k) x_l + (-1)^{k(n-1)} x_k d_n(x_l) \\ &= s x_{k-1} x_l + (-1)^{k(n-1)} x_k s x_{l-1} = \left(\alpha_{k-1,l} + (-1)^{k(n-1)} \alpha_{k,l-1}\right) s x_{k+l-1}. \end{aligned}$$

Clearly we have that $\alpha_{k,l}$ satisfy the identity (7). □

5. Ranks of homotopy groups of spheres

Let X be a CW-complex, and $\pi = \pi_n X$ be the first nontrivial homotopy group of X, $n \geq 2$. Recall that in this case $H_n(X;\mathbf{Z}) \cong \pi_n X$ and there is a fundamental class $a_n \in H^n(X;\pi)$. The class a is represented by a map $f : X \longrightarrow K(\pi, n)$, so that $a = f^*(\iota_n)$, where $\iota_n \in H^n(K(\pi, n);\pi)$ is the fundamental class. Consider the following diagram:

$$\begin{tikzcd} X|_{n+1} \arrow[r, "\widetilde{f}"] \arrow[d, "K(\pi,n-1)"'] & P(K(\pi,n)) \arrow[d, "K(\pi,n-1)"'] \\ X \arrow[r, "f"] & K(\pi,n) \end{tikzcd} \tag{8}$$

where $E \xrightarrow{K(\pi,n-1)} K(\pi,n)$ is the fiber bundle with $\Omega K(\pi,n)=K(\pi,n-1)$, $P(K(\pi,n))$ is the space of paths, and $X|_{n+1}$ is a pull-back of the fiber bundle $P(K(\pi,n)) \longrightarrow K(\pi,n)$.

Lemma 5.1. *The space $X|_n$ has the following property:*

$$\pi_q\left(X|_{n+1}\right) = \begin{cases} 0 & \text{if } \ i \leq n, \\ \pi_q X & \text{if } \ i \geq n+1 \end{cases}$$

Proof. We consider the exact sequences in homotopy groups:

$$\begin{tikzcd} \pi_i X|_{n+1} \arrow[r] \arrow[d, "\widetilde{f}"'] & \pi_i X \arrow[r, "\partial"] \arrow[d, "f_*"'] & \pi_{i-1}K(\pi,n-1) \arrow[r] \arrow[d, "Id"'] & \pi_{i-1}X|_{n+1} \arrow[d, "\widetilde{f}_*"'] \\ \pi_i P(K(\pi,n)) \arrow[r] & \pi_i K(\pi,n) \arrow[r, "\partial"] & \pi_{i-1}K(\pi,n-1) \arrow[r] & \pi_{i-1}P(X) \end{tikzcd} \tag{9}$$

Clearly the boundary homomorphism $\partial : \pi_n K(\pi,n) \longrightarrow \pi_{n-1}K(\pi,n-1)$ is an isomorphism in the bottom row, consequently $\partial : \pi_n X \longrightarrow \pi_{n-1}K(\pi,n-1)$ is an isomorphism since $f_* : \pi_n X \longrightarrow \pi_n K(\pi,n)$ is an isomorphism. Then $\pi_i K(\pi,n-1)=0$ for $i \neq n$, so the homomorphism $\pi_i X|_{n+1} \longrightarrow \pi_i X$ is an isomorphism for $i \neq n$. □

Remark. We did not compute yet the cohomology ring $H^*(\mathbf{CP}^\infty)$. We will do this shortly, however I would like to use the fact that $H^*(\mathbf{CP}^\infty;\mathbf{Z}) \cong \mathbf{Z}[x]$ with $\deg x = 2$.

