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STABLE MODULI SPACES OF HIGH DIMENSIONAL MANIFOLDS

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Dedicated to Ib Madsen on the occasion of his 70th birthday

Abstract. We prove an analogue of the Madsen–Weiss theorem for high dimensional manifolds. For example, we explicitly describe the ring of characteristic classes of smooth fibre bundles whose fibres are connected sums of g copies of $S^n \times S^n$, in the limit $g \to \infty$. Rationally it is a polynomial ring in certain explicit generators, giving a high dimensional analogue of Mumford's conjecture.

More generally, we study a moduli space $\mathcal{N}(P)$ of those null-bordisms of a fixed $(2n-1)$-dimensional manifold P which are highly connected relative to P. We determine the homology of $\mathcal{N}(P)$ after stabilisation using certain self-bordisms of P. The stable homology is identified with that of a certain infinite loop space.

1. Introduction and statement of results

This paper proves a generalisation of Madsen–Weiss' theorem [MW07] to manifolds of any even dimension above 4, building on the methods of our previous paper [GRW10] in dimension 2. Recall that Madsen–Weiss' theorem says that there is a map

$$\operatorname*{hocolim}_{g\to\infty} B\mathrm{Diff}(\Sigma_g, D^2) \longrightarrow \Omega^\infty_\bullet MTSO(2)$$

inducing an isomorphism in any homology theory. Here, Σ_g is an oriented closed 2-manifold of genus g and $D^2 \subset \Sigma_g$ is a disc. The maps in the direct system are induced by taking connected sum with a torus inside the disc. We will recall the definition of $\Omega^\infty_\bullet MTSO(2)$ below, but it has rational cohomology $\mathbb{Q}[\kappa_1, \kappa_2, \dots]$, the polynomial algebra in infinitely many variables with degrees $|\kappa_i| = 2i$. As a corollary, they determined

$$\varprojlim H^*(B\mathrm{Diff}(\Sigma_g, D^2);\mathbb{Q}) = \mathbb{Q}[\kappa_1, \kappa_2, \dots],$$

settling Mumford's conjecture ([Mum83]). The goal of the present paper is to prove analogues of the Madsen–Weiss theorem and Mumford's conjecture for manifolds of higher dimension. We have results for manifolds of any even dimension except 4. As an interesting special case of our results, we completely determine the stable rational cohomology ring

$$\varprojlim H^*(B\mathrm{Diff}(W_g, D^{2n});\mathbb{Q}),$$

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where $W_g = \#^g S^n \times S^n$ denotes the connected sum of g copies of $S^n \times S^n$. To state our results more precisely, we need some definitions.

1.1. **Definitions and recollections.** Let us first recall some facts about classifying spaces of diffeomorphism groups. For a closed manifold W, the group $\mathrm{Diff}(W)$ is given the C^∞ topology and $B\mathrm{Diff}(W)$ denotes its classifying space. This can be defined in a number of ways, for example using Milnor's construction [Mil56], or more geometrically as the quotient space $\mathrm{Emb}(W, \mathbb{R}^\infty)/\mathrm{Diff}(W)$, where $\mathrm{Emb}(W, \mathbb{R}^\infty)$ is given the C^∞ topology. Either way, it classifies smooth fiber bundles: for smooth X, there is a bijection between the set $[X, B\mathrm{Diff}(W)]$ of homotopy classes of maps, and the set of isomorphism classes of smooth fiber bundles $E \to X$ with fiber W.

In the case where W has boundary, we will first pick an embedding $\partial W \to \mathbb{R}^\infty = \{0\} \times \mathbb{R}^\infty$ and let $\mathrm{Emb}^\partial(W, [0,\infty) \times \mathbb{R}^\infty)$ denote the space of all extensions to an embedding of W (required to be standard on a collar neighbourhood of ∂W). Let $\mathrm{Diff}(W, \partial W)$ denote the group of diffeomorphisms that restrict to the identity on (a neighbourhood of) the boundary, and let $B\mathrm{Diff}(W, \partial W) = \mathrm{Emb}^\partial(W, [0,\infty) \times \mathbb{R}^\infty)/\mathrm{Diff}(W, \partial W)$ be the orbit space. If W is closed and $A \subset W$ is a codimension 0 submanifold, we write $B\mathrm{Diff}(W, A) = B\mathrm{Diff}(W - \mathrm{int} A, \partial A)$. The construction of $B\mathrm{Diff}(W, \partial W)$ has the following naturality property: any inclusion $W \subset W'$ of a codimension 0 submanifold induces a continuous map $B\mathrm{Diff}(W, \partial W) \to B\mathrm{Diff}(W', \partial W')$, well defined up to homotopy. (On the point-set level it depends on a choice of embedding of the cobordism $W' - \mathrm{int} W$ into $[0,1] \times \mathbb{R}^\infty$.) For example, the inclusion $W_g - \mathrm{int} D^{2n} \to W_{g+1}$ induces a map $B\mathrm{Diff}(W_g, D^{2n}) \to B\mathrm{Diff}(W_{g+1}, D^{2n})$.

Secondly, we recall that for any space B and any map $\theta : B \to BO(d)$, where $BO(d) = \mathrm{Gr}_d(\mathbb{R}^\infty)$, there is a Thom spectrum $MT\theta = B^{-\theta}$ constructed in the following way. First, we let $B(\mathbb{R}^n) = \theta^{-1}(\mathrm{Gr}_d(\mathbb{R}^n))$. The Grassmannian $\mathrm{Gr}_d(\mathbb{R}^n)$ carries a $(d-n)$-dimensional bundle of orthogonal complements which we denote $\gamma_n^\perp$. Then the nth space of the spectrum $MT\theta$ is the Thom space $B(\mathbb{R}^n)^{\theta^*\gamma_n^\perp}$. The associated infinite loop space is the direct limit

$$\Omega^\infty MT\theta = \operatorname*{colim}_{n\to\infty} \Omega^n \left(B(\mathbb{R}^n)^{\theta^*\gamma_n^\perp} \right),$$

and we shall write $\Omega^\infty_\bullet MT\theta$ for the basepoint component. The rational cohomology of this space is easy to describe; in the case where the bundle classified by θ is oriented, it is as follows: for each $c \in H^{d+k}(B)$, there is a corresponding "generalised Mumford–Morita–Miller class" $\kappa_c \in H^k(\Omega^\infty MT\theta)$, and $H^*(\Omega^\infty_\bullet MT\theta; \mathbb{Q})$ is the free graded-commutative algebra on the classes κ_c, where c runs through a basis for the vector space $H^{>d}(B; \mathbb{Q})$. We describe the general case in Section 2.5.

Thirdly, recall that for any map $A \to X$ of spaces and any $n \geq 0$, there is a factorisation $A \to B \to X$ with the property that $\pi_i(A) \to \pi_i(B)$ is surjective for $i = n$ and bijective for $i < n$, and $\pi_i(B) \to \pi_i(X)$ is injective for $i = n$ and bijective for $i > n$ (the requirements are imposed for all basepoints). This is the nth stage of the Moore–Postnikov tower for the map $A \to X$, and can be constructed for example by attaching cells (of dimension $> n$) to A. It is well known that a factorisation $A \to B \to X$ with these properties is unique up to weak homotopy equivalence. In the case where A is a point, $B = X\langle n\rangle \to X$ is the n-connective cover of the based space X, characterised by the property that $\pi_i(X\langle n\rangle) = 0$ for $0 \leq i \leq n$, and that $\pi_i(X\langle n\rangle) \to \pi_i(X)$ is an isomorphism for $i > n$.

With these definitions, we can now state our main theorems. Since our most general result (Theorem 1.8) is technical to state, we start by formulating some interesting special cases. (But we remark that the general statement is not much harder to prove than the special case.)

1.2. **Connected sums of copies of $S^n \times S^n$.** The most direct generalisation of Madsen–Weiss' theorem to dimension $2n$ (recall that we assume $2n \neq 4$) concerns the manifolds

$$W_g = \#^g S^n \times S^n,$$

the connected sum of g copies of $S^n \times S^n$.

If we pick a disc $D^{2n} \subset W_g$, there is a classifying space $B\mathrm{Diff}(W_g, D^{2n})$ and there are maps $B\mathrm{Diff}(W_g, D^{2n}) \to B\mathrm{Diff}(W_{g+1}, D^{2n})$ induced by taking connected sum with one more copy of $S^n \times S^n$. Our first main theorem determines the homotopy colimit (mapping telescope) of these maps. To state it, we let $\theta^n : BO(2n)\langle n \rangle \to BO(2n)$ be the n-connective cover, and $MT\theta^n$ the associated Thom spectrum. Let us also say that a continuous map is a *homology equivalence* if it induces an isomorphism in integral homology (and hence in any homology or cohomology theory).

Theorem 1.1. *Let $2n \neq 4$. There is a homology equivalence*

$$\operatorname*{hocolim}_{g\to\infty} B\mathrm{Diff}(W_g, D^{2n}) \longrightarrow \Omega^\infty_\bullet MT\theta^n.$$

More generally, if W is any $(n-1)$-connected closed $2n$-manifold which is parallelisable in the complement of a point, there is a homology equivalence

$$\operatorname*{hocolim}_{g\to\infty} B\mathrm{Diff}(W \# W_g, D^{2n}) \longrightarrow \Omega^\infty_\bullet MT\theta^n.$$

The classes κ_c mentioned in Section 1.1 give characteristic classes of smooth manifold bundles in the following way. Let $\pi : E \to B$ be a smooth bundle of $2n$-dimensional manifolds, with oriented vertical tangent bundle T_vE. Given a characteristic class $c \in H^{2n+k}(BSO(2n))$, we can define the associated *generalised Mumford–Morita–Miller* class by

$$\kappa_c(E) = \pi_!(c(T_vE)) \in H^k(B).$$

In particular we obtain characteristic classes $\kappa_c \in H^k(B\mathrm{Diff}^+(M))$ for any oriented $2n$-manifold M by performing this construction on the universal bundle. Using the known structure of rational cohomology of infinite loop spaces, we get the following higher dimensional analogue of Mumford's conjecture.

Corollary 1.2. *With assumptions as in Theorem 1.1, let $\mathcal{B} \subset H^*(BSO(2n);\mathbb{Q})$ be the set of monomials in the classes $e, p_{n-1}, p_{n-2}, \ldots, p_{\lceil \frac{n+1}{4} \rceil}$ of total degree $> 2n$. Then the natural map*

$$\mathbb{Q}[\kappa_c | c \in \mathcal{B}] \longrightarrow \varprojlim H^*(B\mathrm{Diff}(W \# W_g, D^{2n});\mathbb{Q})$$

is an isomorphism.

1.3. **More general manifolds.** The determination, in Theorem 1.1 and Corollary 1.2 above, of the stable homology and cohomology of $B\mathrm{Diff}(W\#gS^n \times S^n, D^{2n})$ is a special case of a theorem determining the stable homology of $B\mathrm{Diff}(W)$ for more general manifolds W. In the general case, the stabilisation process is more involved than forming connected sum with $S^n \times S^n$.

Theorem 1.3. *Let W be a compact manifold of dimension $2n > 4$, such that $(W, \partial W)$ is $(n-1)$-connected (equivalently, W is homotopy equivalent to a CW complex of dimension n). Let K be a smooth cobordism from ∂W to itself, with the properties that*

(i) $(K, \partial W)$ *is $(n-1)$-connected,*
(ii) $\mathrm{Ker}(\pi_{n-1}(\partial W) \to \pi_{n-1}(W)) \subset \mathrm{Ker}(\pi_{n-1}(\partial W) \to \pi_{n-1}(K))$,
(iii) $\mathrm{Im}(\pi_n(W) \to \pi_n(BO)) \subset \mathrm{Im}(\pi_n(K) \to \pi_n(BO))$, *where $K \to BO$ and $W \to BO$ classify the stable normal bundles.*

(iv) For some $g > 0$, the iterated composition gK contains a submanifold diffeomorphic to $S^n \times S^n - \mathrm{int}(D^{2n})$. (In case K is not connected, we require that this holds for each path component of gK.)

In (i) and (ii) we consider $\partial W \subset K$ to be the incoming boundary. Then there is a homology equivalence

$$\operatorname*{hocolim}_{g\to\infty} B\mathrm{Diff}(W \cup_{\partial W} gK, \partial W) \longrightarrow \Omega^\infty_\bullet MT\theta_K,$$

where $\theta_K : B_K \to BO(2n)$ is the nth stage in the Moore–Postnikov tower for the Gauss map $K \to BO(2n)$.

There is a further way of interpreting our results. If (W, ℓ_W) and (K, ℓ_K) are as in the statement of this theorem, then $W \cup \infty K = W \cup_{\partial W} K \cup_{\partial W} K \cup_{\partial W} \cdots$ is a non-compact manifold and the theorem implies that there is a homology equivalence

$$B\mathrm{Diff}_c(W \cup \infty K) \longrightarrow \Omega^\infty_\bullet MT\theta_K$$

from the classifying space of the group of compactly-supported diffeomorphisms of $W \cup \infty K$.

In the case $\pi_{n-1}(\partial W) \to \pi_{n-1}(W)$ is injective, there is a standard choice of K, namely the connected sum of $[0,1]\times\partial W$ with the closed manifold K_f, defined as the total space of the linear S^n-bundle over S^n classified by a map $f : S^n \to BO(n+1)$ whose class in $\pi_n(BO(n+1)) = \pi_n(BO)$ is a generator of $\mathrm{Im}(\pi_n(W) \to \pi_n(BO))$. In that case, the direct limit in the theorem can be interpreted as taking connected sum of W with more and more copies of K_f. For example, the theorem determines the stable homology of $B\mathrm{Diff}(\#^g K_f, D^{2d})$.

1.4. **Tangential structures.** Next, we present a version of Theorem 1.3 where all manifolds are equipped with a *tangential structure*, defined as follows.

Definition 1.4. Let $\theta : B \to BO(2n)$ be a map. A θ-structure on a $2n$-dimensional manifold W is a bundle map $\ell : TW \to \theta^*\gamma$, i.e. a fiberwise linear isomorphism. Such a pair (W, ℓ) will be called a θ-manifold. A θ-structure on a $(2n-1)$-dimensional manifold M is a bundle map $\varepsilon^1 \oplus TM \to \theta^*\gamma$. If ℓ is a θ-structure on W, the *induced* structure on ∂W is obtained by composing with a certain isomorphism $\varepsilon^1 \oplus T(\partial W) \to TW|_{\partial W}$. In fact, there are two such isomorphisms: One comes from a collar $[0,1) \times \partial W \to W$ of ∂W. Differentiating this gives an isomorphism $\varepsilon^1 \oplus T(\partial W) \to TW|_{\partial W}$, and the resulting θ-structure on ∂W will be called the *incoming* restriction. Another comes from a collar $(-1, 0] \times \partial W \to W$; this is the *outgoing* restriction. When W is a cobordism, we will generally use the incoming restriction to induce a θ-structure on the source of W and the outgoing restriction on the target.

Definition 1.5. Let W be a compact $2n$-dimensional manifold, and $\ell_0 : TW|_{\partial W} \to \theta^*\gamma$ be a θ-structure on ∂W. Let $\mathrm{Bun}^\partial(TW, \theta^*\gamma; \ell_0)$ denote the space of all bundle maps $\ell : TW \to \theta^*\gamma$ which restrict to ℓ_0 over ∂W. The group $\mathrm{Diff}(W, \partial W)$ of diffeomorphisms of W which restrict to the identity near ∂W acts on $\mathrm{Bun}^\partial(TW, \theta^*\gamma; \ell_0)$ in a obvious way, and we shall write

$$B\mathrm{Diff}^\theta(W, \ell_0) = (E\mathrm{Diff}(W, \partial W) \times \mathrm{Bun}^\partial(TW, \theta^*\gamma; \ell_0))/\mathrm{Diff}(W, \partial W)$$

for the homotopy orbit space of this action. If $\ell : TW \to \theta^*\gamma$ is a particular extension, we shall write $B\mathrm{Diff}^\theta(W, \ell_0)_\ell \subset B\mathrm{Diff}^\theta(W, \ell_0)$ for the path component containing ℓ.

Theorem 1.6. *Let $\theta : B \to BO(2n)$ be such that any θ-structure on D^{2n} extends to a structure on S^{2n}. Let W be a compact $2n$-dimensional manifold with θ-structure $\ell_W : TW \to \theta^*\gamma$, such that $(W, \partial W)$ is $(n-1)$-connected. Let (K, ℓ_K) be a compact* *manifold with θ-structure, which is a cobordism of θ-manifolds from ∂W to itself, satisfying the following conditions.*

(i) $(K, \partial W)$ *is* $(n-1)$*-connected,*
(ii) $\mathrm{Ker}(\pi_{n-1}(\partial W) \to \pi_{n-1}(W)) \subset \mathrm{Ker}(\pi_{n-1}(\partial W) \to \pi_{n-1}(K))$*,*
(iii) $\mathrm{Im}(\pi_n(W) \to \pi_n(B)) \subset \mathrm{Im}(\pi_n(K) \to \pi_n(B))$*,*
(iv) *For some $g > 0$, the iterated composition gK contains a submanifold diffeomorphic to $S^n \times S^n - \mathrm{int}(D^{2n})$. (In case K is not connected we require that this holds for each path component of gK.)*

In (i) and (ii) we consider $\partial W \subset K$ to be the incoming boundary. Then there is a homology equivalence

$$\operatorname*{hocolim}_{g\to\infty} B\mathrm{Diff}^\theta(W \cup_{\partial W} gK, \ell_0)_{\ell_W \cup g\ell_K} \longrightarrow \Omega^\infty_\bullet MT\theta',$$

where $\theta' : B' \to BO(2n)$ is the nth stage of the Moore–Postnikov tower for $K \to B$.

Our most general theorem about stable homology is Theorem 1.8 below. By considering all possible W's at once, it includes information about π_0. The information about π_0 given in Theorem 1.8 below is very similar to the results of Kreck [Kre99], and parts of our work have been inspired by his results and techniques.

Definition 1.7. Let P be a closed $(2n-1)$-dimensional manifold with θ-structure $\ell_P : \varepsilon^1 \oplus TP \to \theta^*\gamma$, and let $P \hookrightarrow \{0\} \times \mathbb{R}^\infty$ be a fixed embedding. The *moduli space of highly connected null-bordisms* of (P, ℓ) is the set $\mathcal{N}^\theta(P, \ell_P)$ of pairs (W, ℓ_W) where $W \subset (-\infty, 0] \times \mathbb{R}^\infty$ is a smooth $2n$-dimensional manifold with boundary $P = W \cap (\{0\} \times \mathbb{R}^\infty)$, W is collared near its boundary, and the map $\ell_W : TW \to \theta^*\gamma$ restricts to ℓ_P on the boundary. Furthermore, we require that (W, P) is $(n-1)$-connected.

If $K \subset [0, 1] \times \mathbb{R}^\infty$ is a cobordism with collared boundary $\partial K = (\{0\} \times P_0) \cup (\{1\} \times P_1)$ which is $(n-1)$-connected with respect to both P_0 and P_1 and is equipped with a θ-structure ℓ_K restricting to ℓ_0 and ℓ_1 on boundaries, there is an induced map $K_* : \mathcal{N}^\theta(P_0, \ell_0) \to \mathcal{N}^\theta(P_1, \ell_1)$ defined by taking union with K and subtracting 1 from the first coordinate.

The set $\mathcal{N}^\theta(P, \ell_P)$ is topologised as the disjoint union of spaces of the form $(\mathrm{Emb}^\partial(W, (-\infty, 0] \times \mathbb{R}^\infty) \times \mathrm{Bun}^\partial(TW, \theta^*\gamma; \ell_P))/\mathrm{Diff}(W, \partial W)$. Using the model $E\mathrm{Diff}(W, \partial W) = \mathrm{Emb}^\partial(W, (-\infty, 0] \times \mathbb{R}^\infty)$, we arrive at the description

$$\mathcal{N}^\theta(P, \ell_P) = \coprod_W B\mathrm{Diff}^\theta(W, \ell_P),$$

where the disjoint union is over compact manifolds W with $\partial W = P$ for which (W, P) is $(n-1)$-connected, one of each diffeomorphism class.

Theorem 1.8. *Let $2n > 4$ and let $\theta : B \to BO(2n)$ be a map such that all θ-structures on D^{2n} extend to S^{2n}. Let $K \subset [0, \infty) \times \mathbb{R}^\infty$ be a submanifold equipped with a θ-structure ℓ_K and assume that $x_1 : K \to [0, \infty)$ is a proper map for which the integers are regular values. For $A \subset [0, \infty)$, we let $(K|_A, \ell_K|_A)$ denote the θ-manifold $K \cap x_1^{-1}(A)$. Suppose that*

(i) $\mathcal{N}^\theta(K|_0, \ell_K|_0) \neq \emptyset$*.*
(ii) *For each $i \in \mathbb{N}$, the cobordism $K|_{[i-1,i]}$ is $(n-1)$-connected relative to each of its boundaries.*
(iii) *For each $i \in \mathbb{N}$, $\pi_n(K|_{[i,\infty)}) \to \pi_n(B)$ is surjective.*
(iv) *For each $i \in \mathbb{N}$, the map*

$$\mathrm{Ker}(\pi_{n-1}(K|_i) \to \pi_{n-1}(K_{[i,\infty)})) \longrightarrow \mathrm{Ker}(\pi_{n-1}(K|_i) \to \pi_{n-1}(B))$$

is an isomorphism.