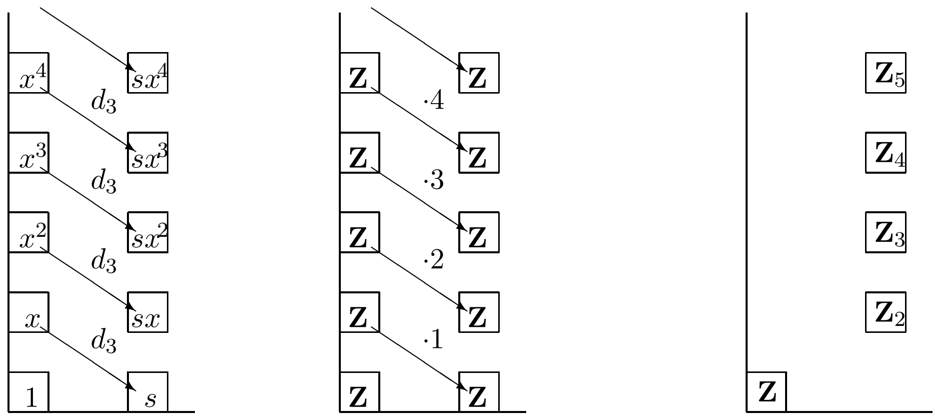
Now we consider $X=S^3$, and let $Y=S^3|_4$, i.e. Y is a pull-back of the fiber bundle $P(K(\mathbf{Z},3)) \longrightarrow K(\mathbf{Z},3)$ with the fiber $K(\mathbf{Z},2)$:

$$\begin{tikzcd} Y \arrow[r, "\widetilde{f}"] \arrow[d, "K(\mathbf{Z},2)"'] & P(K(\mathbf{Z},3)) \arrow[d, "K(\mathbf{Z},2)"'] \\ S^3 \arrow[r, "f"] & K(\mathbf{Z},3) \end{tikzcd}$$

Lemma 5.2. *There is an isomorphism:*

$$H^q(Y;\mathbf{Z}) \cong \begin{cases} \mathbf{Z}_m & \text{if } q=2m+1,\ m=2,3,4,\dots \\ 0 & \text{if } q \neq 0,5,7,\dots \end{cases}$$

Proof. Consider the cohomological spectral sequence for the fiber bundle $Y \longrightarrow S^3$. The E_2-term is shown below, where $x \in H^2(\mathbf{CP}^\infty;\mathbf{Z}) \cong E_2^{0,2}$ is a generator, and s is a generator of the group $H^3(S^3;\mathbf{Z}) \cong E_2^{3,0}$.



$E_2 = E_3$-terms $\qquad\qquad$ $E_4 = E_\infty$-term

We have that the group $E_2^{0,2m}$ is generated by x^m, the group $E_2^{3,2m}$ is generated by sx^m. All other groups $E_2^{p,q}$ are trivial. The only possible differential is d_3, so $E_2 = E_3$, and $E_4 = E_\infty$. On the other hand, the space Y is 3-connected, consequently the groups $E_2^{3,0} = \mathbf{Z}$ and $E_2^{0,2}$ should disappear, so the differential $d_3 : E_3^{0,2} \longrightarrow E_3^{3,0}$ is an isomorphism. We may assume that $d_3(x) = s$. Thus

$$d_3(x^m) = mx^{m-1}d_3(x) = msx^{m-1}.$$

In other words, the differential $d_3 : E_2^{0,2m} \cong \mathbf{Z} \longrightarrow \mathbf{Z} \cong E_3^{3,2m-2}$ is a multiplication by m as it is shown above. The resulting $E_4 = E_\infty$-term is shown above. This completes our proof. $\square$

Remark. The universal coefficient formula implies that

$$H_q(Y;\mathbf{Z}) \cong \begin{cases} \mathbf{Z}_m & \text{if } q = 2m,\ m = 2,3,4,\dots \\ 0 & \text{if } q \neq 0,4,6,8,\dots \end{cases}$$

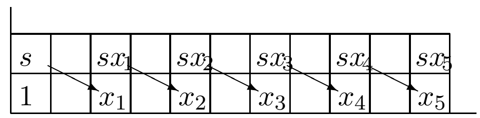
For instance we have that $\mathbf{Z}_2 \cong H_4(Y;\mathbf{Z}) \cong \pi_4(Y) \cong \pi_4 S^3$.