(v) *For each $i \in \mathbb{N}$, each path-component of $K|_{[i,\infty)}$ contains a submanifold diffeomorphic to $S^n \times S^n - \mathrm{int}(D^{2n})$, which in addition has null-homotopic structure map to B.*

Then there is a homology equivalence

$$\operatorname*{hocolim}_{i\to\infty} \mathcal{N}^\theta(K|_i, \ell_K|_i) \longrightarrow \Omega^\infty MT\theta',$$

where $\theta' : B' \to B \xrightarrow{\theta} BO(2n)$ is the $(n-1)$st stage of the Moore–Postnikov tower for $\ell_K : K \to B$.

Remark 1.9. The maps in all the theorems above are induced by the Pontryagin–Thom construction. We shall briefly explain this in the setting of Theorem 1.8, after replacing $\mathcal{N}^\theta(P, \ell_P)$ by a weakly equivalent space, and refer the reader to [MT01, §2.3] for further details. First we say that a submanifold $W \subset (-\infty, 0] \times \mathbb{R}^{q-1}$ with collared boundary is *fatly embedded* if the canonical map from the normal bundle νW to $\mathbb{R}^q$ restricts to an embedding of the disc bundle into $(-\infty, 0] \times \mathbb{R}^{q-1}$. In that case the Pontryagin–Thom collapse construction gives a continuous map from $[-\infty, 0] \wedge S^{q-1}$ to the Thom space of νW. Secondly we replace $\theta' : B' \to BO(2n)$ by a fibration, and redefine $\mathcal{N}^\theta(P, \ell_P)$ as a space of pairs (W, ℓ_W) where $W \subset (-\infty, 0] \times \mathbb{R}^\infty$ is a fatly embedded submanifold, collared near $\partial W = \{0\} \times P$, and $\ell_W : W \to B'$ is a continuous map such that $\theta' \circ \ell_W : W \to BO(2n) = \mathrm{Gr}_{2n}(\mathbb{R}^\infty)$ is *equal* to the Gauss map and whose restriction to ∂W is equal to a specified map $\ell_P : P \to B'$. There is a forgetful map from the space of such pairs to the space in Definition 1.7, and standard homotopy theoretic methods imply that it is a weak equivalence. If $P \subset \mathbb{R}^{q-1} \subset \mathbb{R}^\infty$, the Pontryagin–Thom construction (composed with ℓ_P) gives a point

$$\alpha(P, \ell_P) \in \Omega^{q-1}\big(B'(\mathbb{R}^q)^{(\theta')^*\gamma^\perp}\big) \subset \Omega^{\infty-1} MT\theta',$$

and if $(W, \ell_W) \in \mathcal{N}^\theta(P, \ell_P)$ has $W \subset (-\infty, 0] \times \mathbb{R}^{q-1}$, it gives a path

$$\alpha(W, \ell_W) : [-\infty, 0] \longrightarrow \Omega^{q-1}\big(B'(\mathbb{R}^q)^{(\theta')^*\gamma^\perp}\big) \subset \Omega^{\infty-1} MT\theta',$$

starting at the basepoint and ending at $\alpha(P, \ell_P)$. The space of such paths is homotopy equivalent to the based loop space which is $\Omega^\infty MT\theta'$. Finally, the non-compact manifold $K \subset [0, \infty) \times \mathbb{R}^\infty$ admits a homotopically unique θ'-structure lifting its θ-structure and extending the canonical θ'-structure on $P = K|_0$. The Pontryagin–Thom construction applied to each bordism $K|_{[i,i+1]}$ then gives a path $\alpha(K|_{[i,i+1]}, \ell_K) : [i, i+1] \to \Omega^{\infty-1} MT\theta'$ and the entire process now commutes (strictly) with the stabilisation maps.

1.5. **Algebraic localisation.** There is one final algebraic version of our main theorem. Fix P, a closed $(2n-1)$-manifold with θ-structure $\ell_P : \varepsilon^1 \oplus TP \to \theta^*\gamma$. As explained in Definition 1.7, a bordism (K, ℓ_K) from (P, ℓ_P) to itself with $K \subset [0, 1] \times \mathbb{R}^\infty$, which is $(n-1)$-connected with respect to both boundaries, gives a self-map $(K, \ell_K)_*$ of $\mathcal{N}^\theta(P, \ell_P)$ defined by $W \mapsto W \cup_P K$. We shall write $\mathcal{K}_0$ for the set of isomorphism classes of such (K, ℓ_K), where we identify (K, ℓ_K) with (K', ℓ'_K) if there is a diffeomorphism $\varphi : K \to K'$ which is the identity near ∂K such that $\varphi^*\ell'_K$ is homotopic to ℓ_K relative to ∂K. It is clear that the homotopy class of the self-map $(K, \ell_K)_*$ depends only on the isomorphism class of (K, ℓ_K), and we get an action of the non-commutative monoid $\mathcal{K}_0$ on $H_*(\mathcal{N}^\theta(P, \ell_P))$. Our theorem determines the algebraic localisation

$$H_*(\mathcal{N}^\theta(P, \ell_P))[\mathcal{K}^{-1}]$$

at a certain commutative submonoid $\mathcal{K} \subset \mathcal{K}_0$ which we now describe.

We say that a θ-cobordism $K : P \rightsquigarrow P$ has *support* in a closed subset $A \subset P$ if it contains $[0,1] \times (P - A) : (P - A) \rightsquigarrow (P - A)$ as a sub-cobordism with the product θ-structure. We let $\mathcal{K} \subset \mathcal{K}_0$ consist elements admitting representatives with support in a regular neighbourhood of a simplicial complex of dimension at most $(n-1)$ inside P and prove the following lemma.

Lemma 1.10. *The subset $\mathcal{K} \subset \mathcal{K}_0$ is a commutative submonoid.*

By the lemma, we may localise the $\mathbb{Z}[\mathcal{K}]$-module $H_*(\mathcal{N}^\theta(P, \ell_P))$ at any submonoid $\mathcal{L} \subset \mathcal{K}$. The content of Theorem 1.11 below is an isomorphism

$$H_*(\mathcal{N}^\theta(P, \ell_P))[\mathcal{L}^{-1}] \cong H_*(\Omega^\infty MT\theta')$$

under certain conditions, where $\theta' : B' \to B \xrightarrow{\theta} BO(2n)$ is the $(n-1)$st stage of the Moore–Postnikov tower for $\ell_P : P \to B$. To describe the isomorphism explicitly, recall that we in Remark 1.9 described a map

$$\mathcal{N}^\theta(P, \ell_P) \longrightarrow \Omega^\infty MT\theta',$$

compatible with gluing highly connected bordisms of (P, ℓ_P). An obstruction theoretic argument (which we explain in more detail in Section 7.6) shows that if $\mathcal{K}'$ is defined like $\mathcal{K}$ but using θ' instead of θ, then the natural map $\mathcal{K}' \to \mathcal{K}$ is a bijection, and hence the induced map

$$H_*(\mathcal{N}^\theta(P, \ell_P)) \longrightarrow H_*(\Omega^\infty MT\theta') \tag{1.1}$$

is naturally a homomorphism of $\mathbb{Z}[\mathcal{K}]$-modules.

Theorem 1.11. *Let $2n > 4$ and let $\theta : B \to BO(2n)$ be a map such that all θ-structures on D^{2n} extend to S^{2n}. Let P be a closed $(2n-1)$-manifold with θ-structure $\ell_P : \varepsilon^1 \oplus TP \to \theta^*\gamma$, such that $\mathcal{N}^\theta(P, \ell_P)$ is non-empty. Then the morphism* (1.1) *induces an isomorphism*

$$H_*(\mathcal{N}^\theta(P, \ell_P))[\mathcal{K}^{-1}] \longrightarrow H_*(\Omega^\infty MT\theta').$$

Furthermore, localisation at a submonoid $\mathcal{L} \subset \mathcal{K}$ agrees with localisation at $\mathcal{K}$, provided $\mathcal{L}$ satisfies the following conditions.

(i) The group $\pi_n(B)$ is generated by the subgroups $\mathrm{Im}(\pi_n(K) \to \pi_n(B))$, $K \in \mathcal{L}$.
(ii) The subgroup of $\pi_{n-1}(P)$ generated by $\mathrm{Ker}(\pi_{n-1}(P) \to \pi_{n-1}(K))$, $K \in \mathcal{L}$, contains $\mathrm{Ker}(\pi_{n-1}(P) \to \pi_{n-1}(B))$.
(iii) Some element of $\mathcal{L}$ contains a submanifold diffeomorphic to $S^n \times S^n - \mathrm{int}(D^{2n})$.

1.6. **Examples and applications.** Recall that the connective cover $BO(d)\langle k\rangle$ is $BSO(d)$ if $k = 1$, $B\mathrm{Spin}(d)$ if $k = 2$ or $k = 3$, and is often called $B\mathrm{String}(d)$ if $k = 4, 5, 6, 7$. We write $MTSO(d)$, $MT\mathrm{Spin}(d)$ and $MT\mathrm{String}(d)$ for the corresponding Thom spectra. As special cases of Theorem 1.3, we have the following maps, which become homology equivalences in the limit $g \to \infty$.

$$\begin{aligned}
B\mathrm{Diff}(gS^3 \times S^3, D^6) &\longrightarrow \Omega^\infty_\bullet MT\mathrm{Spin}(6)\\
B\mathrm{Diff}(g(\mathbb{H}P^2 \# \overline{\mathbb{H}P}^2), D^8) &\longrightarrow \Omega^\infty_\bullet MT\mathrm{Spin}(8)\\
B\mathrm{Diff}(gS^4 \times S^4, D^8) &\longrightarrow \Omega^\infty_\bullet MT\mathrm{String}(8)\\
B\mathrm{Diff}(gS^5 \times S^5, D^{10}) &\longrightarrow \Omega^\infty_\bullet MT\mathrm{String}(10)\\
B\mathrm{Diff}(gS^6 \times S^6, D^{12}) &\longrightarrow \Omega^\infty_\bullet MT\mathrm{String}(12)\\
B\mathrm{Diff}(gS^7 \times S^7, D^{14}) &\longrightarrow \Omega^\infty_\bullet MT\mathrm{String}(14)\\
B\mathrm{Diff}(g(\mathbb{O}P^2 \# \overline{\mathbb{O}P}^2), D^{16}) &\longrightarrow \Omega^\infty_\bullet MT\mathrm{String}(16)
\end{aligned}$$

An example of a different flavour is $B\mathrm{Diff}(\mathbb{C}P^3\#gS^3\times S^3;\mathcal{O}p\mathbb{C}P^1)$, whose stable homology is that of $\Omega^\infty_\bullet MT\mathrm{Spin}^c(6)$, where $B\mathrm{Spin}^c(6)$ is the homotopy fiber of the map $\beta w_2 : BSO(6) \to K(\mathbb{Z},3)$.

An example where we need a more complicated stabilisation (not induced by connected sum) comes from $\mathbb{R}P^6$. The standard self-indexing Morse function $f : \mathbb{R}P^6 \to [0,6]$ given by

$$f(x_0;\cdots;x_6) = \sum_{i=0}^{6} i\cdot x_i^2$$

has one critical point of each index, and we let $W = f^{-1}([0,2.5]) \cong \mathbb{R}P^2 \times D^4$. Then $K = f^{-1}([2.5,3.5])$ is a cobordism from $\partial W = \mathbb{R}P^2 \times S^3$ to itself, and in this situation Theorem 1.3 gives a stable homology equivalence

$$B\mathrm{Diff}((\mathbb{R}P^2\times D^4)\cup_\partial gK,\partial) \longrightarrow \Omega^\infty_\bullet MT\mathrm{Pin}^-(6),$$

where $B\mathrm{Pin}^-(6)$ is the homotopy fiber of $w_2 + w_1^2 : BO(6) \to K(\mathbb{Z}/2,2)$.

An interesting special case of Theorem 1.6 concerning the manifolds $W_g = \#^g S^n \times S^n$ is the following. Let (Y,y) be a pointed space such that $\pi_n(Y,y)$ is finitely generated. The stable homology of each component of the homotopy orbit space

$$\mathcal{S}^n_g(Y,y) = E\mathrm{Diff}(W_g,D^{2n}) \times_{\mathrm{Diff}} \mathrm{Map}((W_g,D^{2n}),(Y,y))$$

is determined as the homology of the space $\Omega^\infty_\bullet(Y\langle n-1\rangle_+ \wedge MT\theta^n)$, where θ^n is as in Theorem 1.1. In this case, we stabilise by connect summing with a manifold $K = W_k$ equipped with a based map $K \to Y$ inducing a surjection $\pi_n(K,D^{2n}) \to \pi_n(Y,y)$. This result is a generalisation of the result of Cohen and Madsen [CM09], who proved the special case where $2n = 2$ and Y simply connected. (The case $2n = 2$ was generalised to non-simply connected Y in [GRW10].)

Finally, the case $\theta : BSO(2n) \to BO(2n)$ implies the following strengthening of the detection result of [Ebe11].

Corollary 1.12. *Let k be an abelian group and $c \in H^*(\Omega^\infty_\bullet MTSO(2n);k)$ be a non-zero class. Then there exists an oriented smooth closed $2n$-manifold N such that the Gauss map $N \to BSO(2n)$ is n-connected, and an oriented smooth fiber bundle $p : E \to B$ with fibers N such that the characteristic class associated to c is non-vanishing in $H^*(B;k)$. If $k = \mathbb{Q}$ there exists a bundle where B is an oriented closed manifold and $\int_B c \neq 0$*

Proof. We explain how to deduce the result from Theorem 1.3. We first build an oriented compact $2n$-manifold W with boundary whose stable normal bundle $W \to BSO$ is n-connected. Such a manifold can be constructed as a handlebody with handles of dimension $\leq n$, and the associated structure $\theta_W : B_W \to BO(2n)$ will be equivalent to $BSO(2n) \to BO(2n)$. Pick a self-indexing Morse function $f : W \to [0, n+.5]$ and set $K_0 = f^{-1}([n-.5, n+.5])$. The cobordism $\overline{K_0}$, obtained by reversing orientation of K_0, is a bordism from ∂W to $f^{-1}(n-.5)$, and we can let $K = (I\times\partial W)\#(\overline{K_0}K_0)\#(S^n\times S^n)$. This gives a bordism from ∂W to itself, satisfying the assumptions in Theorem 1.3, so the theorem gives a map

$$\operatorname*{hocolim}_{g\to\infty} B\mathrm{Diff}(W\cup gK,\partial) \longrightarrow \Omega^\infty_\bullet MTSO(2n)$$

inducing an isomorphism in integral homology. The case where k is a field follows easily from this. (The $\lim^1$ contribution to the cohomology of the hocolim vanishes when k is a field, and any cohomology class is detected on a homology class, which lives on a finite complex.)

For a general abelian group k, we first prove that for some g there exists a manifold B' and a map $f' : B' \to \Sigma B\mathrm{Diff}(W\cup gK,\partial)$ with $(f')^*(\sigma c) \neq 0$ where Σ

denotes unreduced suspension and σ denotes the suspension isomorphism. To see this, we take unreduced suspension of the above homology equivalence to turn it into a weak equivalence

$$\operatorname*{hocolim}_{g\to\infty} \Sigma B\mathrm{Diff}(W \cup gK, \partial) \longrightarrow \Sigma\Omega^\infty_\bullet MTSO(2n).$$

Standard homotopy theoretic arguments imply that there exists a CW approximation $X \to \Omega^\infty_\bullet MTSO(2n)$ with only finitely many cells in each dimension. Any non-zero class $c \in H^p(\Omega^\infty_\bullet MTSO(2n))$ is then detected on the p-skeleton X^p, and since X^p is a finite CW complex, the map $\Sigma X^p \to \Sigma\Omega^\infty_\bullet MTSO(2n)$ is homotopic to a map which up to homotopy factors through $\Sigma B\mathrm{Diff}(W \cup gK, \partial)$ for some finite g. If we let B' be a manifold homotopy equivalent to ΣX^p (for example a regular neighbourhood of some embedding of ΣX^p into Euclidean space), we have produced a map $f' : B' \to \Sigma B\mathrm{Diff}(W \cup gK, \partial)$ which detects the cohomology class σc.

To finish the proof of the first part of the corollary, we consider the map $\lambda : B' \to [0, 1]$ induced by the suspension coordinate and let $B = \lambda^{-1}((0, 1))$. The restriction of f' then gives a map $f : B \to B\mathrm{Diff}(W \cup gK, \partial)$ which by a Mayer-Vietoris argument detects c. To get a fibre bundle with closed manifold fibres, we let $N = W \cup gK \cup \overline{W}$; then the composition $B \to B\mathrm{Diff}(N)$ classifies an oriented smooth fiber bundle with fiber N, for which the characteristic class c is non-zero.

In case $k = \mathbb{Q}$, we can say more. The first part of the statement shows that for any non-zero $c \in H^*(\Omega^\infty_\bullet MTSO(2n); \mathbb{Q})$ there exists a g and a class $x \in H_p(B\mathrm{Diff}(W \cup gK, \partial); \mathbb{Q})$ such that $\langle x, c\rangle = 1$. By a theorem of Thom [Tho54], some integer multiple of x is represented by a map from an oriented smooth closed manifold B. □

Remark 1.13. The argument in this proof also shows that in the situation of Theorem 1.6, the natural map

$$H^*(\Omega^\infty_\bullet MT\theta'; k) \longrightarrow \varprojlim H^*(B\mathrm{Diff}^\theta(W \cup_{\partial W} gK, \ell); k)$$

is an isomorphism for any coefficient group k, provided the space B' is of finite type. Similarly for Theorem 1.8. The cohomology version of Theorem 1.11, again valid when B' is of finite type, is an isomorphism

$$H^*(\Omega^\infty MT\theta'; k) \longrightarrow \mathrm{Hom}_{\mathbb{Z}[\mathcal{K}]}(\mathbb{Z}[\mathcal{K}][\mathcal{K}^{-1}], H^*(\mathcal{N}^\theta(P, \ell_P); k)),$$

and hence $H^*(\Omega^\infty_\bullet MT\theta'; k)$ is isomorphic to the ring of invariant characteristic classes $H^*(\mathcal{N}^\theta(P, \ell_P); k)^{\mathcal{K}} = \mathrm{Hom}_{\mathbb{Z}[\mathcal{K}]}(\mathbb{Z}, H^*(\mathcal{N}^\theta(P, \ell_P); k))$.

1.7. **Cobordism categories and outline of proof.** Finally, let us say a few words about our method of proof, which follows the strategy in [GRW10] and [GMTW09]. A central object is the *cobordism category* $\mathcal{C}_\theta(\mathbb{R}^N)$, whose objects are closed $(d-1)$-dimensional manifolds $M \subset \mathbb{R}^N$ and whose morphisms are d-dimensional cobordisms $W \subset [0, t] \times \mathbb{R}^N$, both equipped with tangential structures.

Remark 1.14. The applications described above use only the case where $d = 2n$ is even. Our results about cobordism categories are valid for odd d as well, but we do not know an interpretation in terms of stable homology in that case. In fact, Ebert [Ebe09] has shown that there are non-trivial classes in $H^*(\Omega^\infty_\bullet MTSO(2n+1); \mathbb{Q})$ which are trivial when restricted to any $B\mathrm{Diff}^+(M^{2n+1})$. Thus there can be no analogue of e.g. Theorem 1.3, expressing $H_*(\Omega^\infty_\bullet MTSO(2n+1))$ as the direct limit of $H_*(B\mathrm{Diff}(W, \partial W))$ over some direct system of W's. It is an interesting question to find an odd-dimensional analogue of our results.

In the limit $N \to \infty$, the main result of [GMTW09] gives a weak equivalence

$$\Omega B\mathcal{C}_\theta \simeq \Omega^\infty MT\theta. \tag{1.2}$$

As in [GRW10], our strategy will be to find subcategories $C \subset \mathcal{C}_\theta$, as small as possible, such that the inclusion induces a weak equivalence $\Omega BC \to \Omega B\mathcal{C}_\theta$. The proof of Theorem 1.8 will consist of applying a version of the "group completion theorem" to a very small subcategory of $\mathcal{C}_\theta$.

Let P be a $(2n-1)$-dimensional manifold with θ-structure $\ell_P : TP \to \theta^*\gamma_{2n}$, and suppose the corresponding map $P \to B$ is $(n-1)$-connected. We pick a self-indexing Morse function $f : P \to [0, 2n-1]$ and set $L = f^{-1}([0, n-\frac{1}{2}])$. The restriction $L \to B$ is then still $(n-1)$-connected. Then we pick a (collared) embedding $L \to (-\infty, 0] \times \mathbb{R}^\infty$, and consider the subcategory $\mathcal{C}_{\theta,L} \subset \mathcal{C}_\theta$ where objects $M \subset \mathbb{R} \times \mathbb{R}^\infty$ satisfy $M \cap \big((-\infty, 0] \times \mathbb{R}^\infty\big) = L$ and morphisms $W \subset [0,t] \times \mathbb{R} \times \mathbb{R}^\infty$ satisfy $W \cap \big([0,t] \times (-\infty, 0] \times \mathbb{R}^\infty\big) = [0,t] \times L$. For both objects and morphisms, these identities are required to hold as θ-manifolds. (For later purposes, we note that the category only depends on ∂L: If $\partial L_1 = \partial L_2$, then there is an isomorphism of categories which cuts out $\mathrm{int}(L_1)$ and replaces it with L_2. In fact, it is convenient to mentally cut out $\mathrm{int}(L)$ and think of objects as manifolds with boundary, and morphisms as manifolds with corners.) In Section 2 we prove that the inclusion map induces a weak equivalence

$$B\mathcal{C}_{\theta,L} \longrightarrow B\mathcal{C}_\theta. \tag{1.3}$$

Secondly, we filter $\mathcal{C}_{\theta,L}$ by *connectivity of morphisms*: for $\kappa \geq -1$, the subcategory $\mathcal{C}^\kappa_{\theta,L}$ has the same objects, but a morphism W from M_0 to M_1 is required to satisfy that the inclusion $M_1 \to W$ is κ-connected, i.e. that any map $(D^i, \partial D^i) \to (W, M_1)$ is homotopic to one with image in M_1, for $i \leq \kappa$. In Section 3 we prove that the inclusion map induces a weak equivalence

$$B\mathcal{C}^\kappa_{\theta,L} \longrightarrow B\mathcal{C}_{\theta,L}, \tag{1.4}$$

as long as $\kappa \leq (d-2)/2$. (In the case where $\kappa = 0$ this is the "positive boundary subcategory", and this case was proved in [GMTW09].)

Thirdly, we filter $\mathcal{C}^\kappa_{\theta,L}$ by *connectivity of objects*: for $l \geq -1$, the subcategory $\mathcal{C}^{\kappa,l}_{\theta,L} \subset \mathcal{C}^\kappa_{\theta,L}$ is the full subcategory on those objects where the structure map $M \to B$ induces an injection $\pi_i(M) \to \pi_i(B)$ for all $i \leq l$ and all basepoints, or equivalently the inclusion $L \to M$ is l-connected. In Section 4 we prove that the inclusion map induces a weak equivalence

$$B\mathcal{C}^{\kappa,l}_{\theta,L} \longrightarrow B\mathcal{C}^\kappa_{\theta,L}, \tag{1.5}$$

provided $l \leq (d-3)/2$ and $l \leq \kappa$. (In the case where $l = 0$ and B is connected, this is the full subcategory on objects which are path connected, and this case was proved in [GRW10].)

Fourthly, we focus on the case where $d = 2n > 4$, where we have now reduced to $\mathcal{C}^{n-1,n-2}_{\theta,L}$, the full subcategory on those objects for which the inclusion $L \to M$ is $(n-2)$-connected. In the final step we let $\mathcal{C}$ denote the full subcategory on those objects M which can be obtained from L by attaching handles of index n and above. (This is equivalent to the condition that $M - \mathrm{int}(L)$ is diffeomorphic to a handlebody with handles of dimension $\leq (n-1)$ only, which if $n > 3$ is in turn equivalent to the inclusion $L \to M$ being $(n-1)$-connected.) In Section 5 we prove that the inclusion map induces a weak equivalence

$$\Omega B\mathcal{C} \longrightarrow \Omega B\mathcal{C}^{n-1,n-2}_{\theta,L}, \tag{1.6}$$

provided that any θ-structure on D^{2n} extends to S^{2n}.

Using the above weak equivalences, Theorem 1.8 is proved as follows. The category $\mathcal{C}$ has two special objects which will be important for us, both associated to the manifold P in Theorem 1.8. The first is obtained by extending the embedding of $L \to (-\infty, 0] \times \mathbb{R}^\infty$ to an embedding $P \to \mathbb{R} \times \mathbb{R}^\infty$; we call the corresponding object P. The second is obtained by gluing L to its "mirror image" $\overline{L} \subset [0, \infty) \times \mathbb{R}^\infty$ to obtain an embedding of the double $D(L) \to \mathbb{R} \times \mathbb{R}^\infty$'; we call the corresponding object $D(L)$. Morphisms from P to $D(L)$ in this category can be viewed as θ-manifolds with corners. In fact, if we cut out $\mathrm{int}(L)$ everywhere and smooth corners, the morphisms are $(n-1)$-connected null-bordisms of P, and

$$\mathcal{C}(P, D(L)) \simeq \mathcal{N}^\theta(P, \ell_P),$$

is a model for the moduli space of all highly connected null-bordisms.

The weak equivalences (1.2), (1.3), (1.4), (1.5) and (1.6) establish the homotopy equivalence

$$\Omega B\mathcal{C} \simeq \Omega^\infty MT\theta,$$

and the proof of Theorem 1.8 will be completed using a suitable version of the "group completion" theorem ([MS74]) to the canonical map $\mathcal{C}(P, D(L)) \to \Omega B\mathcal{C}$.

The weak equivalences (1.4), (1.5) and (1.6) are established using a parametrised surgery procedure, and the proof depends on the contractibility of certain spaces of surgery data. Contractibility is proved in a similar way in all three cases, and we defer this to Section 6. Finally, in Section 7 we explain how to use a version of the group completion theorem to prove Theorem 1.8 and tie things together.

2. Definitions and recollections

2.1. **Tangential structures.** Throughout this paper, an important role will be played by the notion of a tangential structure on manifolds.

Definition 2.1. A *tangential structure* is a map $\theta : B \to BO(d)$. A *θ-structure* on a d-manifold W is a bundle map (i.e. fibrewise linear isomorphism) $\ell : TW \to \theta^*\gamma$. A θ-manifold is a pair (W, ℓ). More generally, a θ-structure on a k-manifold M (with $k \leq d$) is a bundle map $\ell : \varepsilon^{d-k} \oplus TM \to \theta^*\gamma$.

Given vector bundles U and V of the same dimension, but not necessarily over the same space, we write $\mathrm{Bun}(U, V)$ for the subspace of $\mathrm{map}(U, V)$ (with the compact-open topology) consisting of the bundle maps. Thus, $\mathrm{Bun}(TW, \theta^*\gamma)$ is the space of θ-structures on W.

2.2. **Spaces of manifolds.** We recall the definition and main properties of spaces of submanifolds, from [GRW10]. Fix a tangential structure $\theta : B \to BO(d)$.

Definition 2.2. For an open subset $U \subset \mathbb{R}^n$, we denote by $\Psi_\theta(U)$ the set of pairs (M^d, ℓ) where $M^d \subset U$ is a smooth d-dimensional submanifold that is a closed as a topological subspace, and ℓ is a θ-structure on M.

We denote by $\Psi_{\theta_{d-m}}(U)$ the set of pairs (M, ℓ) where $M \subset U$ is a smooth $(d-m)$-dimensional submanifold that is closed as a topological subspace, and ℓ is a bundle map $\varepsilon^m \oplus TM \to \theta^*\gamma$.

In [GRW10, §2] we have defined a topology on these sets so that $U \mapsto \Psi_{\theta_{d-m}}(U)$ defines a continuous sheaf of topological spaces on the site of open subsets of $\mathbb{R}^n$. We will not give full details of the topology again here, but remind the reader that the topology is "compact-open" in flavour: disregarding tangential structures, points nearby to M are those which near some large compact subset $K \subset U$ look like small normal deformations of M. In particular, a typical neighbourhood of the empty manifold $\emptyset \in \Psi_\theta(U)$ consists of all those manifolds in U disjoint from some compact K.

Definition 2.3. We define $\psi_\theta(n,k) \subset \Psi_\theta(\mathbb{R}^n)$ to be the subspace of those θ-manifolds (M,ℓ) such that $M \subset \mathbb{R}^k \times (-1,1)^{n-k}$. We make the analogous definition of $\psi_{\theta_{d-m}}(n,k)$.

2.3. **Semi-simplicial spaces and non-unital categories.** Let Δ denote the category of finite non-empty totally ordered sets and monotone maps, the simplicial indexing category. Let $\Delta_{\mathrm{inj}} \subset \Delta$ denote the subcategory of finite ordered sets and injective maps. For a category $\mathcal{C}$, a *simplicial object in* $\mathcal{C}$ is a contravariant functor $X : \Delta \to \mathcal{C}$, and a *semi-simplicial object in* $\mathcal{C}$ is a contravariant functor $X : \Delta_{\mathrm{inj}} \to \mathcal{C}$. A map of (semi-)simplicial objects is a natural transformation of functors.

We call a semi-simplicial object in the category of topological spaces a *semi-simplicial space*. More concretely, it consists of a space $X_n = X(0 < 1 < \cdots < n)$ for each $n \geq 0$ and *face maps* $d_i : X_n \to X_{n-1}$ defined for $i = 0, ..., n$ satisfying the simplicial identities $d_i d_j = d_{j-1} d_i$ for $i < j$. We often denote a semi-simplicial space by $X_\bullet$, where we treat $\bullet$ as a place-holder for the simplicial degree.

The *geometric realisation* of a semi-simplicial space $X_\bullet$ is defined to be

$$|X_\bullet| = \coprod_{n \geq 0} X_n \times \Delta^n / \sim$$

where Δ^n denotes the standard topological n-simplex and the equivalence relation is $(d_i(x), y) \sim (x, d^i(y))$ where $d^i : \Delta^n \to \Delta^{n+1}$ the inclusion of the ith face. This space is given the quotient topology.

The *k-skeleton* of $|X_\bullet|$ is

$$|X_\bullet|^{(k)} = \coprod_{n=0}^{k} X_n \times \Delta^n / \sim$$

with the quotient topology, and one easily checks that $|X_\bullet| = \cup_{k \geq 0} |X_\bullet|^{(k)}$ with the direct limit topology. A useful consequence of this is the following: a map from a compact space to $|X_\bullet|$ lands in a finite skeleton.

Lemma 2.4. *If $X_\bullet \to Y_\bullet$ is a map of semi-simplicial spaces such that each $X_n \to Y_n$ is a weak homotopy equivalence, then $|X_\bullet| \to |Y_\bullet|$ is too.*

Proof. There is a push-out square

$$\begin{tikzcd} X_k \times \partial\Delta^k \arrow[r] \arrow[d, hook] & {|X_\bullet|^{(k-1)}} \arrow[d, hook] \\ X_k \times \Delta^k \arrow[r] & {|X_\bullet|^{(k)}} \end{tikzcd}$$

where the vertical maps are cofibrations. Hence it is also a homotopy push-out, which inductively shows that each of the $|X_\bullet|^{(k)} \to |Y_\bullet|^{(k)}$ are weak homotopy equivalences. The direct limit characterisation of $|X_\bullet|$ gives the result. □

Remark 2.5. The term *semi-simplicial object* we have defined above is not quite standard (though is gaining popularity) and deserves some justification. Our justification is that it agrees with Eilenberg and Zilber's original usage of "semi-simplicial complex" [EZ50]. Another is that the alternative used in the literature is Δ-*space*, but as Δ is the indexing category for full simplicial objects this seems counterintuitive.

A *non-unital topological category* $\mathcal{C}$ consists of a pair of spaces $(\mathcal{O}, \mathcal{M})$ of objects and morphisms, equipped with source and target maps $s, t : \mathcal{M} \to \mathcal{O}$. We let $\mathcal{M} \times_{t\mathcal{O}s} \mathcal{M}$ denote the fibre product made with the maps t and s, and require in addition a composition map $\mu : \mathcal{M} \times_{t\mathcal{O}s} \mathcal{M} \to \mathcal{M}$ which satisfies the evident associativity requirement.

A non-unital topological category $\mathcal{C}$ has a semi-simplicial nerve, generalising the simplicial nerve of a topological category [Seg68]. Define $N_\bullet\mathcal{C}$ by $N_0\mathcal{C} = \mathcal{O}$ and

$$N_k\mathcal{C} = \mathcal{M} \times_{t\mathcal{O}s} \mathcal{M} \times_{t\mathcal{O}s} \cdots \times_{t\mathcal{O}s} \mathcal{M} \quad k > 0$$

being the space of k-tuples of composable morphisms, and let the face maps be given by composing and forgetting morphisms, as in the simplicial nerve of a topological category. We define the *classifying space* of a non-unital topological category by

$$B\mathcal{C} = |N_\bullet\mathcal{C}|.$$

2.4. **Definition of the cobordism categories.** For convenience in the rest of the paper, we introduce the following notation. All of our constructions will take place inside $\mathbb{R} \times \mathbb{R}^N$, and we write $x_1 : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ for the projection to the first coordinate. Given a manifold $W \subset \mathbb{R} \times \mathbb{R}^N$ and a set $A \subset \mathbb{R}$, we write

$$W|_A = W \cap x_1^{-1}(A),$$

and we also write $\ell|_A$ for the restriction of a θ-structure ℓ on W to this manifold.

Our definition of the cobordism category of θ-manifolds is similar to that of [GRW10] (the only difference is that here will we only define a non-unital category); it follows that of [GMTW09] in spirit, but is different in some technical points. We use the spaces of manifolds of the last section in order to describe the point-set topology of these categories.

Definition 2.6. For each $\varepsilon > 0$ we let the non-unital topological category $\mathcal{C}_\theta(\mathbb{R}^N)_\varepsilon$ have space of objects $\psi_{\theta_{d-1}}(N, 0)$. The space of morphisms from (M_0, ℓ_0) to (M_1, ℓ_1) is the subspace of those $(t, (W, \ell)) \in \mathbb{R} \times \psi_\theta(N + 1, 1)$ such that $t > 0$ and

$$W|_{(-\infty,\varepsilon)} = \mathbb{R} \times M_0|_{(-\infty,\varepsilon)} \in \Psi_\theta((-\infty, \varepsilon) \times \mathbb{R}^N)$$

and

$$W|_{(t-\varepsilon,\infty)} = \mathbb{R} \times M_1|_{(t-\varepsilon,\infty)} \in \Psi_\theta((t - \varepsilon, \infty) \times \mathbb{R}^N).$$

Here $\mathbb{R} \times M_i$ denotes the θ-manifold with underlying manifold $\mathbb{R} \times M_i \subset \mathbb{R} \times \mathbb{R}^N$ and θ-structure

$$T(\mathbb{R} \times M_i) \longrightarrow \varepsilon^1 \oplus TM_i \xrightarrow{\ell_i} \theta^*\gamma.$$

Composition in this category is defined by

$$(t, W) \circ (t, W') = (t + t', W|_{(-\infty,t]} \cup (W' + t \cdot e_1)|_{[t,\infty)})$$

where $W' + t \cdot e_1$ denotes the manifold W' translated by t in the first coordinate. We topologise the total space of morphisms as a subspace of $(0, \infty) \times \psi_\theta(N + 1, 1)$.

If $\varepsilon < \varepsilon'$ there is an inclusion $\mathcal{C}_\theta(\mathbb{R}^N)_{\varepsilon'} \subset \mathcal{C}_\theta(\mathbb{R}^N)_\varepsilon$, and we define $\mathcal{C}_\theta(\mathbb{R}^N)$ to be the colimit over all $\varepsilon > 0$.

Note that a morphism $(t, (W, \ell))$ in this category is uniquely determined by the restriction $(t, (W|_{[0,t]}, \ell|_{[0,t]}))$. We often think of morphisms in this category as being given by such restricted manifolds, but the topology on the space of morphisms is best described as we did above.

As explained in the introduction, we will also require a version of this category where the objects and morphisms contain a fixed codimension zero submanifold. In order to define this, we let

$$L \subset (-1/2, 0] \times (-1, 1)^{N-1}$$

be a compact $(d - 1)$-manifold which near $\{0\} \times \mathbb{R}^{N-1}$ agrees with $(-1, 0] \times \partial L$. Furthermore, we let $\ell|_L : \varepsilon^1 \oplus TL \to \theta^*\gamma$ be a θ-structure on L. Near ∂L we require that the structure is a product (i.e. that translation in the collar direction preserves the structure). Such an ℓ makes $\mathbb{R} \times L$ into a θ-manifold with boundary, and we make the following definition.

Definition 2.7. The topological subcategory $\mathcal{C}_{\theta,L}(\mathbb{R}^N) \subset \mathcal{C}_\theta(\mathbb{R}^N)$ has space of objects those (M, ℓ) such that

$$M \cap \big((-\infty, 0] \times \mathbb{R}^{N-1}\big) = L$$

as θ-manifolds. It has space of morphisms from (M_0, ℓ_0) to (M_1, ℓ_1) given by those $(t, (W, \ell))$ such that

$$W \cap \big(\mathbb{R} \times (-\infty, 0] \times \mathbb{R}^{N-1}\big) = \mathbb{R} \times L$$

as θ-manifolds.

Remark 2.8. The category $\mathcal{C}_{\theta,L}(\mathbb{R}^N)$ does not really depend on L, but only on ∂L. It is sometimes convenient to think of the interior of L as being cut out, so that objects in the category are manifolds with boundary ∂L and morphisms are cobordisms between manifolds with boundary which are trivial along the boundary.

If we take $L = D^{d-1}$ then the category $\mathcal{C}_{\theta,L}(\mathbb{R}^N)$ is equivalent to the category of "manifolds with basepoint" defined in [GRW10, Definition 4.2].

The subject of our main technical theorem, from which we shall show how to obtain results on diffeomorphism groups in Section 7, is certain subcategories of $\mathcal{C}_{\theta,L}(\mathbb{R}^N)$ where we require the morphisms to have certain connectivities relative to their outgoing boundaries, and objects to be those (M, ℓ_M) whose Gauss map $M \to B$ (i.e. the map underlying $\ell_M : \varepsilon^1 \oplus TM \to \theta^*\gamma$) has a certain injectivity range on homotopy groups.

Definition 2.9. The topological subcategory $\mathcal{C}^\kappa_{\theta,L}(\mathbb{R}^N) \subset \mathcal{C}_{\theta,L}(\mathbb{R}^N)$ has the same space of objects. It has space of morphisms from (M_0, ℓ_0) to (M_1, ℓ_1) given by those $(t, (W, \ell))$ such that the pair

$$(W|_{[0,t]}, W|_t)$$

is κ-connected, i.e. such that $\pi_i(W|_{[0,t]}, W|_t) = 0$ for all basepoints and all $i \leq \kappa$. Thus this is the subcategory on those morphisms which are κ-connected relative to their outgoing boundary.

The category $\mathcal{C}^0_\theta$ is the "positive boundary category" as in [GMTW09], where each path-component of a cobordism is required to have non-empty outgoing boundary.

Definition 2.10. The topological subcategory $\mathcal{C}^{\kappa,l}_{\theta,L}(\mathbb{R}^N) \subset \mathcal{C}^\kappa_{\theta,L}(\mathbb{R}^N)$ is the full subcategory on those objects (M, ℓ) such that the map

$$\ell_* : \pi_i(M) \longrightarrow \pi_i(B)$$

is injective for all $i \leq l$ and all basepoints. (In our main application in Section 7, the map $L \to B$ will be $(l+1)$-connected. In the case the requirement is equivalent to (M, L) being l-connected.)

For our final definition we specialise to the case $d = 2n$. Let

$$\mathcal{A} \subset \pi_0\left(\mathrm{ob}(\mathcal{C}^{n-1,n-2}_{\theta,L}(\mathbb{R}^N))\right)$$

be a collection of path-components of the space of objects, containing at least one element of each path-component of $B\mathcal{C}^{n-1,n-2}_{\theta,L}(\mathbb{R}^N)$.

Definition 2.11. The topological subcategory $\mathcal{C}^{n-1,\mathcal{A}}_{\theta,L}(\mathbb{R}^N) \subset \mathcal{C}^{n-1,n-2}_{\theta,L}(\mathbb{R}^N)$ is the full subcategory on the subspace of those objects in $\mathcal{A}$.

For $N = \infty$, we shall often denote $\mathcal{C}_\theta(\mathbb{R}^\infty) = \mathrm{colim}_N\, \mathcal{C}_\theta(\mathbb{R}^N)$ by $\mathcal{C}_\theta$, and similarly with any decorations.

2.5. **The homotopy type of the cobordism category.** The main theorem of [GMTW09] identifies the homotopy type $\Omega B\mathcal{C}_\theta$ in terms of the infinite loop space of a certain Thom spectrum $MT\theta$.

Recall from the introduction that given a map $\theta : B \to BO(d) = \mathrm{Gr}_d(\mathbb{R}^\infty)$ we let $B(\mathbb{R}^n) = \theta^{-1}(\mathrm{Gr}_d(\mathbb{R}^n))$ and define $\gamma_n^\perp \to \mathrm{Gr}_d(\mathbb{R}^n)$ to be the orthogonal complement of the tautological bundle. The canonical map $B(\mathbb{R}^n) \to B(\mathbb{R}^{n+1})$ pulls back $\theta^*\gamma_{n+1}^\perp$ to $\varepsilon^1 \oplus \theta^*\gamma_n^\perp$ and hence we obtain pointed maps

$$S^1 \wedge \left(B(\mathbb{R}^n)^{\theta^*\gamma_n^\perp}\right) \longrightarrow B(\mathbb{R}^{n+1})^{\theta^*\gamma_{n+1}^\perp}$$

of Thom spaces, which form a spectrum $MT\theta$. Its associated infinite loop space is

$$\Omega^\infty MT\theta = \operatorname*{colim}_{n\to\infty} \Omega^n \left(B(\mathbb{R}^n)^{\theta^*\gamma_n^\perp}\right).$$

Theorem 2.12 (Galatius–Madsen–Tillmann–Weiss [GMTW09])**.** *There is a canonical map*

$$\Omega B\mathcal{C}_\theta \longrightarrow \Omega^\infty MT\theta$$

which is a weak homotopy equivalence.