Corollary 5.3. $\pi_{n+1}S^n \cong \mathbf{Z}_2$ *for* $n \geq 3$.

Now we compute the cohomology ring $H^*(\mathbf{CP}^\infty;\mathbf{Z})$.

Lemma 5.4. *There is an isomorphism of rings:* $H^*(\mathbf{CP}^\infty;\mathbf{Z}) \cong \mathbf{Z}[x]$, *where* $x \in H^2(\mathbf{CP}^\infty;\mathbf{Z})$.

Proof. Consider the Hopf fiber bundle $S^\infty \longrightarrow \mathbf{CP}^\infty$ with the fiber S^1. The cohomological spectral sequence has the following E_2 term:



Let $x_i \in H^{2i}(\mathbf{CP}^\infty;\mathbf{Z}) \cong E_2^{0,2i} \cong \mathbf{Z}$ be a generator, $s \in H^1(S^1;\mathbf{Z}) \cong E_2^{1,0} \cong \mathbf{Z}$ be a generator as well. The differential d_2 is nontrivial since the groups $E_2^{1,0}$, $E_2^{0,2i}$ have to disappear: $d_2(s) = x_1$. The element sx_1 is a generator of $E_2^{1,2}$, and $d_2(sx_1) = x_2$. On the other hand, $d_2(sx_1) = d_2(s)x_1 = x_1^2$, i.e. $x_1^2 = x_2$. An obvious induction completes the argument. □

It is already clear how important is to compute the cohomology rings $H^*(K(\pi,n);G)$. This problem was solved completely in the case of an abelian group π by Borel, Cartan and Serre. Here is the first general result.

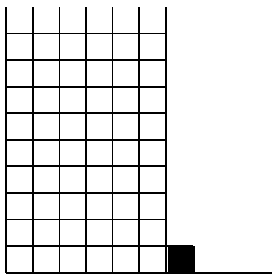
Theorem 5.5. *Let π and G be a finite (finitely generated) abelian groups. Then for any $q>0$, $n>0$ the group $H^q(K(\pi,n);G)$ is finite (finitely generated).*

Proof. First we note that it is enough to prove the statement in the case $G=\mathbf{Z}$: the universal coefficient theorem will imply the result for any finitely generated abelian group.

Let $n=1$. Then the statement of the theorem holds. We know well the spaces $K(\mathbf{Z},1) = S^1$, $K(\mathbf{Z}_2,1) = \mathbf{RP}^\infty$, $K(\mathbf{Z}_m,1) = L_m^\infty$ if $m \geq 3$, the last one is the infinite lens space. We also know their cohomology with coefficients in $\mathbf{Z}$:

q	0	1	2	3	4	5	6	$\ldots$	$2k-1$	$2k$	$\ldots$
S^1	$\mathbf{Z}$	$\mathbf{Z}$	0	0	0	0	0	$\ldots$	0	0	$\ldots$
$\mathbf{RP}^\infty$	$\mathbf{Z}$	0	$\mathbf{Z}_2$	0	$\mathbf{Z}_2$	0	$\mathbf{Z}_2$	$\ldots$	0	$\mathbf{Z}_2$	$\ldots$
L_m^∞	$\mathbf{Z}$	0	$\mathbf{Z}_m$	0	$\mathbf{Z}_m$	0	$\mathbf{Z}_m$	$\ldots$	0	$\mathbf{Z}_m$	$\ldots$

We assume by induction that the statement of the theorem holds for $K(\pi,n-1)$. Assume that it fails for $K(\pi,n)$. Let m be the first index so that the group $H^m(K(\pi,n);\mathbf{Z})$ is infinite (infinitely generated) group. Consider the cohomological spectral sequence of the fiber bundle $* \xrightarrow{K(\pi,n-1)} K(\pi,n)$, where $*$ denotes the space of paths $P(K(\pi,n))$, which is a contractable space. The inductive assumption and the universal coefficients formula imply that the groups $E_2^{p,q}$ are finite (finitely generated) if $p<m$. The groups $E_r^{p,q}$ are obtained out of the groups $E_2^{p,q}$ by means of taking subgroups and factor-groups, consequently the groups $E_r^{p,q}$ are finite (finitely generated) if $p<m$, see the picture below:



Here the light boxes are finite (finitely generated) groups, and the black box is the infinite (infinitely generated) group. Clearly the factor group of infinite (infinitely generated) group by a finite (finitely generated) subgroup is infinite (infinitely generated). We have that the groups

$$E_3^{m,0} = E_2^{m,0}/\mathrm{Im}\ d_3,$$
$$E_4^{m,0} = E_3^{m,0}/\mathrm{Im}\ d_4,$$
$$\dots$$
$$E_{r+1}^{m,0} = E_r^{m,0}/\mathrm{Im}\ d_r$$

are infinite (infinitely generated). In particular, the group $E_\infty^{m,0}$ is infinite (infinitely generated). It contradicts to the fact that $E_\infty^{m,0} = 0$. $\square$

Remark. There is important generalization of Theorem 5.5. Let $\mathcal{C}$ be a *class of abelian groups.* It means that

(1) each abelian group G either belongs to $\mathcal{C}$ or does not;
(2) the trivial group is in $\mathcal{C}$;
(3) two isomorphic groups either both belong to $\mathcal{C}$ or do not;
(4) if a group G belongs to $\mathcal{C}$ then each subgroup H of G also belongs to $\mathcal{C}$;
(5) if a subgroup $H \subset G$ and G/H belongs to $\mathcal{C}$, then G belongs to $\mathcal{C}$.

This definition due to Serre. The examples of such classes are the class $\mathcal{C}_p$ of abelian p-groups, the class $\mathcal{C}_f$ of finite abelian groups or the class $\mathcal{C}_{fg}$ of finitely generated groups.

Exercise. Prove the following theorem:

Theorem 5.6. *Let $\mathcal{C}$ be a class of abelian groups, and $\pi \in \mathcal{C}$, and G be any finitely-generated group. Then the groups $H^q(K(\pi, n); G)$ belong to the class $\mathcal{C}$ for all $q, n > 0$.*

We have the following interesting application:

Theorem 5.7. *Let X be a simply-connected CW-complex, such that the homology groups $H_q(X; \mathbf{Z})$ are finite (finitely generated). Then the homotopy groups $\pi_q X$ are finite (finitely generated) for all $q > 0$.*

Proof. First we prove the following result.

Lemma 5.8. *Let $E \longrightarrow B$ be a fiber bundle with a fiber F over a simply-connected space B. Then if the groups $H^q(B; \mathbf{Z})$ and $H^p(F; \mathbf{Z})$ are finite (finitely generated) for all $p, q > 0$ then the groups $H^n(E; \mathbf{Z})$ are finite (finitely generated) for all $n > 0$.*

Proof of Lemma. Consider the cohomological spectral sequence for this bundle. $\square$

Proof of Theorem 5.7. We apply Lemma 5.8 for the fiber bundles and the universal coefficient formula

$$X|_3 \xrightarrow{K(H_2(X|_2))} X, \quad X|_4 \xrightarrow{K(H_3(X|_3,1))} X|_3, \quad X|_5 \xrightarrow{K(H_4(X|_4,1))} X|_4, \ldots$$

to conclude that the homology groups $H_q(X|_n;\mathbf{Z})$ are finite (finitely generated). Then we have that $H_n(X|_n;\ Z) \cong \pi_n(X|_n) \cong \pi_n(X)$ are finite (finitely generated). $\square$

Corollary 5.9. *The groups $\pi_q(S^3)$ are finite for $q \geq 4$.*

Indeed, we may apply Theorem 5.7 for the space $Y = S^3|_4$. $\square$

Exercise. Let G be a finite group. Prove that $H^*(K(G,n);\mathbf{Q}) \cong H^*(pt;\mathbf{Q})$.

Let R be a commutative ring. We denote $\Lambda_R(x_1,\ldots,x_k)$ the exterior algebra on the generators $x_1,\ldots,x_k$.