We write $\Omega^\infty_\bullet MT\theta$ for the basepoint component of $\Omega^\infty MT\theta$, and now describe the rational cohomology of this space. The map $B \overset{\theta}{\to} BO(d) \overset{\det}{\to} BO(1)$ on fundamental groups defines a character $w_1 : \pi_1(B) \to \mathbb{Z}^\times$, and we write $H^*(B;\mathbb{Q}^{w_1})$ for the rational cohomology of B with local coefficients given by this character. For each n there are evaluation maps

$$ev : \Sigma^n\Omega^n\left(B(\mathbb{R}^n)^{\theta^*\gamma_n^\perp}\right) \longrightarrow B(\mathbb{R}^n)^{\theta^*\gamma_n^\perp}$$

and so maps

$$\begin{array}{ccc} H^{*+d}(B(\mathbb{R}^n);\mathbb{Q}^{w_1}) & \longrightarrow & H^*(\Omega^n(B(\mathbb{R}^n)^{\theta^*\gamma_n^\perp});\mathbb{Q}) \\ \text{Thom iso.}\Big\| & & \Big\|\text{Suspension iso.} \\ \widetilde{H}^{*+n}(B(\mathbb{R}^n)^{\theta^*\gamma_n^\perp};\mathbb{Q}) & \xrightarrow{ev^*} & \widetilde{H}^{*+n}(\Sigma^n\Omega^n(B(\mathbb{R}^n)^{\theta^*\gamma_n^\perp});\mathbb{Q}). \end{array}$$

Taking limits and restricting to the basepoint component, we obtain a map

$$\sigma : H^{*+d}(B;\mathbb{Q}^{w_1}) \longrightarrow H^*(\Omega^\infty_\bullet MT\theta;\mathbb{Q})$$

and the right-hand side is a graded-commutative algebra, so this extends to the free graded-commutative algebra on the part of $H^{*+d}(B;\mathbb{Q}^{w_1})$ of degree > 0,

$$\Lambda(H^{*+d>0}(B;\mathbb{Q}^{w_1})) \longrightarrow H^*(\Omega^\infty_\bullet MT\theta;\mathbb{Q}).$$

This is an isomorphism of graded-commutative algebras.

2.6. **Poset models.** A key step in the proofs of [GMTW09] and [GRW10] identifying the infinite loop space $B\mathcal{C}_\theta$ is to first identify this classifying space with the classifying space of a certain topological poset. The result holds for all variations of the cobordism category mentioned above; we prove the general result in Proposition 2.14 below.

Definition 2.13. Let $\mathcal{C} \subset \mathcal{C}_\theta(\mathbb{R}^N)$ be a subcategory. Let

$$D_\theta^{\mathcal{C}} \subset \mathbb{R} \times \mathbb{R}_{>0} \times \psi_\theta(N+1,1)$$

denote the subspace of triples $(t,\varepsilon,(W,\ell))$ such that $[t-\varepsilon, t+\varepsilon]$ consists of regular values for $x_1 : W \to \mathbb{R}$, and $W|_t \in \mathrm{ob}(\mathcal{C})$. Define a partial order on $D_\theta^{\mathcal{C}}$ by

$$(t,\varepsilon,(W,\ell)) < (t',\varepsilon',(W',\ell'))$$

if and only if $(W,\ell) = (W',\ell')$, $t+\varepsilon < t'-\varepsilon$ and $W|_{[t,t']} \in \mathrm{Mor}\,\mathcal{C}$.

Proposition 2.14. *Let $\mathcal{C} \subset \mathcal{C}_{\theta,L}(\mathbb{R}^N) \subset \mathcal{C}_\theta(\mathbb{R}^N)$ be a subcategory which consists of entire path components of the object and morphism spaces of $\mathcal{C}_{\theta,L}(\mathbb{R}^N)$. Then there is a weak homotopy equivalence*

$$B\mathcal{C} \simeq BD_\theta^{\mathcal{C}}.$$

Proof. We introduce an auxiliary topological poset $D_\theta^{\mathcal{C},\perp}$ which maps to both $D_\theta^{\mathcal{C}}$ and $\mathcal{C}$. It is the subposet of $D_\theta^{\mathcal{C}}$ consisting of $(t, \varepsilon, (W, \ell)$ such that (W, ℓ) is a product over $(t-\varepsilon, t+\varepsilon)$. This conditions means that if we write $W|_t = \{t\} \times M$ and give M the inherited θ-structure, then

$$W|_{(t-\varepsilon,t+\varepsilon)} = (t-\varepsilon, t+\varepsilon) \times M$$

as θ-manifolds. Then there is a zig-zag of functors

$$D_\theta^{\mathcal{C}} \longleftarrow D_\theta^{\mathcal{C},\perp} \longrightarrow \mathcal{C},$$

where the first arrow is the inclusion of the subposet and the second is the functor that sends a morphism $(a < b, W, \ell)$ to the manifold $(W|_{[a,b]} - a \cdot e_1)$ extended cylindrically in $(-\infty, 0] \times \mathbb{R}^N$ and $[b-a, \infty) \times \mathbb{R}^N$. This induces a zig-zag diagram

$$N_k D_\theta^{\mathcal{C}} \longleftarrow N_k D_\theta^{\mathcal{C},\perp} \longrightarrow N_k\mathcal{C},$$

and we prove that both maps are weak equivalence for all k in the same way as in [GRW10, Theorem 3.9]. □

Applying the above construction to the categories $\mathcal{C}_{\theta,L}^{\kappa,l}(\mathbb{R}^N)$ we obtain topological posets $D_{\theta,L}^{\kappa,l}(\mathbb{R}^N)$ and weak homotopy equivalences

$$B\mathcal{C}_{\theta,L}^{\kappa,l}(\mathbb{R}^N) \simeq BD_{\theta,L}^{\kappa,l}(\mathbb{R}^N). \tag{2.1}$$

Similarly, when we specialise to the case $d = 2n$ and let $\mathcal{A} \subset \pi_0(\mathrm{ob}(\mathcal{C}_{\theta,L}^{n-1,n-2}(\mathbb{R}^N)))$ be a collection of path-components of objects, we obtain weak homotopy equivalences

$$B\mathcal{C}_{\theta,L}^{n-1,\mathcal{A}}(\mathbb{R}^N) \simeq BD_{\theta,L}^{n-1,\mathcal{A}}(\mathbb{R}^N). \tag{2.2}$$

2.7. **The homotopy type of $\mathcal{C}_{\theta,L}(\mathbb{R}^N)$.** In [GRW10, Theorems 3.9 and 3.10] we proved that there are weak homotopy equivalences $BD_\theta(\mathbb{R}^N) \simeq \psi_\theta(N+1, 1)$, which combined with Proposition 2.14 gives

$$B\mathcal{C}_\theta(\mathbb{R}^N) \simeq BD_\theta(\mathbb{R}^N) \simeq \psi_\theta(N+1, 1). \tag{2.3}$$

(Strictly speaking, in that paper we worked with a version of $D_\theta(\mathbb{R}^N)$ where $\varepsilon = 0$, but the obvious map induces a levelwise weak equivalence of nerves.) For the purposes of this paper we require a slightly stronger version of this result, taking into account the submanifold L. The proof of [GRW10, Theorem 3.10] applies verbatim to prove

Proposition 2.15. *There are weak homotopy equivalences*

$$B\mathcal{C}_{\theta,L}(\mathbb{R}^N) \simeq BD_{\theta,L}(\mathbb{R}^N) \simeq \psi_{\theta,L}(N+1, 1)$$

where $\psi_{\theta,L}(N+1,1) \subset \psi_\theta(N+1,1)$ is the subspace consisting of those (W, ℓ) such that $W \cap (\mathbb{R} \times (-\infty, 0] \times \mathbb{R}^{N-1}) = \mathbb{R} \times L$ as θ-manifolds.

Proposition 2.16. *The inclusion*

$$i : \psi_{\theta,L}(N+1, 1) \longrightarrow \psi_\theta(N+1, 1)$$

is a weak homotopy equivalence.

Proof. This is similar to [GRW10, Lemma 4.6], which is essentially the case $L = D^{d-1}$. It requires careful analysis of θ-structures, so let us, for this proof only, denote the θ_{d-1}-structure on L by $\ell_L : \varepsilon^1 \oplus TL \to \theta^*\gamma$. We first want to construct the *double* $D(L)$ of L as a θ_{d-1}-manifold, and a canonical θ-null-bordism of it. Recall that L is a submanifold of $(-1/2, 0] \times (-1,1)^{N-1}$ which we identify with $\{0\} \times (-1/2, 0] \times (-1,1)^{N-1} \subset [0,1) \times (-1/2, 0] \times (-1,1)^{N-1}$. Let V denote the volume swept out by rotating L around $(0,0)$ in the plane $[0,1) \times (-1,1)$. As L was collared, this volume is naturally a d-manifold with boundary and L lies in its boundary. We define $D(L) = \partial V$, and $\overline{L} = D(L) - L$. The inclusion $L \hookrightarrow V$ is a homotopy equivalence, so there is a unique extension up to homotopy

$$\begin{tikzcd} \varepsilon^1 \oplus TL \arrow[r, "\ell_L"] \arrow[d] & \theta^*\gamma \\ TV \arrow[ur, dotted] & \end{tikzcd}$$

where the vertical map sends ε^1 to the outwards pointing vector. This restricts to a θ-structure on $D(L)$, and hence on $\overline{L}$, and V gives a θ-cobordism $V : \emptyset \rightsquigarrow D(L)$.

Similarly, we can rotate around the point $(0, -1/2)$ to obtain a θ-cobordism $U : [0,1] \times \partial L \rightsquigarrow L \sqcup \overline{L}$ relative boundary.

These θ-manifolds give us the tools we need. The collared embedding of L in $(-1/2, 0] \times (-1,1)^{N-1}$ gives a smooth embedding of $D(L)$ in $(-1/2, 1/2) \times (-1,1)^{N-1}$, so of the θ-manifold $\mathbb{R} \times D(L)$ in $\mathbb{R} \times (-1/2, 1/2) \times (-1,1)^{N-1}$. We define a map

$$r : \psi_\theta(N+1, 1) \longrightarrow \psi_{\theta,L}(N+1, 1)$$

which given $(W, \ell) \subset \mathbb{R} \times (-1,1) \times (-1,1)^{N-1}$ applies the affine diffeomorphism $(-1,1) \cong (1/2, 1)$ to its second coordinate, and then takes the (disjoint) union with $\mathbb{R} \times D(L)$.

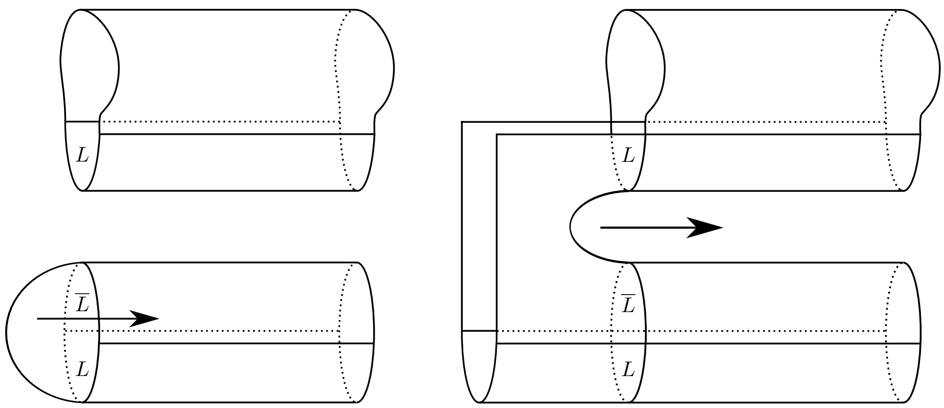

FIGURE 1. Adding and removing L.

The composition $i \circ r$ is homotopic to the identity as the θ-null-bordism V of $(D(L), D(\ell))$ may be used to push the cylinder $\mathbb{R} \times D(L)$ off to the right. The composition $r \circ i$ is homotopic to the identity as the relative θ-cobordism U may be brought on from the left to give a path from $\mathbb{R} \times (L \sqcup D(L))$ to $\mathbb{R} \times L$. Figure 1 shows how. $\square$

Combining this proposition with Proposition 2.15 and the homotopy equivalence (2.3) gives the following corollary.

Corollary 2.17. *For any pair (L, ℓ_L) as in Definition 2.7, the inclusion*
$$B\mathcal{C}_{\theta,L}(\mathbb{R}^N) \longrightarrow B\mathcal{C}_\theta(\mathbb{R}^N)$$
is a weak homotopy equivalence.

2.8. **A more flexible model.** From the poset models of Section 2.6 we construct the semi-simplicial spaces
$$D^{\kappa,l}_{\theta,L}(\mathbb{R}^N)_\bullet = N_\bullet D^{\kappa,l}_{\theta,L}(\mathbb{R}^N).$$
The remarks of Section 2.6 and Proposition 2.15 show that the geometric realisations of these semi-simplicial spaces are a model for the classifying spaces of the categories $\mathcal{C}^{\kappa,l}_{\theta,L}(\mathbb{R}^N)$ in which we are interested. The benefit of working with these semi-simplicial spaces instead of the cobordism categories is that we can often make constructions which are *not* functorial, yet give well-defined maps between geometric realisations of the semi-simplicial spaces involved.

To make this technique easier to apply, we will define an auxiliary semi-simplicial space $X^{\kappa,l}_\bullet$. We will prove that its geometric realisation is weakly equivalent to $B\mathcal{C}^{\kappa,l}_{\theta,L}(\mathbb{R}^N)$, but it will be easier to construct a simplicial map into $X^{\kappa,l}_\bullet$ than into $N_\bullet\mathcal{C}^{\kappa,l}_{\theta,L}(\mathbb{R}^N)$ or $D^{\kappa,l}_{\theta,L}(\mathbb{R}^N)_\bullet$. The space $X^{\kappa,l}_\bullet$ also depends on N, but we omit that from the notation.

Definition 2.18. Let $\theta : B \to BO(d)$, N and L be as before. Let $-1 \leq \kappa \leq \frac{d-1}{2}$, $-1 \leq l \leq \kappa$ and $-1 \leq l \leq d-\kappa-2$. Define $X^{\kappa,l}_\bullet$ to be the semi-simplicial space with p-simplices consisting of certain tuples $(a, \varepsilon, (W, \ell))$ such that $a = (a_0, \dots, a_p) \in \mathbb{R}^{p+1}$, $\varepsilon = (\varepsilon_0, \dots, \varepsilon_p) \in (\mathbb{R}_{>0})^{p+1}$, and $(W, \ell) \in \Psi_\theta((a_0 - \varepsilon_0, a_p + \varepsilon_p) \times \mathbb{R}^N)$, satisfying

(i) $W \subset (a_0 - \varepsilon_0, a_p + \varepsilon_p) \times (-1, 1)^N$,
(ii) W and $(a_0 - \varepsilon_0, a_p + \varepsilon_p) \times L$ agree as θ-manifolds on the subspace $x_2^{-1}(-\infty, 0]$,
(iii) $a_0 < a_1 < \cdots < a_p$,
(iv) the intervals $[a_i - \varepsilon_i, a_i + \varepsilon_i]$ are pairwise disjoint,
(v) $x_1 : W|_{(a_i-\varepsilon_i, a_i+\varepsilon_i)} \to (a_i - \varepsilon_i, a_i + \varepsilon_i)$ has only isolated critical values,
(vi) for each pair of regular values $t_0 < t_1 \in \cup_i(a_i - \varepsilon_i, a_i + \varepsilon_i)$, the cobordism $W|_{[t_0,t_1]}$ is κ-connected relative to its outgoing boundary,
(vii) for each regular value $t \in (a_i - \varepsilon_i, a_i + \varepsilon_i)$, the map
$$\pi_j(W|_t) \longrightarrow \pi_j(B),$$
induced by ℓ, is injective for all basepoints and all $j \leq l$.

We topologise this set as a subspace of $\mathbb{R}^{p+1} \times (\mathbb{R}_{>0})^{p+1} \times \Psi_\theta((-1, 1) \times \mathbb{R}^N)$, where we use the standard affine diffeomorphism $(-1, 1) \cong (a_0 - \varepsilon_0, a_p + \varepsilon_p)$ to identify $\Psi_\theta((a_0 - \varepsilon_0, a_p + \varepsilon_p) \times \mathbb{R}^N)$ with $\Psi_\theta((-1, 1) \times \mathbb{R}^N)$. The jth face map is given by forgetting a_j and ε_j, and if $j = 0$ composing with the restriction map $\Psi_\theta((a_0 - \varepsilon_0, a_p + \varepsilon_p) \times \mathbb{R}^N) \to \Psi_\theta((a_1 - \varepsilon_1, a_p + \varepsilon_p) \times \mathbb{R}^N)$, and similarly if $j = p$.

There are semi-simplicial maps $D^{\kappa,l}_{\theta,L}(\mathbb{R}^N)_\bullet \to X^{\kappa,l}_\bullet$, which on p-simplices are given by sending $(a, \varepsilon, (W, \ell))$ with $(W, \ell) \in \Psi_\theta(\mathbb{R} \times \mathbb{R}^N)$ to the same thing restricted down to $\Psi_\theta((a_0 - \varepsilon_0, a_p + \varepsilon_p) \times \mathbb{R}^N)$.

The semi-simplicial space $X^{\kappa,l}_\bullet$ is easier to map into (by a semi-simplicial map) than $D^{\kappa,l}_{\theta,L}(\mathbb{R}^N)_\bullet$ for two reasons. Firstly, we do not require that the intervals $(a_i - \varepsilon_i, a_i + \varepsilon_i)$ consist entirely of regular values: instead, condition (v) allows critical values, and conditions (vi)–(vii) ensure that the critical values do not affect the essential properties of the space. Secondly, we discard those parts of the manifold outside of $(a_0 - \varepsilon_0, a_p + \varepsilon_p)$, and so do not need to worry about controlling parts of the manifold outside of the region.

Definition 2.19. In the case $d = 2n$, with $\mathcal{A} \subset \pi_0(\mathrm{ob}(\mathcal{C}_{\theta,L}^{n-1,n-2}(\mathbb{R}^N)))$ a collection of path-components of objects, we make the entirely analogous definition of $X_\bullet^{n-1,\mathcal{A}}$. Precisely, in Definition 2.18 we replace condition (vii) by

(vii′) for each regular value $t \in (a_i - \varepsilon_i, a_i + \varepsilon_i)$, the $(d-1)$-manifold $W|_t$ lies in $\mathcal{A}$.

The following is our main result concerning these models, and together with (2.1) and (2.2) provides weak homotopy equivalences $B\mathcal{C}_{\theta,L}^{\kappa,l}(\mathbb{R}^N) \simeq |X_\bullet^{\kappa,l}|$ and, in the case $d = 2n$, $B\mathcal{C}_{\theta,L}^{n-1,\mathcal{A}}(\mathbb{R}^N) \simeq |X_\bullet^{n-1,\mathcal{A}}|$.

Proposition 2.20. *Let κ and l satisfy the inequalities in Definition 2.18. The semi-simplicial map $D_{\theta,L}^{\kappa,l}(\mathbb{R}^N)_\bullet \to X_\bullet^{\kappa,l}$, and in the case $d = 2n$ also the map $D_{\theta,L}^{n-1,\mathcal{A}}(\mathbb{R}^N)_\bullet \to X_\bullet^{n-1,\mathcal{A}}$, induce weak homotopy equivalences after geometric realisation.*

Proof. For the proof we introduce an auxiliary semi-simplicial space $\bar{X}_\bullet^{\kappa,l}$. Its p-simplices are those tuples

$$(a, \varepsilon, (W, \ell)) \in \mathbb{R}^{p+1} \times (\mathbb{R}_{>0})^{p+1} \times \psi_\theta(N+1, 1)$$

satisfying the conditions of Definition 2.18, except that the interval $(a_0 - \varepsilon_0, a_p + \varepsilon_p)$ is replaced with $\mathbb{R}$ in (i) and (ii). We can regard $D_{\theta,L}^{\kappa,l}(\mathbb{R}^N)_\bullet$ as a subspace of $\bar{X}_\bullet^{\kappa,l}$, and we have a factorisation

$$D_{\theta,L}^{\kappa,l}(\mathbb{R}^N)_\bullet \xrightarrow{i} \bar{X}_\bullet^{\kappa,l} \longrightarrow X_\bullet^{\kappa,l}.$$

The map $\bar{X}_\bullet^{\kappa,l} \to X_\bullet^{\kappa,l}$ is a weak homotopy equivalence in each simplicial degree, by methods similar to [GRW10, Theorem 3.9]. Briefly, in simplicial degree p choose—continuously in the data $(a_0, a_p, \varepsilon_0, \varepsilon_p)$—diffeomorphisms $(a_0 - \varepsilon_0, a_p + \varepsilon_p) \cong \mathbb{R}$ which are the identity on $[a_0, a_p]$. Using this family of diffeomorphisms to stretch gives a map $X_p^{\kappa,l} \to \bar{X}_p^{\kappa,l}$, which is homotopy inverse to the restriction map $\bar{X}_p^{\kappa,l} \to X_p^{\kappa,l}$.

To show that the first map induces a weak homotopy equivalence on geometric realisation, we use a technique which we shall use many times in this paper. That is, we consider a map

$$f : (D^n, \partial D^n) \longrightarrow (|\bar{X}_\bullet^{\kappa,l}|, |D_{\theta,L}^{\kappa,l}(\mathbb{R}^N)_\bullet|)$$

representing an element of the nth relative homotopy group, and show that it may be homotoped through maps of pairs to a map with image in $|D_{\theta,L}^{\kappa,l}(\mathbb{R}^N)_\bullet|$.

For each $x \in D^n$ the point $f(x)$ is a tuple $(t, a, \varepsilon, (W(x), \ell))$, and we may choose a pair (a^x, ε^x) such that $[a^x - \varepsilon^x, a^x + \varepsilon^x] \subset \cup_i((a_i - \varepsilon_i, a_i + \varepsilon_i) - \{a_i\})$ and that $[a^x - \varepsilon^x, a^x + \varepsilon^x]$ consists of regular values of $x_1 : W(x) \to \mathbb{R}$. By properness of $x_1 : W(x) \to \mathbb{R}$, there is a neighbourhood $U_x \ni x$ for which $[a^x - \varepsilon^x, a^x + \varepsilon^x]$ still consists of regular values. The U_x's cover D^n and we let $\{U_j\}_{j \in J}$ be a finite subcover. We may suppose that $a^j \neq a^k$, as otherwise we may change the cover by letting $U_j' = U_j \cup U_k$ with $(a^j)' = a^j = a^k$ and $(\varepsilon^j)' = \min(\varepsilon^j, \varepsilon^k)$. Once the a^j are distinct, we may shrink the ε^j so that the intervals $[a^j + \varepsilon^j, a^j - \varepsilon^j]$ are pairwise disjoint, and so that no a_i lies in such an interval.

As the intervals $[a^j + \varepsilon^j, a^j - \varepsilon^j]$ are chosen to consist of regular values, the data $\{(U_j, a^j, \varepsilon^j)\}_{j \in J}$ determines a map $\hat{f} : D^n \to |D_{\theta,L}^{-1,-1}(\mathbb{R}^N)_\bullet|$ with the same underlying family of θ-manifolds. As $[a^j - \varepsilon^j, a^j + \varepsilon^j] \subset \cup_i(a_i - \varepsilon_i, a_i + \varepsilon_i)$, this new family satisfies conditions (vi) and (vii) of Definition 2.18 (as the old family did) so $\hat{f}$ actually has image in the subspace $|D_{\theta,L}^{\kappa,l}(\mathbb{R}^N)_\bullet|$. There is a homotopy H of $p \circ \hat{f}$ to f as follows: on underlying θ-manifolds it is constant, but on the interval data we first use the straight-line homotopy from the data $\{(a^j, \varepsilon^j)\}$ to the data $\{(a_i, \varepsilon)\}$ where we choose $\varepsilon \leq \min(\varepsilon_i)$ small enough so that $[a_i - \varepsilon, a_i + \varepsilon]$ is disjoint from the $[a^j - \varepsilon^j, a^j + \varepsilon^j]$. This straight-line homotopy is in the barycentric coordinates: as the intervals are all disjoint, the join of the simplices they describe also lies in $|D^{\kappa,l}_{\theta,L}(\mathbb{R}^N)_\bullet|$, and so there is a canonical straight line between them. Then we use the obvious homotopy from the data $\{(a_i, \varepsilon)\}$ to the data $\{(a_i, \varepsilon_i)\}$ that stretches the ε's. The restriction of H to ∂D^n remains in the subspace $|D^{\kappa,l}_{\theta,L}(\mathbb{R}^N)_\bullet|$, and so H gives a relative null-homotopy of f.

The case when $d = 2n$ and $\mathcal{A}$ is chosen is identical. □

3. Surgery on morphisms

In this section we wish to study the filtration

$$\mathcal{C}^{\kappa}_{\theta,L}(\mathbb{R}^N) \subset \cdots \subset \mathcal{C}^{1}_{\theta,L}(\mathbb{R}^N) \subset \mathcal{C}^{0}_{\theta,L}(\mathbb{R}^N) \subset \mathcal{C}^{-1}_{\theta,L}(\mathbb{R}^N) = \mathcal{C}_{\theta,L}(\mathbb{R}^N)$$

and in particular establish the following theorem.

Theorem 3.1. *Suppose that the following conditions are satisfied*

(i) $2\kappa \leq d - 2$,

(ii) $\kappa + 1 + d < N$,

(iii) L admits a handle decomposition only using handles of index $< d - \kappa - 1$.

Then the map

$$B\mathcal{C}^{\kappa}_{\theta,L}(\mathbb{R}^N) \longrightarrow B\mathcal{C}^{\kappa-1}_{\theta,L}(\mathbb{R}^N)$$

is a weak homotopy equivalence.

In order to motivate some of the more technical constructions, let us first give an informal account of the technique we wish to apply. For simplicity, we suppose that $N = \infty$, that we have no tangential structure, that $L = \emptyset$, and that $\kappa = 0$. We first apply the equivalence (2.1) to reduce the problem to studying the map

$$BD^0 \longrightarrow BD^{-1}$$

of classifying spaces of posets. Let

$$\sigma = (t_0, t_1; a_0, a_1; \varepsilon_0, \varepsilon_1; W) \in BD^{-1}$$

be a point on a 1-simplex (for example), where $(t_0, t_1) \in \Delta^1$ are the barycentric coordinates. We will describe a way of producing a path from its image in $|X^{-1}_\bullet|$ into the subspace $|X^0_\bullet|$. The proof of Theorem 3.1 will be a systematic, parametrised version of this construction.

If the cobordism $W|_{[a_0,a_1]}$ is already 0-connected relative to its outgoing boundary, then the image of σ in $|X^{-1}_\bullet|$ already lies in the subspace $|X^0_\bullet|$, and we are done. If not, we may choose distinct points

$$\{f_\alpha : * \to W|_{[a_0,a_1]}\}_{\alpha\in\Lambda}$$

such that the pair $(W|_{[a_0,a_1]}, (W|_{a_1}) \cup \bigcup_\alpha f_\alpha(*))$ is 0-connected. We then choose tubular neighbourhoods of these points to obtain codimension 0 embeddings $\hat{f}_\alpha : D^d \to W|_{[a_0,a_1]}$, which we can extend to an embedding

$$e_\alpha : (S^0, \{+1\}) \times D^d \longrightarrow ([a_0, \infty) \times \mathbb{R}^\infty, [a_1 + \varepsilon_1, \infty) \times \mathbb{R}^\infty).$$

As the original points $f_\alpha(*)$ were distinct, we may suppose the embeddings e_α are disjoint. Now on each $e_\alpha(S^0 \times D^d)$ we do the surgery move shown in Figure 2, a move similar in spirit, though much simpler, than that described in [GMTW09, §6.2].

More precisely, Figure 2 describes a continuous 1-parameter family of d-manifolds $\mathcal{P}_t$, $t \in [0,1]$, depicted (for $d = 2$) by its values at times $t = 0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1$. The family comes equipped with functions to $\mathbb{R}$, depicted in the figure as the height function. The family starts at the manifold $\mathcal{P}_0 = S^0 \times D^d$, and we may cut out each

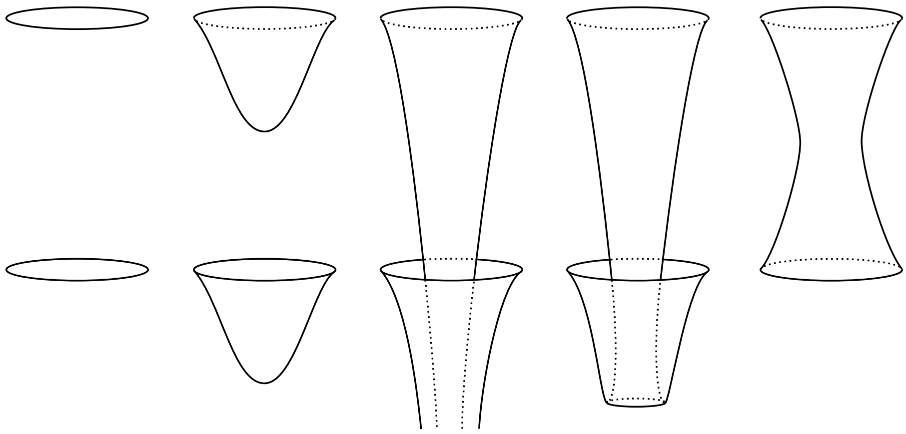

FIGURE 2. The basic move for surgery on morphisms.

$e_\alpha(S^0 \times D^d)$ from W and glue in $\mathcal{P}_t$, to obtain a 1-parameter family of manifolds W_t, each equipped with a height function $W_t \to \mathbb{R}$, with $W_0 = W$. The values $\{a_0, a_1\}$ do not remain regular throughout this move, so this does not describe a path in the space BD^{-1}. However, the intervals $(a_i - \varepsilon_i, a_i + \varepsilon_i)$ do only contain isolated critical values, so it does describe a path in the space $|X_\bullet^{-1}|$. Furthermore, at the end of the move we obtain a manifold $W_1 = \overline{W}$ such that $(\overline{W}|_{[a_0,a_1]}, \overline{W}|_{a_1})$ is 0-connected, and hence a point in $|X_\bullet^0|$.

This surgery move generalises easily to the case when N is finite (but large enough), $L \neq \emptyset$, and $\kappa > 0$ (the analogue of the surgery move will start with $S^\kappa \times D^{d-\kappa}$). However, it does not generalise well to the case of arbitrary tangential structures (to understand how it can fail, we suggest that the reader try to impose a family of framings to the family of 2-manifolds in Figure 2). One way to fix this would be to use the surgery move described in [GMTW09, §6.2], but that does not seem to generalise to $\kappa > 0$. Instead we modify the surgery move in Figure 2 as shown in Figure 3. As we shall see (in the proof of Proposition 3.6, where we also

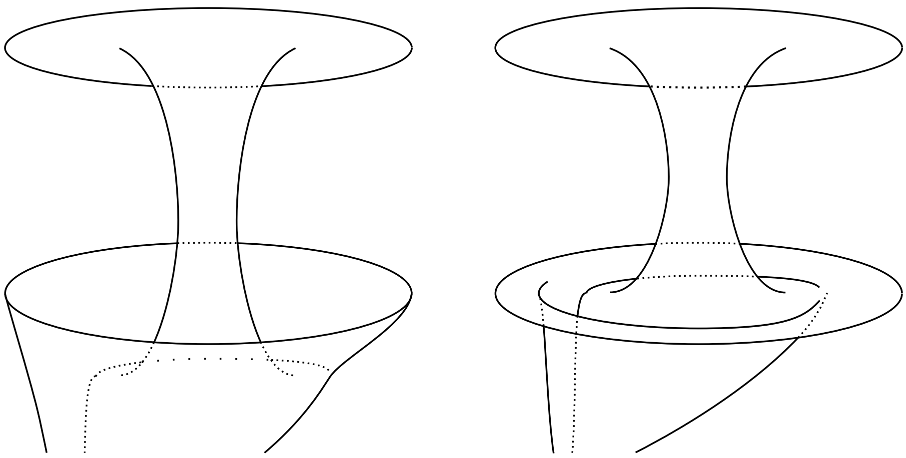

FIGURE 3. In the refined move for surgery on morphisms, these pictures replace the last two frames in Figure 2.

explain the analogous process for $\kappa > 0$) there is a canonical way of extending any tangential structure on $\{-1\} \times D^d$ to the resulting 1-parameter family of manifolds.

3.1. **Surgery data.** We will fatten the semi-simplicial space $D^{\kappa}_{\theta,L}(\mathbb{R}^N)_\bullet$ up to a bi-semi-simplicial space $D^{\kappa}_{\theta,L}(\mathbb{R}^N)_{\bullet,\bullet}$ which includes suitable surgery data, in order to implement the ideas discussed above. The space $D^{\kappa}_{\theta,L}(\mathbb{R}^N)_{\bullet,\bullet}$ is described in Definition 3.3 below, using the following notation. Let $V \subset \overline{V} \subset \mathbb{R}^{\kappa+1} \times \mathbb{R}^{d-\kappa}$ be the subspaces

$$V = (-2,0) \times \mathbb{R}^d \qquad \overline{V} = [-2,0] \times \mathbb{R}^d$$

and let $h : \overline{V} \to [-2,0] \subset \mathbb{R}$ denote projection to the first coordinate, which we call the *height function*. Let $\partial_- D^{\kappa+1} \subset \partial D^{\kappa+1}$ denote the lower hemisphere (i.e. $\partial_- D^{\kappa+1} = \partial D^{\kappa+1} \cap ([-1,0] \times \mathbb{R}^\kappa)$). We shall also use the notation $[p]^\vee = \Delta([p],[1])$ when $[p] \in \Delta_{\mathrm{inj}}$. The elements of $[p]^\vee$ are in bijection with $\{0,\dots,p+1\}$, using the convention that $\varphi : [p] \to [1]$ corresponds to the number i with $\varphi^{-1}(1) = \{i, i+1, \dots, p\}$. Finally, we fix once and for all an uncountable set Ω.

Definition 3.2. Let $(a,\varepsilon,W) \in D^{\kappa-1}_{\theta,L}(\mathbb{R}^N)_p$ and define $Z_q(a,\varepsilon,W)$ to be the set of triples (Λ,δ,e), where $\Lambda \subset \Omega$ is a finite set, $\delta : \Lambda \to [p]^\vee \times [q]$ is a map, and

$$e : \Lambda \times \overline{V} \hookrightarrow \mathbb{R} \times (0,1) \times (-1,1)^{N-1}$$

is an embedding, satisfying the conditions below. We shall write $\Lambda_{i,j} = \delta^{-1}(i,j)$, $e_{i,j} = e|_{\Lambda_{i,j} \times \overline{V}}$ and $D_{i,j} = e_{i,j}\left(\Lambda_{i,j} \times \partial_- D^{\kappa+1} \times \{0\}\right) \subset W$.

(i) On the subset $(x_1 \circ e)^{-1}(a_k - \varepsilon_k, a_k + \varepsilon_k) \subset \Lambda \times \overline{V}$, the height function $x_1 \circ e$ coincides with the height function h (composed with the projection $\Lambda \times \overline{V} \to \overline{V}$) up to an affine transformation.
(ii) e sends $\Lambda \times h^{-1}(0)$ into $x_1^{-1}(a_p + \varepsilon_p, \infty)$.
(iii) For $i > 0$, e sends $\Lambda_{i,j} \times h^{-1}(-3/2)$ into $x_1^{-1}(a_{i-1} + \varepsilon_{i-1}, \infty)$.
(iv) e sends $\Lambda \times h^{-1}(-2)$ into $x_1^{-1}(-\infty, a_0 - \varepsilon_0)$.
(v) $e^{-1}(W) = \Lambda \times \partial_- D^{\kappa+1} \times \mathbb{R}^{d-\kappa}$.
(vi) For each j and each $i \in \{1,...,p\}$, the pair
$$\left(W|_{[a_{i-1},a_i]}, W|_{a_i} \cup D_{i,j}|_{[a_{i-1},a_i]}\right)$$
is κ-connected.

$Z_\bullet(a,\varepsilon,W)$ is a semi-simplicial space: Given an injective map $k : [q] \to [q']$, we replace Λ by the subset $\delta^{-1}([p]^\vee \times \mathrm{Im}(k))$, compose δ with $[p]^\vee \times k^{-1}$, and restrict e. Explicitly, the face map d_j forgets the embeddings $e_{*,j}$

Note that the set $Z_q(a,\varepsilon,W)$ consists of those $(q+1)$-tuples of elements of $Z_0(a,\varepsilon,W)$ which are disjoint.

Definition 3.3. We define a bi-semi-simplicial space $D^{\kappa}_{\theta,L}(\mathbb{R}^N)_{\bullet,\bullet}$ as a set by

$$D^{\kappa}_{\theta,L}(\mathbb{R}^N)_{p,q} = \left\{(a,\varepsilon,W) \in D^{\kappa-1}_{\theta,L}(\mathbb{R}^N)_p, e \in Z_q(a,\varepsilon,W)\right\}$$

topologised as a subspace of

$$D^{\kappa-1}_{\theta,L}(\mathbb{R}^N)_p \times \left(\coprod_{\Lambda \subset \Omega} C^\infty(\Lambda \times \overline{V}, \mathbb{R}^n)\right)^{(p+2)(q+1)}.$$

The space $D^{\kappa}_{\theta,L}(\mathbb{R}^N)_{p,q}$ is functorial in $[p] \in \Delta_{\mathrm{inj}}$ by composing $\delta : \Lambda \to [p]^\vee \times [q]$ with the induced map $[p']^\vee \to [p]^\vee$ and functorial in $[q] \in \Delta_{\mathrm{inj}}$ in the same way as in Definition 3.2. Explicitly, the face map d_i in the q direction forgets the embeddings $e_{*,i}$ and in the p direction takes the union of $e_{i,*}$ and $e_{i+1,*}$. We shall write $D^{\kappa}_{\theta,L}(\mathbb{R}^N)_{p,-1} = D^{\kappa-1}_{\theta,L}(\mathbb{R}^N)_p$, and there is an augmentation map $D^{\kappa}_{\theta,L}(\mathbb{R}^N)_{p,q} \to D^{\kappa}_{\theta,L}(\mathbb{R}^N)_{p,-1}$ which forgets all surgery data.

The main result concerning this bi-semi-simplicial space is the following, whose proof we defer until Section 6.

Theorem 3.4. *Under the assumptions of Theorem 3.1, the augmentation map*

$$D^{\kappa}_{\theta,L}(\mathbb{R}^N)_{\bullet,\bullet} \longrightarrow D^{\kappa-1}_{\theta,L}(\mathbb{R}^N)_{\bullet}$$

induces a weak homotopy equivalence after geometric realisation.

In fact, this theorem is true under slightly weaker hypotheses: when $2\kappa \leq d-1$, $\kappa+1+d < N$, and L admits a handle decomposition only using handles of index $< d-\kappa-1$.

3.2. **The standard family.** We will now construct a one-parameter family of submanifolds of V which formalises the family of manifolds depicted in Figure 3. Let us write coordinates in $\mathbb{R}^{\kappa+1}\times\mathbb{R}^{d-\kappa}$ as (u,v). First define an element $\widetilde{\mathcal{P}}_0 \in \Psi_d(\mathbb{R}\times\mathbb{R}^{\kappa}\times\mathbb{R}^{d-\kappa})$ as

$$\widetilde{\mathcal{P}}_0 = \partial D^{\kappa+1}\times\mathbb{R}^{d-\kappa}.$$

Choose a function $\varphi : [0,\infty)\to[0,\infty)$ that is the identity function on a neighbourhood of $[1/2,\infty)$, takes value $1/4$ near 0, and has $\varphi''\geq 0$. We then define an embedding by

$$\begin{aligned} g' : \mathbb{R}^{\kappa+1}\times\partial D^{d-\kappa} &\longrightarrow D^{\kappa+1}\times\mathbb{R}^{d-\kappa}\\ (u,v) &\longmapsto (u/\varphi(|u|), \varphi(|u|)\cdot v), \end{aligned}$$

and another embedding $g : \mathbb{R}^{\kappa+1}\times\partial D^{d-\kappa}\to[-2,1]\times\mathbb{R}^{\kappa}\times\mathbb{R}^{d-\kappa}$ as

$$g(u,v) = g'(u,v) + \tau(u)\left(\frac{v_1-1}{2}, 0, 0\right)$$

where $\tau : \mathbb{R}^{\kappa+1}\to[0,1]$ is a bump function supported in a small neighbourhood of the point $u_0 = (-1/2, 0)\in\mathbb{R}\times\mathbb{R}^{\kappa}$, having $\tau(u_0)=1$, $\tau(u)<1$ otherwise, and no critical points in $\tau^{-1}((0,1))$. We can arrange that the support of τ be small enough that it is contained in the region where $\varphi(|u|)=|u|$. We let $\widetilde{\mathcal{P}}_1\subset\mathbb{R}^{d+1}$ denote the image of g and $\widetilde{\mathcal{P}}_1'$ be the image of g'. We then define

$$\mathcal{P}_0, \mathcal{P}_1 \in \Psi_d(V)$$

by intersecting the manifolds $\widetilde{\mathcal{P}}_0, \widetilde{\mathcal{P}}_1$ with the open set $V = (-2,0)\times\mathbb{R}^d$.

To construct $\mathcal{P}_t\in\Psi_d(V)$ for intermediate values of $t\in[0,1]$ we first observe that $\widetilde{\mathcal{P}}_0$ and $\widetilde{\mathcal{P}}_1'$ agree on the subset $|v|\geq 1/2$ and that $\widetilde{\mathcal{P}}_1$ agrees with them on a slightly smaller subset $|v|\geq 1/2+\varepsilon$ (where ε depends on the support of the bump function τ but can be made arbitrarily small). Starting with the two submanifolds $\widetilde{\mathcal{P}}_0$ and $\widetilde{\mathcal{P}}_1\subset\mathbb{R}\times\mathbb{R}^{\kappa}\times\mathbb{R}^{d-\kappa}$, we then pull the region $\{(u,v)\mid |v|<1/2+\varepsilon\}$ downwards by decreasing the first coordinate in $\mathbb{R}\times\mathbb{R}^d$, until the region where the submanifolds may disagree is moved completely outside of V. This gives two one-parameter families of submanifolds which, upon restricting to V, give two paths in $\Psi_d(V)$ starting at $\mathcal{P}_0$ and $\mathcal{P}_1$ and ending at the same point in $\Psi_d(V)$. Concatenating one path with the reverse of the other, we get the desired path from $\mathcal{P}_0$ to $\mathcal{P}_1$.

Spelling this process out in a little more detail, we first choose a function $\rho : [0,\infty)\to[0,\infty)$ taking the value 1 near $[0,1/2+\varepsilon]$, the value 0 on $[3/5,\infty)$, and which is strictly decreasing on $\rho^{-1}(0,1)$. We then define embeddings

$$\begin{aligned} H_t : \mathbb{R}\times\mathbb{R}^{\kappa}\times\mathbb{R}^{d-\kappa} &\longrightarrow \mathbb{R}\times\mathbb{R}^{\kappa}\times\mathbb{R}^{d-\kappa}\\ (s,x,y) &\longmapsto (s - t\cdot\rho(|y|), x, y) \end{aligned}$$

which for all t restrict to the identity for $|y| \geq 3/5$. Define one-parameter families of manifolds as

$$\mathcal{P}_t^0 = V \cap H_t(\widetilde{\mathcal{P}}_0) = (H_{-t}|_V)^{-1}(\widetilde{\mathcal{P}}_0)$$
$$\mathcal{P}_t^1 = V \cap H_t(\widetilde{\mathcal{P}}_1) = (H_{-t}|_V)^{-1}(\widetilde{\mathcal{P}}_1).$$

The second description shows that these are closed subsets of V and describe continuous functions $\mathbb{R} \to \Psi_d(V)$. It is easy to see that we have $\mathcal{P}_t^0 = \mathcal{P}_t^1 \in \Psi_d(V)$ for $t \geq 3$, and we then define the path $\mathcal{P}_t$ as the concatenation

$$\mathcal{P}_0 = \mathcal{P}_0^0 \rightsquigarrow \mathcal{P}_3^0 = \mathcal{P}_3^1 \rightsquigarrow \mathcal{P}_0^1 = \mathcal{P}_1$$

in $\Psi_d(V)$, reparametrised so that the path has length 1. We collect the most important properties of this family in Proposition 3.6 below. The following remark partially explains how it relates to an ordinary κ-surgery.