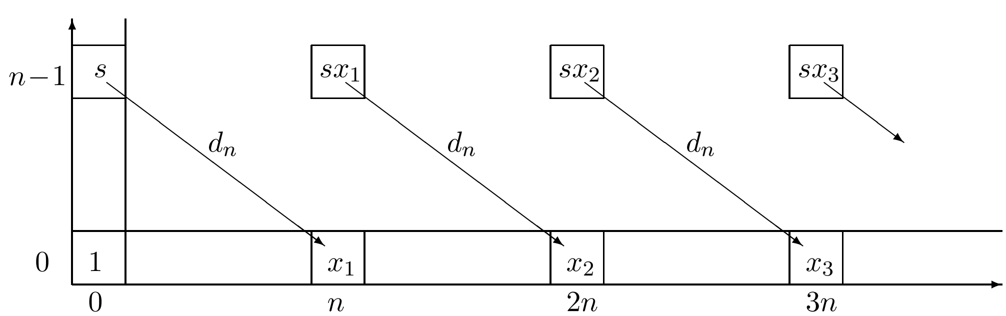
Theorem 5.10. *There is an isomorphism*

$$H^*(K(\mathbf{Z},n);\mathbf{Q}) = \begin{cases} \Lambda_{\mathbf{Q}}(x), & \dim x = n, \quad \text{if } n \text{ is odd,} \\ \mathbf{Q}[x], & \dim x = n, \quad \text{if } n \text{ is even.} \end{cases} \tag{10}$$

Proof. The statement holds if $n=1$. Induction on n. Assume that the statement holds for $K(\mathbf{Z},n-1)$. Consider the cohomological spectral sequence with coefficients in $\mathbf{Q}$ for the fiber bundle

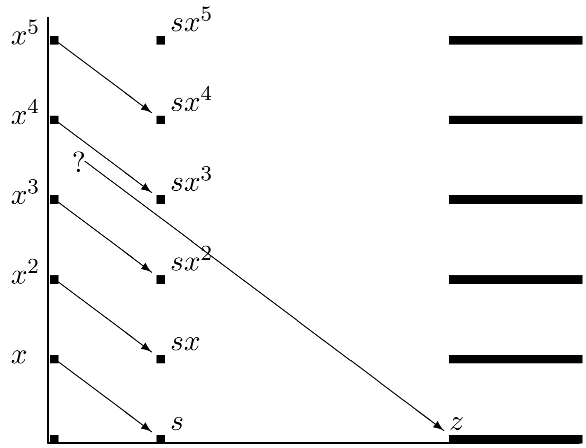
$$* \xrightarrow{K(\mathbf{Z},n-1)} K(\mathbf{Z},n).$$

Let n be even. Then (by induction) $H^*(K(\mathbf{Z},n-1);\mathbf{Q}) = \Lambda_{\mathbf{Q}}(x)$, $\dim x = n-1$. Here is the E_2-term of cohomological spectral sequence:



Here $s \in H^{n-1}(K(\mathbf{Z},n-1);\mathbf{Q}) \cong E_2^{0,n-1}$ is a generator. Then clearly the differential d_n takes s to the generator $x_1 \in H^n(K(\mathbf{Z},n);\mathbf{Q}) = E_2^{n,0}$: $d_n(s) = x_1$. Then the groups $E_2^{kn,0} \cong \mathbf{Q}$, let $x_k \in E_2^{kn,0}$ be generators such that $d_n(sx_1) = x_2$, $d_n(sx_2) = x_3$, $\ldots$ $d_n(sx_{k-1}) = x_k$, ... Then we have that $d_n(sx_1) = x_1^2$, $d_n(sx_1^2) = x_1^3$, $\ldots$ $d_n(sx_1^{k-1}) = x_1^k$, and so on. Thus $H^*(K(\mathbf{Z},n);\mathbf{Q}) = \mathbf{Q}[x]$.