Remark 3.5. Let $Q(u,v) = -|u|^2 + |v|^2$, where as usual $(u,v) \in \mathbb{R}^{\kappa+1} \times \mathbb{R}^{d-\kappa}$. For $t \in [0,3]$ the function $(u,v) \mapsto H_t(u/|u|, v)$ defines a diffeomorphism to $\mathcal{P}_t^0$ from an open subset of $Q^{-1}(t-3)$ (namely the inverse image of V by that function) and similarly the function $(u,v) \mapsto H_t \circ g(u, v/|v|)$ defines a diffeomorphism to $\mathcal{P}_t^1$ from an open subset of $Q^{-1}(3-t)$. The inverses of these diffeomorphisms give smooth embeddings $\mathcal{P}_t^0 \to Q^{-1}(t-3)$ and $\mathcal{P}_t^1 \to Q^{-1}(3-t)$ and it is easy to verify that for $t=3$ the two resulting embedings $\mathcal{P}_3^0 = \mathcal{P}_3^1 \to Q^{-1}(0)$ agree, and so glue to a continuous family of embeddings $\mathcal{P}_t \to Q^{-1}(6t-3)$.

The continuous map $t \mapsto \mathcal{P}_t$ has *graph* given by $\mathcal{P} = \{(t,x) \in [0,1] \times V \mid x \in \mathcal{P}_t\}$. The above remarks give an embedding $\mathcal{P} \to Q^{-1}([-3,3])$ and it is easy to verify that the image is disjoint from the straight lines from 0 to $p_0 = (-1/2, 0, -1/2, 0) \in \mathbb{R} \times \mathbb{R}^\kappa \times \mathbb{R} \times \mathbb{R}^{d-\kappa-1}$ and from p_0 to $p_1 = (-1/2, 0, -\sqrt{13}/2, 0)$. Thus we get a diffeomorphism from $\mathcal{P}$ to an open subset of the contractible set

$$\mathcal{Q} = Q^{-1}([-3,3]) - \big([0,p_0] \cup [p_0,p_1]\big).$$

Proposition 3.6. *For $2\kappa \leq d-1$, the 1-parameter family $\mathcal{P}_t \in \Psi_d(V)$, defined for $t \in [0,1]$, has the following properties.*

(i) The height function, i.e. the restriction of $h : V \to (-2,0)$ to $\mathcal{P}_t \subset V$, has isolated critical values.
(ii) $\mathcal{P}_0 = \mathrm{int}(\partial_- D^{\kappa+1}) \times \mathbb{R}^{d-\kappa}$.
(iii) Independently of $t \in [0,1]$ we have

$$\mathcal{P}_t - (\mathbb{R}^{\kappa+1} \times B_{3/4}^{d-\kappa}(0)) = \mathrm{int}(\partial_- D^{\kappa+1}) \times (\mathbb{R}^{d-\kappa} - B_{3/4}(0)).$$

For ease of notation we write $\mathcal{P}_t^\partial$ for this closed subset of $\mathcal{P}_t$.
(iv) For all t and each pair of regular values $-2 < a < b < 0$ of the height function, the pair

$$(\mathcal{P}_t|_{[a,b]}, \mathcal{P}_t|_b \cup \mathcal{P}_t^\partial|_{[a,b]}) \tag{3.1}$$

is κ-connected.
(v) For each pair of regular values $-2 < a < b < 0$ of the height function, the pair

$$(\mathcal{P}_1|_{[a,b]}, \mathcal{P}_1|_b)$$

is κ-connected.

Furthermore, if $\mathcal{P}_0$ is equipped with a θ-structure ℓ we can upgrade this, continuously in ℓ, to a 1-parameter family $\mathcal{P}_t(\ell) \in \Psi_\theta(V)$ starting from $(\mathcal{P}_0, \ell)$ such that

(iii′) The path $\mathcal{P}_t(\ell)$ is constant as θ-manifolds near $\mathcal{P}_t^\partial$.

Proof. We have seen properties (i)–(iii) during the construction (the statement in (iii) would still be true with $3/4$ replaced by $3/5$, but we wish to emphasize the smaller set). For property (iv) we consider two cases depending on the value of a. In the case $a > -1$, the pair (3.1) is homotopy equivalent to the pair

$$(\mathcal{P}_t|_{[a,b]}, \mathcal{P}_t|_b),$$

using e.g. the gradient flow trajectories of h to deform $\mathcal{P}_t^\partial|_{[a,b]}$ back to $\mathcal{P}_t^\partial|_b$. In the case $a < -1$ we consider the *modified height function*, defined using the coordinates $(u,v) \in \mathbb{R}^{\kappa+1} \times \mathbb{R}^{d-\kappa}$ as $\overline{h}(u,v) = h(u,v) + \lambda(|v|)$, where $\lambda : [0,\infty) \to [0,\infty)$ is a smooth function which is 0 on $[0,4/5]$ and restricts to a diffeomorphism $(4/5,\infty) \to (0,\infty)$. We claim that the inclusion of pairs

$$(3.2) \qquad (\mathcal{P}_t \cap \overline{h}^{-1}([a,b]), \mathcal{P}_t \cap \overline{h}^{-1}(b)) \longrightarrow (\mathcal{P}_t|_{[a,b]}, \mathcal{P}_t|_b \cup \mathcal{P}_t^\partial|_{[a,b]})$$

is a homotopy equivalence. To define a homotopy inverse, we first consider the continuous, piecewise smooth function $\rho_t : [0,\infty) \to (0,\infty)$ defined for $t \leq b$ by

$$\begin{aligned} \rho_t(s) &= 1 && \text{for } s \in [0,3/5], \\ \rho_t(s) &= \frac{\lambda^{-1}(b-t)}{s} && \text{for } s \in [3/4,\infty), \end{aligned}$$

and by linear interpolation for $s \in [3/5,3/4]$. Then the function $(u,v) \mapsto (u, v \cdot \rho_{u_1}(|v|))$ restricts to a homotopy inverse of (3.2), where both homotopies are given by straight lines in $\mathbb{R}^{d+1}$.

In either case, the connectivity question is reduced to studying the inverse image of an interval relative to its outgoing boundary and can be studied as in ordinary Morse theory one critical level at a time. The proof of (iv) will be finished once we establish that for each critical value of $\overline{h} : \mathcal{P}_t \to \mathbb{R}$ in the interval (a,b), the function can be perturbed in a neighbourhood of the critical set contained in $\overline{h}^{-1}((a,b))$ to a Morse function with at most a critical point of index $\leq d-\kappa-1$. (In the case $a > -1$ we have $h = \overline{h}$ near any critical point of h, so it suffices to consider $\overline{h}$.) It is easy to verify that $\overline{h} : \mathcal{P}_t^0 \to \mathbb{R}$ has at most two critical values in $(-2,0)$. One critical value moves with t and is homotopically Morse of index 0 for $0 \leq t < 1$ and index κ for $1 < t < 3$ (meaning that the function can be perturbed to a Morse function with one critical point of that index). The other is at -1 and can be cancelled (meaning that the function can be perturbed to a non-singular function there). Since $2\kappa \leq d-1$ and hence $\kappa \leq d-\kappa-1$, the index is at most $d-\kappa-1$ as claimed. Similarly, one verifies that $\overline{h} : \mathcal{P}_t^1 \to \mathbb{R}$ has at most two critical values in $(-2,0)$, one of which is -1 and can be cancelled, the other of which moves with t and homotopically is Morse of index $d-\kappa-1$.

Property (v) can be proved in a similar way. In the case $a < -1 < b$ the pair is a relative $(d-1)$-cell, so it is $(d-2)$-connected and hence κ-connected (since $d \geq 2$ and $2\kappa \leq d-1$). In all other cases the inclusion $\mathcal{P}_t|_b \to \mathcal{P}_t|_{[a,b]}$ is a homotopy equivalence.

To establish the extra properties which can be obtained given a θ-structure ℓ on $\mathcal{P}_0 = \mathrm{int}(\partial_- D^{\kappa+1}) \times \mathbb{R}^{d-\kappa}$, we again use the graph $\mathcal{P} = \{(t,x) \in [0,1] \times \mathbb{R}^{d+1} \mid x \in \mathcal{P}_t\}$ and its identification with an open subset of the manifold $\mathcal{Q}$ from Remark 3.5. The tangent bundles $T\mathcal{P}_t$ assemble to a d-dimensional vector bundle $T_t\mathcal{P} \to \mathcal{P}$ which then becomes identified with the restriction of the vector bundle $T_t\mathcal{Q} = \mathrm{Ker}(DQ : T\mathcal{Q} \to T[-3,3])$ and since both $\mathcal{P}_0$ and $\mathcal{Q}$ are contractible, there is no obstruction to picking a vector bundle map $r : T_t\mathcal{Q} \to T\mathcal{P}_0$ which is the identity (with respect to the identifications) over $\mathcal{P}_0$ and each $\mathcal{P}_t^\partial = \mathcal{P}_0^\partial \subset \mathcal{P}_0$. We can then restrict r to $r_t : T\mathcal{P}_t \to T\mathcal{P}_0$ and let $\mathcal{P}_t(\ell)$ have the θ-structure $\ell \circ r_t$. $\square$

Let $(a, \varepsilon, (W, \ell), e) \in D^{\kappa}_{\theta,L}(\mathbb{R}^N)_{p,0}$, with $e = \{e_{i,0}\}_{i=0}^{p+1}$. We construct a 1-parameter family of manifolds

$$\mathcal{K}^t_e(W, \ell) \in \Psi_\theta((a_0 - \varepsilon_0, a_p + \varepsilon_p) \times \mathbb{R}^N),$$

$t \in [0, 1]$, by letting it be equal to $W|_{(a_0-\varepsilon_0,a_p+\varepsilon_p)}$ outside of the images of the $e_{i,0}|_{\Lambda_{i,j}\times V}$, and on each $e_{i,0}(\{\lambda\} \times V)$ we let it be given by $e_{i,0}(\{\lambda\} \times \mathcal{P}_t(\ell \circ De_{i,0}))$. This gives a θ-manifold, as by the properties established above, $\mathcal{P}_t(\ell \circ De_{i,0})$ and $\mathcal{P}_0(\ell \circ De_{i,0})$ agree as θ-manifolds near the set $(-2, 0) \times \mathbb{R}^\kappa \times (\mathbb{R}^{d-\kappa} \setminus B_{3/4}(0))$.

Lemma 3.7. *Let $2\kappa \le d - 2$. The tuple $(a, \varepsilon, \mathcal{K}^t_e(W, \ell))$ is an element of $X^{\kappa-1}_p$. If either $t = 1$ or $(W, \ell) \in D^{\kappa}_{\theta,L}(\mathbb{R}^N)_p$, then $(a, \varepsilon, \mathcal{K}^t_e(W, \ell))$ lies in the subspace $X^\kappa_p \subset X^{\kappa-1}_p$.*

Proof. We must verify conditions (i)–(vii) of Definition 2.18. Condition (i) is true by definition, and certainly (ii) is satisfied as the embeddings $e_{i,0}$ are disjoint from $\mathbb{R} \times L$. For (iii)–(v) and (vii) there is nothing to say.

For (vi), consider regular values $a < b \in \cup_i(a_i - \varepsilon_i, a_i + \varepsilon_i)$ of the height function $x_1 : W_t = \mathcal{K}^t_e(W, \ell) \to \mathbb{R}$. The cobordism $W_t|_{[a,b]}$ is obtained from $W|_{[a,b]}$ by cutting out embedded images of cobordisms $\mathcal{P}_0|_{[a_\lambda,b_\lambda]}$ indexed by $\lambda \in \Lambda = \sqcup_i \Lambda_{i,0}$ and gluing in $\mathcal{P}_t|_{[a_\lambda,b_\lambda]}$, where $a_\lambda < b_\lambda$ are regular values of the height function on $\mathcal{P}_0$ and $\mathcal{P}_t$. If we denote by X the complement of the embedded $e_{i,0}(\mathrm{int}(\partial_- D^{\kappa+1}) \times B_{3/4}(0))$ in the manifold $W|_{[a,b]}$, there are homotopy push-out squares

$$\begin{array}{ccc} X|_b & \longrightarrow & W_t|_b \\ \downarrow & & \downarrow \\ X & \longrightarrow & W_t|_b \cup X \end{array}$$

and

$$\begin{array}{ccc} \coprod_{\lambda\in\Lambda} \mathcal{P}_t|_{b_\lambda} \cup \left(\mathcal{P}^\partial_t|_{[a_\lambda,b_\lambda]}\right) & \longrightarrow & W_t|_b \cup X \\ \downarrow & & \downarrow \\ \coprod_{\lambda\in\Lambda} \mathcal{P}_t|_{[a_\lambda,b_\lambda]} & \longrightarrow & W_t|_{[a,b]}. \end{array}$$

The left hand map of the second square is a disjoint union of the maps discussed in property (iv) of Proposition 3.6, so is κ-connected. As this square is a homotopy push-out, the right hand map is also κ-connected.

The pair $(X, X|_b)$ is obtained from the manifold pair $(W|_{[a,b]}, W|_b)$ by cutting out embedded copies of $(D^\kappa, \partial D^\kappa)$. By transversality we see that this does not change relative homotopy groups in dimensions $* \le d - \kappa - 2$, which includes $* \le \kappa$ by our assumption that $2\kappa \le d - 2$. In particular, suppose the pair $(W|_{[a,b]}, W|_b)$ is k-connected, with $k \le \kappa$, then the pair $(X, X|_b)$ is k-connected too. As the first square above is a homotopy push-out square, the inclusion $W_t|_b \to W_t|_b \cup X$ also has this connectivity.

Hence the composition $W_t|_b \to W_t|_b \cup X \to W_t|_{[a,b]}$ has the same connectivity as $W|_b \to W|_{[a,b]}$ has, up to a maximum of κ. This establishes that the tuple $(a, \varepsilon, \mathcal{K}^t_e(W, \ell))$ is an element of $X^{\kappa-1}_p$, and also that it lies in X^κ_p if (W, ℓ) lies in $D^\kappa_{\theta,L}(\mathbb{R}^N)$. When $t = 1$, there is a little more to say.

Step 1. Suppose $a < b \in (a_i - \varepsilon_i, a_i + \varepsilon_i)$. Then $(W|_{[a,b]}, W|_b)$ is ∞-connected and so $(W_1|_{[a,b]}, W_1|_b)$ is κ-connected, by the discussion above.

Step 2. Suppose $a \in (a_{i-1} - \varepsilon_{i-1}, a_{i-1} + \varepsilon_{i-1})$ and $b \in (a_i - \varepsilon_i, a_i + \varepsilon_i)$. We now do the surgeries for $\Lambda_{i,0} \subset \Lambda$ first, giving a manifold $\widetilde{W}_1$. The pair $(\widetilde{W}_1|_{[a,b]}, \widetilde{W}_1|_b)$ is obtained from $(W|_{[a,b]}, W|_b)$ by cutting out copies of $(D^\kappa \times D^{d-\kappa}, S^{\kappa-1} \times D^{d-\kappa})$ which together generate $\pi_\kappa(W|_{[a,b]}, W|_b)$ (by property (vi) in the definition of $Z_0(a,\varepsilon,W)$), and gluing in κ-connected manifolds pairs of the form $(\mathcal{P}_1|_{[a_\lambda,b_\lambda]}, \mathcal{P}_1|_{b_\lambda})$.

We claim that the pair $(\widetilde{W}_1|_{[a,b]}, \widetilde{W}_1|_b)$ is κ-connected. Once this is established, doing the remaining surgeries to obtain W_1 does not change this property, as we have seen above.

The two maps

$$\pi_\kappa(X, X|_b) \longrightarrow \pi_\kappa(\widetilde{W}_1|_b \cup X, \widetilde{W}_1|_b) \longrightarrow \pi_\kappa(\widetilde{W}_1|_{[a,b]}, \widetilde{W}_1|_b)$$

are surjective: the first from homotopy excision on the first push-out square, and the second from the long exact sequence for the triple $(\widetilde{W}_1|_{[a,b]}, \widetilde{W}_1|_b \cup X, \widetilde{W}_1|_b)$. We will show the composition is also zero. We have seen that $\pi_\kappa(X, X|_b) \to \pi_\kappa(W|_{[a,b]}, W|_b)$ is an isomorphism, and so $\pi_\kappa(X, X|_b)$ is generated by the images of the maps

$$\pi_\kappa(D^\kappa \times \partial D^{d-\kappa}, S^{\kappa-1} \times \partial D^{d-\kappa}) \longrightarrow \pi_\kappa(X, X|_b).$$

The composition of each of these with $\iota : (X, X|_b) \to (\widetilde{W}_1|_{[a,b]}, \widetilde{W}_1|_b)$ factors through some $(\mathcal{P}_1|_{[a_\lambda,b_\lambda]}, \mathcal{P}_1|_{b_\lambda})$ which is κ-connected, so $\pi_\kappa(\iota)$ is trivial as required.

Step 3. For general $a < b \in \cup_i(a_i - \varepsilon_i, a_i + \varepsilon_i)$, we may choose regular values in each intermediate interval $(a_j - \varepsilon_j, a_j + \varepsilon_j)$. By the previous case, this expresses $W_1|_{[a,b]}$ as a composition of cobordisms which are each κ-connected relative to their outgoing boundaries, and the hence the composition also has that property. □

3.3. **Proof of Theorem 3.1.** We begin with the composition

$$|D^\kappa_{\theta,L}(\mathbb{R}^N)_{\bullet,\bullet}| \longrightarrow |D^{\kappa-1}_{\theta,L}(\mathbb{R}^N)_\bullet| \longrightarrow |X^{\kappa-1}_\bullet|,$$

where the first map (induced by the augmentation) is a homotopy equivalence by Theorem 3.4 and the second is a homotopy equivalence by Proposition 2.20. We will define a homotopy

$$\mathscr{S} : [0,1] \times |D^\kappa_{\theta,L}(\mathbb{R}^N)_{\bullet,\bullet}| \longrightarrow |X^{\kappa-1}_\bullet|$$

starting from this map so that $\mathscr{S}(1,-)$ has image in the subspace $|X^\kappa_\bullet|$. Furthermore, there is an inclusion

$$|D^\kappa_{\theta,L}(\mathbb{R}^N)_\bullet| \hookrightarrow |D^\kappa_{\theta,L}(\mathbb{R}^N)_{\bullet,0}| \hookrightarrow |D^\kappa_{\theta,L}(\mathbb{R}^N)_{\bullet,\bullet}|$$

as manifolds equipped with no surgery data, and $\mathscr{S}$ will be constant on this subspace. The existence of a homotopy with these properties establishes Theorem 3.1, as follows: there is a diagram

$$\begin{tikzcd}
|D^\kappa_{\theta,L}(\mathbb{R}^N)_\bullet| \arrow[r] \arrow[d] & |X^{\kappa-1}_\bullet| \arrow[d] \\
|D^\kappa_{\theta,L}(\mathbb{R}^N)_{\bullet,\bullet}| \arrow[r, "\mathscr{S}(0,-)"] \arrow[ur, "\mathscr{S}(1,-)"] & |X^\kappa_\bullet|
\end{tikzcd}$$

where the square commutes, the horizontal maps are weak homotopy equivalences, the top triangle commutes exactly and the bottom triangle commutes up to the homotopy $\mathscr{S}$. Taking homotopy groups we see that the vertical maps are also weak equivalences. Under the equivalence $B\mathcal{C}^\kappa_{\theta,L}(\mathbb{R}^N) \simeq |X^\kappa_\bullet|$, and similarly for $\kappa-1$, we obtain Theorem 3.1.

To define the surgery map $\mathscr{S}$ we will give a collection of maps

$$\mathscr{S}_{p,q} : [0,1] \times D^\kappa_{\theta,L}(\mathbb{R}^N)_{p,q} \times \Delta^p \times \Delta^q \longrightarrow X^{\kappa-1}_p \times \Delta^p$$

compatible on their faces. The construction of the last section gives a 1-parameter family

$$\begin{aligned}\mathcal{K}^r : D^\kappa_{\theta,L}(\mathbb{R}^N)_{p,0} &\longrightarrow X^{\kappa-1}_p \\ (a,\varepsilon,W,e) &\longmapsto (a,\varepsilon,\mathcal{K}^r_e(W)),\end{aligned}$$

for $r \in [0,1]$, such that $\mathcal{K}^1$ lands in X_p^κ. When $q = 0$, we set

$$\mathscr{S}_{p,0}(r, (a, \varepsilon, W, e), t) = ((a, \varepsilon, \mathcal{K}_e^r(W)), t) \in X_p^{\kappa-1} \times \Delta^p,$$

where $t = (t_0, \dots, t_p) \in \Delta^p$. More generally, for $q \geq 0$ we have $e = \{e_{i,j}\}$, and for each j we get an element $(a, \varepsilon, W, e_{*,j}) \in D_{\theta,L}^\kappa(\mathbb{R}^N)_{p,0}$. We then set

$$\mathscr{S}_{p,q}(r, (a, \varepsilon, W, e), t, s) = ((a, \varepsilon, \mathcal{K}_{e_{*,q}}^{\bar{s}_q \cdot r} \circ \cdots \circ \mathcal{K}_{e_{*,0}}^{\bar{s}_0 \cdot r}(W)), t),$$

where $\bar{s}_j = s_j / \max_k\{s_k\}$. Note that some $\bar{s}_j$ is always equal to 1, so when $r = 1$, some $\mathcal{K}_{e_{*,j}}^1$ is applied to W making each morphism κ-connected relative to its outgoing boundary. The remaining $\mathcal{K}_{e_{*,k}}^{\bar{s}_k \cdot r}$ do not change this property, by Lemma 3.7, and so the resulting manifold lies in X_p^κ.

We will now show that the maps $\mathscr{S}_{p,q}$ fit together. Let us write d_i for the p direction face maps, and $\bar{d}_i$ for the q direction ones. If $t_i = 0$ then $t = d^i t'$ and so $d_i \mathscr{S}_{p,q}(r, x, t, s)$ is $d_i \mathscr{S}_{p,q}(r, x, d^i t', s)$. On the other hand it is equal to $\mathscr{S}_{p-1,q}(r, d_i x, t', s)$ as the value of t does not affect how much surgery $\mathscr{S}$ does. If $s_j = 0$, so $s = \bar{d}^j s'$, then $\bar{s}_j = 0$ also and so the operation $\mathcal{K}_{e_{*,j}}^{\bar{s}_j \cdot r}$ is the identity. Thus $\mathscr{S}_{p,q}(r, x, t, \bar{d}^j s') = \mathscr{S}_{p,q-1}(r, \bar{d}^j x, t, s')$, and so together these maps define $\mathscr{S} : [0,1] \times |D_{\theta,L}^\kappa(\mathbb{R}^N)_{\bullet,\bullet}| \to | X_\bullet^{\kappa-1}|$. On the subspace $|D_{\theta,L}^\kappa(\mathbb{R}^N)_\bullet| \hookrightarrow |D_{\theta,L}^\kappa(\mathbb{R}^N)_{\bullet,0}|$, the homotopy is constant as there is no surgery data. At $r = 1$ it has image in $|X_\bullet^\kappa|$.

4. Surgery on objects below the middle dimension

In this section we wish to study the filtration

$$\mathcal{C}_{\theta,L}^{\kappa,l}(\mathbb{R}^N) \subset \cdots \subset \mathcal{C}_{\theta,L}^{\kappa,1}(\mathbb{R}^N) \subset \mathcal{C}_{\theta,L}^{\kappa,0}(\mathbb{R}^N) \subset \mathcal{C}_{\theta,L}^{\kappa,-1}(\mathbb{R}^N) = \mathcal{C}_{\theta,L}^{\kappa}(\mathbb{R}^N)$$

and in particular establish the following theorem.

Theorem 4.1. *Suppose that the following conditions are satisfied*

(i) $2(l+1) < d$,
(ii) $l \leq \kappa$,
(iii) $l \leq d - \kappa - 2$,
(iv) $l + 2 + d < N$,
(v) *L admits a handle decomposition only using handles of index* $< d - l - 1$,
(vi) *the map $\ell_L : L \to B$ is $(l+1)$-connected.*

Then the map

$$B\mathcal{C}_{\theta,L}^{\kappa,l}(\mathbb{R}^N) \longrightarrow B\mathcal{C}_{\theta,L}^{\kappa,l-1}(\mathbb{R}^N)$$

is a weak homotopy equivalence.

The proof will be similar in spirit to that of the last section, insofar as we will describe a surgery move which compresses $B\mathcal{C}_{\theta,L}^{\kappa,l-1}(\mathbb{R}^N)$ into the subspace $B\mathcal{C}_{\theta,L}^{\kappa,l}(\mathbb{R}^N)$. In the same way that the surgery move of the last section was a refinement of that of [GMTW09], the surgery move we use in this and the next section is a refinement of that of [GRW10]. Let us first give an informal account of this move, and for simplicity suppose that $N = \infty$, that we have no tangential structure (i.e. we consider $\theta : BO(d) \to BO(d)$), that $L = \emptyset$, and that $d > 2$, $l = 0$ and $\kappa = 0$. We first apply the equivalence (2.1) to reduce the problem to studying the map

$$BD^{0,0} \longrightarrow BD^{0,-1}$$

of classifying spaces of posets. Let

$$\sigma = (t_0, t_1; a_0, a_1; \varepsilon_0, \varepsilon_1; W) \in BD^{0,-1}$$

be a point on a 1-simplex (for example), and let us suppose that $W|_{a_1}$ is already connected (so its path-components inject into $\pi_0(BO(d))$). We will describe a way of producing a path from the image of this point in $|X_\bullet^{0,-1}|$ into the subspace $|X_\bullet^{0,0}|$.

If $W|_{a_0}$ is already connected, then the point σ already lies in $|X_\bullet^{0,0}|$ and there is nothing to say. Otherwise, let us choose disjoint embeddings

$$\{f_\alpha : S^0 \hookrightarrow W|_{a_0}\}_{\alpha\in\Lambda}$$

such that if we perform 0-surgery along all of these embeddings, the resulting $(d-1)$-manifold is connected. As $\kappa = 0$, the cobordism $W|_{[a_0,a_1]}$ is path-connected relative to its top, and so we can extend the f_α to smooth maps

$$\hat{f}_\alpha : (a_0 - \varepsilon_0, a_1 + \varepsilon_1) \times S^0 \longrightarrow W$$

such that the standard height function (i.e. the projection to $(a_0 - \varepsilon_0, a_1 + \varepsilon_1)$) and $x_1 \circ \hat{f}_\alpha$ agree inside $(x_1 \circ \hat{f}_\alpha)^{-1}(\cup(a_i - \varepsilon_i, a_i + \varepsilon_i))$. As we have supposed that $d > 2$, we may assume that these $\hat{f}_\alpha$ are mutually disjoint embeddings. By taking a tubular neighbourhood, we extend the $\hat{f}_\alpha$ to embeddings

$$\hat{e}_\alpha : (a_0 - \varepsilon_0, a_1 + \varepsilon_1) \times D^{d-1} \times S^0 \hookrightarrow W$$

which are still mutually disjoint, and extend these further to disjoint embeddings

$$e_\alpha : (a_0 - \varepsilon_0, a_1 + \varepsilon_1) \times D^{d-1} \times D^1 \hookrightarrow \mathbb{R} \times \mathbb{R}^\infty$$

such that $e_\alpha^{-1}(W) = (a_0 - \varepsilon_0, a_1 + \varepsilon_1) \times D^{d-1} \times S^0$. It is clear that we can arrange the same relationship between the standard height function on $(a_0 - \varepsilon_0, a_1 + \varepsilon_1) \times D^{d-1} \times D^1$ and the function $x_1 \circ e_\alpha$ as we have over $(a_0 - \varepsilon_0, a_1 + \varepsilon_1) \times D^{d-1} \times S^0$.

The surgery move is then given by gluing the following figure,

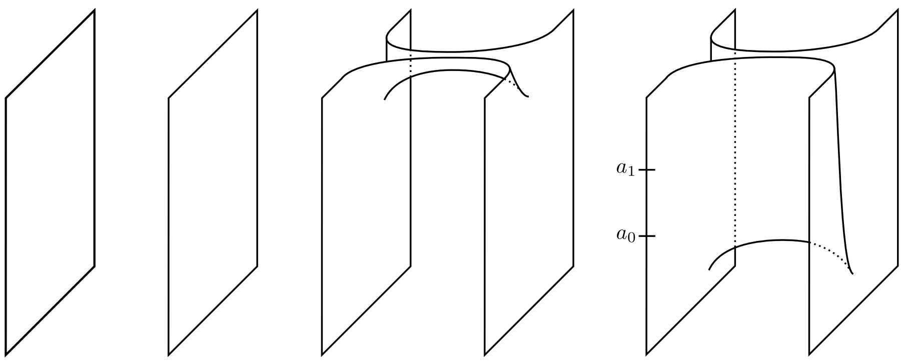

FIGURE 4. The surgery move for surgery on objects below the middle dimension.

the trace of a 0-surgery on $D^{d-1} \times S^0$ inside of $(a_0 - \varepsilon_0, a_1 + \varepsilon_1) \times D^{d-1} \times D^1$, into $\mathbb{R} \times \mathbb{R}^\infty$ using each of the embeddings e_α. This does not define a path in $BD^{0,-1}$, as $(a_1 - \varepsilon_1, a_1 + \varepsilon_1)$ will contain a critical value at some points during the path. However, it does define a path in $|X_\bullet^{0,-1}|$. Furthermore, if we let $\overline{W}$ be the manifold obtained at the end of the path, then $\overline{W}|_{a_0}$ is obtained from $W|_{a_0}$ by doing 0-surgery along the data $\{\hat{e}_\alpha|_{\{a_0\}\times D^{d-1}\times S^0}\}_{\alpha\in\Lambda}$ and so is connected. Also, $\overline{W}|_{a_1}$ is obtained from $W|_{a_1}$ by doing 0-surgery along the data $\{\hat{e}_\alpha|_{\{a_1\}\times D^{d-1}\times S^0}\}_{\alpha\in\Lambda}$, and as it was connected to start with (and $d > 2$), it remains connected. Hence $(t_0, t_1; a_0, a_1; \varepsilon_0, \varepsilon_1; \overline{W}) \in |X_\bullet^{0,0}|$, as required.

This surgery move generalises well to $l > 0$, to finite (but large enough) N, and to non-empty L, but to make it work with general tangential structures θ we must equip the surgery data $\{e_\alpha\}_{\alpha\in\Lambda}$ with extra data describing how to induce a

θ-structure on the surgered manifold. We will first give a definition of θ-surgery, then describe the standard family, and finally go on to describe the semi-simplicial space of surgery data analogous to that of Section 3.1.

4.1. **θ-surgery.** Consider a θ-manifold (M, ℓ_M) and an embedding $e : \mathbb{R}^{d-l-1} \times S^l \hookrightarrow M$, and let C be the d-dimensional cobordism obtained as the trace of the surgery along e. Thus $\partial_{in} C = M$ and $\partial_{out} C = \overline{M}$ is the result of the surgery.

The data of a *θ-surgery* on M is an embedding e as above along with a θ-structure ℓ on C which agrees with ℓ_M on M. This induces a θ-structure on $\overline{M}$. We will typically give the data of a θ-surgery extending an embedding e by giving an extension of the θ-structure $\ell_M \circ De$ on $\mathbb{R}^{d-l-1} \times S^l$ to $\mathbb{R}^{d-l-1} \times D^{l+1}$. Up to homotopy, this is the same as specifying a null-homotopy of the map $S^l \to B$ underlying the θ-structure on $\mathbb{R}^{d-l-1} \times S^l$.

4.2. **The standard family.** Let us construct the one-parameter family of manifolds depicted in Figure 4. Choose a function $\rho : \mathbb{R} \to \mathbb{R}$ which is the identity on $(-\infty, 1/2)$, has nowhere negative derivative, and has $\rho(t) = 1$ for all $t \geq 1$. We define

$$K = \{(x, y) \in \mathbb{R}^{d-l} \times \mathbb{R}^{l+1} |\; |y|^2 = \rho(|x|^2 - 1)\},$$

a smooth d-dimensional submanifold, contained in $\mathbb{R}^{d-l} \times D^{l+1}$, which outside of the set $B^{d-l}_{\sqrt{2}}(0) \times D^{l+1}$ is identically equal to $\mathbb{R}^{d-l} \times S^l$. Let

$$h = x_1 : K \longrightarrow \mathbb{R}$$

denote the first of the x coordinates, which is the height function we will consider on K. This function is Morse with precisely two critical points: $(-1, 0, ..., 0)$ of index $l+1$ and $(1, 0, ..., 0)$ of index $d-l-1$.

We now define a 1-parameter family of d-dimensional submanifolds $\mathcal{P}_t$ inside $(-6,-2) \times \mathbb{R}^{d-l-1} \times D^{l+1}$ in the following way. Pick a smooth one-parameter family of embeddings $\lambda_s : (-6,-2) \to (-6, 0)$, such that $\lambda_0 = \mathrm{id}$, that $\lambda_s|_{(-6,-4)} = \mathrm{id}$ for all s, and that $\lambda_1(-3) = -1$. Then we get an embedding $\lambda_t \times 1 : (-6,-2) \times \mathbb{R}^d \to (-6, 0) \times \mathbb{R}^d$ and define

$$\mathcal{P}_t = (\lambda_t \times 1)^{-1}(K) \in \Psi_d((-6,-2) \times \mathbb{R}^{d-l-1} \times \mathbb{R}^{l+1}).$$

It is easy to verify that $\mathcal{P}_t$ agrees with $(-6,-2) \times \mathbb{R}^{d-l-1} \times S^l$ outside $(-4,-2) \times B^{d-l-1}_{\sqrt{2}}(0) \times D^{l+1}$, independently of t.

We shall also need a tangentially structured version of this construction, given a structure $\ell : TK|_{(-6,0)} \to \theta^*\gamma$. For this purpose, let $\omega : [0, \infty) \to [0, 1]$ be a smooth function such that $\omega(r) = 0$ for $r \geq 2$ and $\omega(r) = 1$ for $r \leq \sqrt{2}$. We define a 1-parameter family of embeddings by

$$\begin{aligned} \psi_t : (-6,-2) \times \mathbb{R}^{d-l-1} \times \mathbb{R}^{l+1} &\longrightarrow (-6, 0) \times \mathbb{R}^{d-l-1} \times \mathbb{R}^{l+1} \\ (s, x, y) &\longmapsto (\lambda_{t\omega(|x|)}(s), x, y), \end{aligned}$$

It is easy to see that we also have $\psi_t^{-1}(K) = (\lambda_t \times 1)^{-1}(K) = \mathcal{P}_t$, and we define a θ-structure on $\mathcal{P}_t$ by pullback along ψ_t. This gives a family

$$\mathcal{P}_t(\ell) \in \Psi_\theta((-6,-2) \times \mathbb{R}^{d-l-1} \times \mathbb{R}^{l+1}),$$

and we record some important properties in the following proposition. We will omit ℓ from the notation when it is unimportant.

Proposition 4.2. *The elements $\mathcal{P}_t(\ell) \in \Psi_\theta((-6,-2) \times \mathbb{R}^{d-l-1} \times \mathbb{R}^{l+1})$ are θ-submanifolds of $(-6,-2) \times \mathbb{R}^{d-l-1} \times D^{l+1}$ satisfying*

(i) $\mathcal{P}_0(\ell) = K|_{(-6,-2)} = (-6,-2) \times \mathbb{R}^{d-l-1} \times S^l$ as θ-manifolds.

(ii) For all t, $\mathcal{P}_t(\ell)$ agrees with $K|_{(-6,-2)}$ as θ-manifolds outside of $(-4,-2) \times B^{d-l-1}_2(0) \times D^{l+1}$.

(iii) *For all t and each pair of regular values $-6 < a < b < -2$ of the height function $h : \mathcal{P}_t \to \mathbb{R}$, the pair*

$$(\mathcal{P}_t|_{[a,b]}, \mathcal{P}_t|_b)$$

is $(d-l-2)$-connected.

(iv) *For each regular value a of $h : \mathcal{P}_t \to (-6,-2)$, the manifold $\mathcal{P}_t|_a$ is either isomorphic to $\mathcal{P}_0|_a$ or is obtained from it by l-surgery.*

(v) *The only critical point of $h : \mathcal{P}_1 \to (-6,-2)$ is -3, and for $a \in (-3,-2)$, $\mathcal{P}_1|_a$ is obtained by l-surgery from $\mathcal{P}_0|_a = \mathbb{R}^{d-l-1} \times S^l$ along the standard embedding.*

In (iv) and (v), the θ-structure on the surgered manifold is determined (up to homotopy, cf. Section 4.1) by the θ-structure on $K|_{(-6,0)}$.

The precise meaning of the word isomorphic in (iv) above is the following: By (ii) we know that the manifolds are *equal* outside $(-4,-2) \times B_2^{n+1}(0) \times D^n$. Being isomorphic means that the identity extends to a diffeomorphism which preserves θ-structures up to a homotopy of bundle maps which is constant outside $(-4,-2) \times B_2^{n+1}(0) \times D^n$.

Proof. (i) and (ii) follow easily from the properties of λ_t and ψ_t, and the fact that K agrees with $\mathbb{R}^{d-l} \times S^l$ outside $B_{\sqrt{2}}^{d-l} \times \mathbb{R}^{l+1}$. It follows from the properties of ω that the θ-structures agree outside $B_2^{d-l} \times \mathbb{R}^{l+1}$. For (iii), the Morse function $\mathcal{P}_t \to (-6,-2)$ has at most one critical point, and that has index $l+1$. If the critical value is in (a,b), then the pair is $(d-l-2)$-connected, otherwise $\mathcal{P}|_{[a,b]}$ deformation retracts to $\mathcal{P}|_b$. The fact that the Morse function has at most one critical point, of index $l+1$, also implies (iv) by definition of surgery (and θ-surgery, cf. Section 4.1). Finally, the property that $\lambda_1(-3) = -1$ and $\lambda_1(-4) = -4$ implies that $h : \mathcal{P}_1 \to (-6,-2)$ does have a critical point of index $l+1$, with critical value in -3, which proves (v). □

4.3. **Surgery data.** We can now describe the semi-simplicial space of surgery data out of which we will construct a "perform surgery" map. In the following section we will describe how to construct this map.

Before doing so, we choose once and for all, smoothly in the data $(a_i, \varepsilon_i, a_p, \varepsilon_p)$, increasing diffeomorphisms

$$\varphi = \varphi(a_i, \varepsilon_i, a_p, \varepsilon_p) : (-6,-2) \cong (a_i - \varepsilon_i, a_p + \varepsilon_p) \tag{4.1}$$

sending -3 to $a_i - \frac{1}{2}\varepsilon_i$ and -4 to $a_i - \frac{3}{4}\varepsilon_i$.

Definition 4.3. Let $(a, \varepsilon, (W, \ell_W)) \in D_{\theta,L}^{\kappa,l-1}(\mathbb{R}^N)_p$, and write $M_i = W \cap x_1^{-1}(a_i)$. Define the set $Y_0(a, \varepsilon, (W, \ell_W))$ to consist of $(p+1)$-tuples of pairs (e_i, ℓ_i) defined for $i \in \{0, ..., p\}$. Each e_i is an embedding

$$e_i : \Lambda_i \times (a_i - \varepsilon_i, a_p + \varepsilon_p) \times \mathbb{R}^{d-l-1} \times D^{l+1} \hookrightarrow \mathbb{R} \times (0,1) \times (-1,1)^{N-1}$$

where $\Lambda_i \subset \Omega$ is a finite set, and are required to satisfy

(i) The e_i have disjoint images.

(ii) $e_i^{-1}(W) = \Lambda_i \times (a_i - \varepsilon_i, a_p + \varepsilon_p) \times \mathbb{R}^{d-l-1} \times S^l$ for each i. We let

$$\partial e_i : \Lambda_i \times (a_i - \varepsilon_i, a_p + \varepsilon_p) \times \mathbb{R}^{d-l-1} \times S^l \hookrightarrow W$$

denote the embedding restricted to the boundary.

(iii) For $t \in \cup_i (a_i - \varepsilon_i, a_i + \varepsilon_i)$, we have $(x_1 \circ e_i)^{-1}(t) = \Lambda_i \times \{t\} \times \mathbb{R}^{d-l-1} \times D^{l+1}$.

Each ℓ_i is a bundle map

$$\ell_i : T(\Lambda_i \times K|_{(-6,0)}) \longrightarrow \theta^* \gamma$$

and under the chosen diffeomorphism

$$K|_{(-6,-2)} = (-6,-2) \times \mathbb{R}^{d-l-1} \times S^l \cong_\varphi (a_i - \varepsilon_i, a_p + \varepsilon_p) \times \mathbb{R}^{d-l-1} \times S^l$$

we insist that ℓ_i and $\ell_W \circ D\partial e_i$ are equal. The data (e_i, ℓ_i) is enough to perform θ-surgery on M_i, and we further insist that

(iv) The resulting θ_{d-1}-manifold $\overline{M}_i$ has the property that $\pi_j(\overline{M}_i) \to \pi_j(B)$ is injective for $j \leq l$.

We let $Y_q(a, \varepsilon, (W, \ell_W))$ be the set of $(q+1)$-tuples of elements in $Y_0(a, \varepsilon, (W, \ell_W))$ which are all disjoint. Then $Y_\bullet(a, \varepsilon, (W, \ell_W))$ forms a semi-simplicial set.

Definition 4.4. We define a bi-semi-simplicial space $D^{\kappa,l}_{\theta,L}(\mathbb{R}^N)_{\bullet,\bullet}$ (augmented in the second semi-simplicial direction) as a set by

$$D^{\kappa,l}_{\theta,L}(\mathbb{R}^N)_{p,q} = \{(a, \varepsilon, (W, \ell_W)) \in D^{\kappa,l-1}_{\theta,L}(\mathbb{R}^N)_p, \{(e_{i,j}, \ell_{i,j})\} \in Y_q(a, \varepsilon, (W, \ell_W))\},$$

and topologise it as a subspace of

$$D^{\kappa,l-1}_{\theta,L}(\mathbb{R}^N)_p \times \left(\coprod_{\Lambda \subset \Omega} C^\infty(\Lambda \times V, \mathbb{R}^{N+1}) \times \mathrm{Bun}\left(T(\Lambda \times K|_{(-6,0)}), \theta^*\gamma\right)\right)^{(p+1)(q+1)}$$

where V denotes the manifold $(0,1) \times \mathbb{R}^{d-l-1} \times D^{l+1}$. The face map d_k in the q direction is by forgetting the surgery data $\{(e_{i,k}, \ell_{i,k})\}_{i=0}^p$, and the face map d_k in the p is by forgetting the surgery data $\{(e_{k,j}, \ell_{k,j})\}_{j=0}^q$ and the kth interval of regular values.