Now let let n be odd. By induction we have that $H^*(K(\mathbf{Z}, n-1); \mathbf{Q}) = \mathbf{Q}[x]$, where $\dim x = n-1$. We have that $E_2^{p,0} = H^p(K(\mathbf{Z}, n); \mathbf{Q}) = 0$ for $0 < p < n$, and let $s \in E_2^{n,0} = H^n(K(\mathbf{Z}, n); \mathbf{Q}) = \mathbf{Q}$ be a generator. We have that $d_n(x) = s$, and $d_n(x^k) = ksx^{k-1} \in E_2^{n,(n-1)k}$ or $d_n\left(\dfrac{x^k}{k}\right) = sx^{k-1}$.



Now we assume that there is a nontrivial element $z \in H^m(K(\mathbf{Z}, n); \mathbf{Q}) = E_2^{m,0}$, where $m > n$, and let m be a minimal index. However there is no element in $E_{n+1}^{p,q}$ which may support a nontrivial differential which would kill z. Thus $E_\infty^{m,0} = H^m(K(\mathbf{Z}, n); \mathbf{Q})$ is not zero. Contradiction. $\square$

Corollary 5.11. *Let π be an abelian group, and rank $\pi = r$. Then*

$$H^*(K(\pi, n); \mathbf{Q}) = \begin{cases} \Lambda_{\mathbf{Q}}(x_1, \ldots, x_r), & \text{if } n \text{ is odd,} \\ \mathbf{Q}[x_1, \ldots, x_r], & \text{if } n \text{ is even,} \end{cases} \tag{11}$$

where $\deg x_i = n$, $i = 1, \ldots, r$.

Finally we prove the following result concerning the homotopy groups of spheres.

Theorem 5.12.

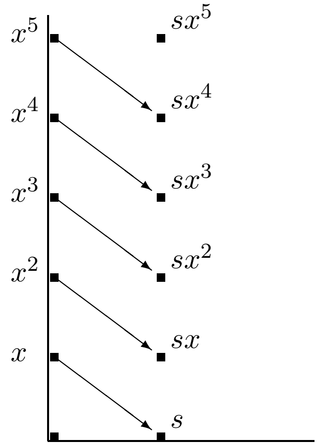
$$\operatorname{rank} \pi_q S^n = \begin{cases} 1 & \text{if } q = n \text{ or} \\ & \text{if } n = 2k \text{ and } q = 4k-1, \\ 0 & \text{otherwise.} \end{cases} \tag{12}$$

Proof. Let n be odd. Consider the cohomological spectral sequence with coefficients in $\mathbf{Q}$ of the fiber bundle

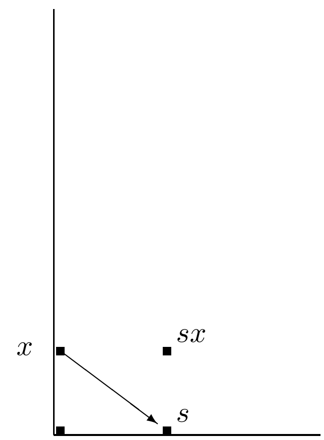
$$S^n|_{n+1} \xrightarrow{K(\mathbf{Z}, n-1)} S^n$$

(see the picture below). We have that $H^*(K(\mathbf{Z}, n-1); \mathbf{Q}) \cong \mathbf{Q}[x]$, so the groups $E_2^{k(n-1),0} \cong \mathbf{Q}$ with generators x^k, all othe groups $E_2^{p,0} = 0$ if $p \neq k(n-1)$. Let s

be a generator of $H^n(S^n;\mathbf{Q}) = E_2^{0,n}$. We already know the argument to show that $dx^k = kx^{k-1}dx = ksx^{k-1}$ which implies that $E_\infty^{p,q} = 0$ unless $p = q = 0$. Consequently the homotopy groups $\pi_q(S^n|_{n+1})$ are finite, so the homotopy groups $\pi_q S^n$ are finite.



n is odd



n is even

Now let n be even. Again we consider the fiber bundle $S^n|_{n+1} \xrightarrow{K(\mathbf{Z},n-1)} S^n$ and the cohomological spectral sequence for this bundle with coefficients in $\mathbf{Q}$. Here we have that $H^*(K(\mathbf{Z},n-1);\mathbf{Q}) = \Lambda_{\mathbf{Q}}(x)$, so the E_2-term looks as it is shown above, where $d_n(x) = s$ and $d_n(sx) = sd_n(x) = s^2 = 0$. We conclude that $H^*(S^n|_{n+1};\mathbf{Q}) \cong H^*(S^{2n-1};\mathbf{Q})$. We obtain that the homology groups $H_q(S^n|_{n+1};\mathbf{Z})$ are finite for $n+1 \leq q \leq 2n-2$. In particular, the group $H_{n+1}(S^n|_{n+1};\mathbf{Z}) \cong \pi_{n+1}S^n|_{n+1} = \pi_{n+1}S^n$ is finite (here we assume that $n > 2$, otherwise we should take the next step, see below). Now consider the fiber bundle

$$S^n|_{n+2} \xrightarrow{K(\pi_{n+1}S^n,n)} S^n|_{n+1}$$

and examine the cohomological spectral sequence of this fiber bundle with coefficients in $\mathbf{Q}$. We already know that $H^*(K(\pi_{n+1}S^n,n);\mathbf{Q}) \cong H^*(pt;\mathbf{Q})$, consequently the spectral sequence gives that $H_q(S^n|_{n+1};\mathbf{Q}) \cong H_q(S^n|_{n+2};\mathbf{Q})$ for $q > 0$ (again if $n > 2$). In particular, $H_q(S^n|_{n+2};\mathbf{Q})$ is a finite group. Repeating the above argument we derive the following:

- The group $\pi_{n+2}(S^n|_{n+2}) = \pi_{n+2}S^n$ is finite.
- The group $\pi_{n+3}(S^n|_{n+3}) = \pi_{n+3}S^n$ is finite.

. . .

- The group $\pi_{2n-2}(S^n|_{2n-2}) = \pi_{2n-2}S^n$ is finite.
- There is an isomorphism $H^*(S^n|_{2n-1};\mathbf{Q}) \cong H^*(S^n|_{2n-2};\mathbf{Q})$.

Thus we conclude:

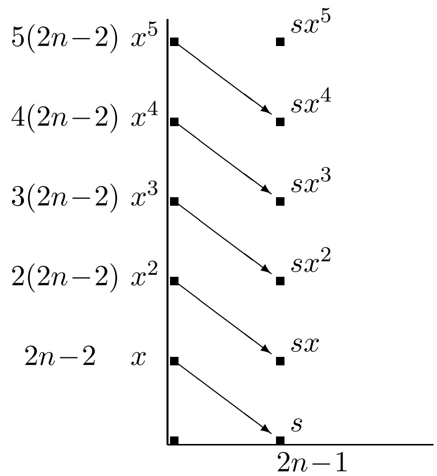
$$H^*(S^n|_{2n-1};\mathbf{Q}) \cong H^*(S^n|_{n+1};\mathbf{Q}) \cong H^*(S^{2n-1};\mathbf{Q}),$$

$$H_{2n-1}(S^n|_{2n-1};\mathbf{Z}) \cong \pi_{2n-1}(S^n|_{2n-1}) \cong \pi_{2n-1}S^n = \mathbf{Z} \oplus \text{finite group}$$

Then the spectral sequence of the fiber bundle

$$S^n|_{2n} \xrightarrow{K(\pi_{2n-1}S^n,2n-2)} S^n|_{2n-1}$$

gives the isomorphism $H^*(S^n|_{2n};\mathbf{Q}) = H^*(pt;\mathbf{Q})$.



Hence the homology groups $H_q(S^n|_{2n};\mathbf{Z})$ are finite for $q \geq 2n$, which implies the the homotopy groups $\pi_q(S^n|_{2n}) \cong \pi_q S^n$ are finite for $q \geq 2n$. $\square$