The main result about this bi-semi-simplicial space of manifolds equipped with surgery data is the following, whose proof we defer until Section 6.

Theorem 4.5. *Under the assumptions of Theorem 4.1, the augmentation map*

$$D^{\kappa,l}_{\theta,L}(\mathbb{R}^N)_{\bullet,\bullet} \longrightarrow D^{\kappa,l-1}_{\theta,L}(\mathbb{R}^N)_\bullet$$

induces a weak homotopy equivalence after geometric realisation.

The inclusion

$$|D^{\kappa,l}_{\theta,L}(\mathbb{R}^N)_{\bullet,0}| \longrightarrow |D^{\kappa,l}_{\theta,L}(\mathbb{R}^N)_{\bullet,\bullet}|$$

is a weak homotopy equivalence under the same conditions.

In fact, this theorem is true under slightly weaker hypotheses: when $2(l+1) < d$, $l \leq \kappa$, $l + 2 + d < N$, L admits a handle decomposition only using handles of index $< d - l - 1$, and the map $\ell_L : L \to B$ is $(l+1)$-connected.

4.4. **Proof of Theorem 4.1.** We now go on to prove Theorem 4.1, so suppose that the inequalities in the statement of that theorem are satisfied: $2(l+1) < d$, $l \leq \kappa$, $l \leq d - \kappa - 2$, $l + 2 + d < N$, and L admits a handle decomposition only using handles of index $< d - l$.

Let $(a, \varepsilon, (W, \ell_W), \{(e_i, \ell_i)\}_{i=0}^p) \in D^{\kappa,l}_{\theta,L}(\mathbb{R}^N)_{p,0}$. We construct a 1-parameter family of elements $\mathcal{K}^t_{e_i,\ell_i}(W, \ell_W) \in \Psi_\theta((a_0 - \varepsilon_0, a_p + \varepsilon_p) \times \mathbb{R}^N)$, $t \in [0,1]$ as follows. Changing the first coordinate of the manifolds $\mathcal{P}_t(\ell_i)$ by composing with the reparametrisation functions of (4.1), we get a family of manifolds

$$\overline{\mathcal{P}}_t(\ell_i) \in \Psi_\theta((a_i - \varepsilon_i, a_p + \varepsilon_p) \times \mathbb{R}^{d-l-1} \times \mathbb{R}^{l+1})$$

having all the properties of Proposition 4.2, where (v) now holds for all regular values in $(a_i - \frac{1}{2}\varepsilon_i, a_p + \varepsilon_p)$. Then for $t \in [0,1]$, let

$$\mathcal{K}^t_{e_i,\ell_i}(W, \ell_W) \in \Psi_\theta((a_0 - \varepsilon_0, a_p + \varepsilon_p) \times \mathbb{R}^N)$$

be equal to $W|_{(a_0 - \varepsilon_0, a_p + \varepsilon_p)}$ outside the image of e_i, and on $e_i(\Lambda_i \times (a_i - \varepsilon_i, a_p + \varepsilon_p) \times \mathbb{R}^{d-l-1} \times D^{l+1})$ be given by $e_i(\Lambda_i \times \overline{\mathcal{P}}_t(\ell_i))$. This gives a θ-manifold, because

$\Lambda_i \times \overline{\mathcal{P}}_t(\ell_i)$ and $\Lambda_i \times \overline{\mathcal{P}}_0(\ell_i)$ agree as θ-manifolds outside of $(a_i - \frac{3}{4}\varepsilon_i, a_p + \varepsilon_p) \times B_2^{d-l-1}(0) \times D^{l+1}$.

As the embeddings e_i are all disjoint, this procedure can be iterated, and for a tuple $t = (t_0, ..., t_p) \in [0,1]^{p+1}$ we let

$$\mathcal{K}^t_{e,\ell}(W, \ell_W) = \mathcal{K}^{t_p}_{e_p,\ell_p} \circ \cdots \circ \mathcal{K}^{t_0}_{e_0,\ell_0}(W, \ell_W) \in \Psi_\theta((a_0 - \varepsilon_0, a_p + \varepsilon_p) \times \mathbb{R}^N).$$

Lemma 4.6. *Firstly, the tuple $(a, \frac{1}{2}\varepsilon, \mathcal{K}^t_{e,\ell}(W, \ell_W))$ is an element of $X_p^{\kappa,l-1}$. Secondly, if $t_i = 1$—so the surgery for the regular value a_i is fully done—then for any regular value b of $x_1 : \mathcal{K}^t_{e,\ell}(W, \ell_W) \to \mathbb{R}$ in the interval $(a_i - \frac{1}{2}\varepsilon_i, a_i + \frac{1}{2}\varepsilon_i)$ we have that*

$$\pi_j(\mathcal{K}^t_{e,\ell}(W, \ell_W)|_b) \longrightarrow \pi_j(B)$$

is injective for $j \leq l$.

Proof. For the first part we must verify the conditions of Definition 2.18. Conditions (i)–(v) are immediate from the properties of (a, ε) that we start with, the disjointness of the surgery data from $\mathbb{R} \times L$, and the fact that the standard family $\mathcal{P}_t$ has isolated critical values.

For condition (vi) we proceed exactly as in the proof of Lemma 3.7, using property (iii) of the standard family, that the pair $(\mathcal{P}_t|_{[a,b]}, \mathcal{P}_t|_b)$, and hence the homotopy equivalent pair

$$\big(\mathcal{P}_t|_{[a,b]}, (\mathcal{P}_t|_b) \cup ([a,b] \times (\mathbb{R}^{d-l-1} - B_2^{d-l-1}(0)) \times S^l)\big)$$

is $(d-l-2)$-connected, and so in particular κ-connected as we have supposed that $l \leq d - \kappa - 2$.

For condition (vii), let $b \in (a_i - \frac{1}{2}\varepsilon_i, a_i + \frac{1}{2}\varepsilon_i)$ be a regular value of the height function on $\mathcal{K}^t_{e,\ell}(W, \ell_W)$, and define θ_{d-1}-manifolds

$$\begin{aligned} \overline{M} &= \mathcal{K}^t_{e,\ell}(W, \ell_W)|_b \\ M &= W|_b. \end{aligned}$$

By property (iv) of the standard family, the θ_{d-1}-manifold $\overline{M}$ is obtained from M by performing θ-l-surgery. Let $C : M \rightsquigarrow \overline{M}$ be the θ-cobordism given by the trace of this surgery. We have the commutative diagram

$$\begin{tikzcd} \pi_j(M) \arrow[r] \arrow[dr] & \pi_j(C) \arrow[d] & \pi_j(\overline{M}) \arrow[l] \arrow[dl] \\ & \pi_j(B) & \end{tikzcd} \tag{4.2}$$

and C is obtained by attaching an $(l+1)$-cell to M or by attaching a $(d-l-1)$-cell to $\overline{M}$. Hence as long as $j \leq \min(l-1, d-l-2)$ the two horizontal maps are isomorphisms, and as long as $j \leq l-1$, the left diagonal map is injective (as $(a, \varepsilon, (W, \ell_W)) \in D^{\kappa,l-1}_{\theta,L}(\mathbb{R}^N)$). As we have supposed $2(l+1) < d$, these two conditions are satisfied as long as $j \leq l-1$, and so the right hand diagonal map is also injective in this range as required.

We now prove the second part, so suppose $t_i = 1$. We construct the manifold $\mathcal{K}^t_{e,\ell}(W, \ell_W)$ by first taking $\mathcal{K}^1_{e_i,\ell_i}(W, \ell_W)$ and then performing the remaining surgeries to it. Let $\widetilde{M} = \mathcal{K}^1_{e_i,\ell_i}(W, \ell_W)|_b$, so that $\overline{M}$ is obtained from $\widetilde{M}$ by l-surgery.

We first show that $\pi_j(\widetilde{M}) \to \pi_j(B)$ is injective for $j \leq l$. By property (iv) of the complex of surgery data, $\{(e_i, \ell_i)\}$ is enough surgery data on $W \cap x_1^{-1}(b)$ to make the map on π_l be injective after performing it. By property (v) of the standard family, as $b > a_i - \frac{1}{2}\varepsilon_i$ the manifold $\widetilde{M}$ has all of this surgery done, and so $\pi_j(\widetilde{M}) \to \pi_j(B)$ is injective for $j \leq l$.

We now claim that the remaining surgeries necessary to get from $\widetilde{M}$ to $\overline{M}$ do not change this injectivity property. This is because $\overline{M}$ is obtained from $\widetilde{M}$ by performing l-surgery along elements in the kernel of $\pi_l(\widetilde{M}) \to \pi_l(B)$, but this map is injective: thus we perform surgery along null-homotopic embeddings $S^l \times D^{d-l-1} \hookrightarrow \widetilde{M}$. As $2(l+1) < d$, by Whitney's embedding theorem the surgery data can be represented by l-spheres embedded inside a small ball in $\widetilde{M}$, and so $\overline{M} \cong \widetilde{M} \# n(S^{l+1} \times S^{d-l-2})$. Taking the trace of this surgery, the two horizontal maps in the corresponding diagram (4.2) are also isomorphisms for $j = l$, and so $\pi_j(\overline{M}) \to \pi_j(B)$ is also injective for $j \leq l$. □

In the composition

$$|D^{\kappa,l}_{\theta,L}(\mathbb{R}^N)_{\bullet,\bullet}| \longrightarrow |D^{\kappa,l-1}_{\theta,L}(\mathbb{R}^N)_{\bullet}| \longrightarrow |X^{\kappa,l-1}_{\bullet}|$$

both maps are homotopy equivalences by Theorem 4.5 and Proposition 2.20 respectively (the inequalities for these two results are implied by those we have assumed). There is also an inclusion

$$|D^{\kappa,l}_{\theta,L}(\mathbb{R}^N)_{\bullet}| \longrightarrow |D^{\kappa,l}_{\theta,L}(\mathbb{R}^N)_{\bullet,0}| \longrightarrow |D^{\kappa,l}_{\theta,L}(\mathbb{R}^N)_{\bullet,\bullet}|$$

as the subspace of manifolds equipped with no surgery data, and the second map is a weak homotopy equivalence by Theorem 4.5.

We define a map

$$\begin{array}{rcl} \mathscr{S}_p : [0,1]^{p+1} \times D^{\kappa,l}_{\theta,L}(\mathbb{R}^N)_{p,0} & \longrightarrow & X^{\kappa,l-1}_p \\ (t, (a, \varepsilon, (W, \ell_W), \{(e_i, \ell_i)\}_{i=0}^p)) & \longmapsto & (a, \tfrac{1}{2}\varepsilon, \mathcal{K}^t_{e,\ell}(W, \ell_W)) \end{array}$$

which has the desired range by the first part of Lemma 4.6, and furthermore sends $(1, ..., 1) \times D^{\kappa,l}_{\theta,L}(\mathbb{R}^N)_{p,0}$ into $X^{\kappa,l}_p$. On the boundary of the cube this map has further distinguished properties: one is given by the second part of Lemma 4.6. The second is that, by Proposition 4.2 (i), we have an equality $\mathcal{K}^0_{e_i,\ell_i}(W') = W'$ of θ-submanifolds of $(a_0 - \varepsilon_0, a_p + \varepsilon_p) \times \mathbb{R}^N$. Thus we obtain the formula

$$d_i \mathscr{S}_p(d^i t, x) = \mathscr{S}_{p-1}(t, d_i(x)) \tag{4.3}$$

where $d^i : [0,1]^p \to [0,1]^{p+1}$ adds a zero in the ith position, and the d_i are the face maps of the semi-simplicial spaces $D^{\kappa,l}_{\theta,L}(\mathbb{R}^N)_{\bullet,0}$ and $X^{\kappa,l-1}_{\bullet}$.

We wish to assemble the maps $\mathscr{S}_p$ to a homotopy $\mathscr{S} : [0,1] \times |D^{\kappa,l}_{\theta,L}(\mathbb{R}^N)_{\bullet,0}| \to |X^{\kappa,l-1}_{\bullet}|$. Hence we define $\lambda, \psi : \Delta^p \to [0,1]^{p+1}$ by the formulæ

$$\begin{array}{rcl} \lambda_i(t) & = & \min\left(1, 2\bar{t}_i\right) \\ \psi_i(t) & = & \max\left(0, 2\bar{t}_i - 1\right), \end{array}$$

where again $\bar{t}_i = t_i / \max(t_j)$, and a map $H : [0,1] \times \Delta^p \to [0,1]^{p+1} \times \Delta^p$ by

$$H(s,t) = \left(s \cdot \lambda(t), \frac{\psi(t)}{\sum_j \psi_j(t)} \right).$$

These may be used to form the composition

$$F_p : [0,1] \times D^{\kappa,l}_{\theta,L}(\mathbb{R}^N)_{p,0} \times \Delta^p \xrightarrow{H} D^{\kappa,l}_{\theta,L}(\mathbb{R}^N)_{p,0} \times [0,1]^{p+1} \times \Delta^p \xrightarrow{\mathscr{S}_p \times \Delta^p} X^{\kappa,l-1}_p \times \Delta^p.$$

Lemma 4.7. *These maps glue to a homotopy $\mathscr{S} : [0,1] \times |D^{\kappa,l}_{\theta,L}(\mathbb{R}^N)_{\bullet,0}| \to | X^{\kappa,l-1}_{\bullet}|$.*

Proof. The points $F_p(s, x, d^i t)$ and $F_p(s, d_i(x), t)$ are identified under the usual face maps among the $X^{\kappa,l-1}_p \times \Delta^p$. This follows immediately from the formula (4.3) and the observation that $\lambda(d^i(t)) = d^i(\lambda(t))$, $\psi(d^i(t)) = d^i(\psi(t))$ and $\sum_j \psi_j(d^i(t)) = \sum_j \psi_j(t)$. □

By construction, the map $\mathscr{S}(1,-) : |D^{\kappa,l}_{\theta,L}(\mathbb{R}^N)_{\bullet,0}| \to | X^{\kappa,l-1}_\bullet|$ has image in the subspace $|X^{\kappa,l}_\bullet|$. This may be seen at the level of the maps F_p, as for each i either $\lambda_i(t) = 1$, and hence the surgery near the regular value a_i is completely done (and so by the second part of Lemma 4.6 it does not matter what the remaining surgeries do near the regular value a_i), or else $\lambda_i(t) < 1$ and hence $\psi_i(t) = 0$ and the regular value a_i is not required to satisfy any conditions as it is equivalent under face identifications to the element obtained by forgetting this regular value.

The homotopy $\mathscr{S}$ is constant on the subspace $|D^{\kappa,l}_{\theta,L}(\mathbb{R}^N)_\bullet| \hookrightarrow |D^{\kappa,l}_{\theta,L}(\mathbb{R}^N)_{\bullet,0}|$, and so it gives the required homotopy equivalence $|X^{\kappa,l}_\bullet| \simeq |X^{\kappa,l-1}_\bullet|$. Theorem 4.1 then follows from the equivalence $B\mathcal{C}^{\kappa,l-1}_{\theta,L}(\mathbb{R}^N) \simeq |X^{\kappa,l-1}_\bullet|$ and its analogue for l.

5. Surgery on objects in the middle dimension

We now restrict our attention to even dimensions, and write $d = 2n$. Given a collection of path components $\mathcal{A} \subset \pi_0(\mathrm{ob}(\mathcal{C}^{n-1,n-2}_{\theta,L}(\mathbb{R}^N)))$ which hit every path-component of $B\mathcal{C}^{n-1,n-2}_{\theta,L}(\mathbb{R}^N)$, in Definition 2.11 we defined

$$\mathcal{C}^{n-1,\mathcal{A}}_{\theta,L}(\mathbb{R}^N) \subset \mathcal{C}^{n-1,n-2}_{\theta,L}(\mathbb{R}^N)$$

to be the full subcategory on this collection of objects. To state our main theorem concerning these subcategories, we first need a definition.

Definition 5.1. We say a tangential structure θ is *reversible* if whenever there is a morphism $C : M \rightsquigarrow N$ in $\mathcal{C}_{\theta,L}$, there also exists a morphism $\overline{C} : N \rightsquigarrow M$ in this category whose underlying manifold is the reflection of C.

Proposition 5.6 gives another characterisation of this property, which is the one that can usually be checked in practice. In particular, it implies that any once-stable tangential structure is reversible. We can now state our main theorem concerning these subcategories, analogous to Theorem 4.1 but in the middle dimension.

Theorem 5.2. *Suppose that*

(i) $2n \geq 6$,
(ii) $3n + 1 < N$,
(iii) θ *is reversible,*
(iv) L *admits a handle decomposition only using handles of index* $< n$,
(v) $\ell_L : L \to B$ *is* $(n-1)$*-connected.*

Then

$$B\mathcal{C}^{n-1,\mathcal{A}}_{\theta,L}(\mathbb{R}^N) \longrightarrow B\mathcal{C}^{n-1,n-2}_{\theta,L}(\mathbb{R}^N)$$

is a weak homotopy equivalence.

The surgery move that we will employ is similar to that of the last section, but has a crucial difference. In the last section, when we performed the surgery move to make a_i be a good regular value, we glued a family of manifolds having the effect of performing l-surgery on the level sets $W|_{a_i}$, but at the same time performing l-surgery on all higher level sets. In Section 4, $l < (d-2)/2 = n-1$, and therefore performing l-surgery on a $(2n-1)$-manifold which is l-connected preserves its l-connectedness. In this section, we will need to change level sets by doing $(n-1)$-surgery on $(2n-1)$-manifolds, and this is much more delicate. For example, any 1-manifold can be made connected by performing 0-surgeries, but performing further 0-surgeries will disconnect it again.

Instead we use a modified surgery move, which will let us perform $(n-1)$-surgery on a level set $W|_a$ and leave all other level sets $W|_b$ unchanged, except when b is very close to a. For $n = 1$, this was done in [GRW10], and the construction there generalises to higher n. Let us briefly recall and depict the case $n = 1$. We start with the same surgery data as in Section 4, a collection of embeddings

$$\{e_\alpha : (a_0 - \varepsilon_0, a_1 + \varepsilon_1) \times D^1 \times D^1 \hookrightarrow \mathbb{R} \times \mathbb{R}^\infty\}_{\alpha \in \Lambda},$$

but glue in to the image of each e_α the path of manifolds shown in Figure 5. This

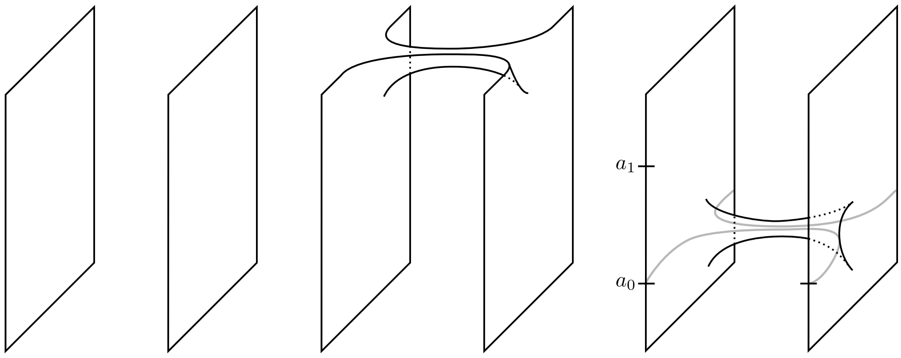

FIGURE 5. The surgery move on objects in the middle dimension. In the last frame we have indicated the level set at a_0 in light grey, to emphasize that it *is* modified by the surgery.

defines a path in the space $|X_\bullet^{0,-1}|$, and if the handle in Figure 5 which we have moved into the manifold is "thin" enough (with respect to the height function) then the manifold $\overline{W}$ obtained at the end of the path has $\overline{W}|_{a_0}$ and $\overline{W}|_{a_1}$ both connected, and so lies in $|X_\bullet^{0,0}|$.

In order to make sense of this surgery move in the presence of θ-structures, we must equip the 1-parameter family of manifolds shown in Figure 5 with θ-structures which start at a given structure, are constant near the vertical boundaries, and at the end of the path the level sets above and below the handle should be isomorphic as θ-manifolds to the level sets before the handle was added. This last property does not hold in general: for example, if we equip the original manifold in Figure 5 with a framing, one may easily see (using the Poincaré–Hopf theorem) that there is no framing on the final manifold consistent with these requirements. As we will see, this problem goes away when θ is assumed to be reversible. Let us first discuss the reversibility condition in more detail.

5.1. **Reversibility.** Let us first discuss some related conditions on tangential structures $\theta : B \to BO(d)$.

Definition 5.3. A tangential structure $\theta : B \to BO(d)$ is *once-stable* if there exists a map $\bar{\theta} : \bar{B} \to BO(d+1)$ and a commutative diagram

$$\begin{tikzcd} B \arrow[r] \arrow[d, "\theta"'] & \bar{B} \arrow[d, "\bar{\theta}"] \\ BO(d) \arrow[r] & BO(d+1) \end{tikzcd}$$

which is homotopy pullback.

A tangential structure θ is *weakly once-stable* if there exists such a diagram for which $\pi_i(BO(d), B) \to \pi_i(BO(d+1), \bar{B})$ is surjective for $i = d+1$ and bijective for $i \leq d$, for all basepoints.

From the commutative diagram in the definition, there is a bundle map $\varepsilon^1 \oplus \theta^*\gamma \to \bar{\theta}^*\gamma$. Hence a θ-structure $TW \to \theta^*\gamma$ on a d-manifold W induces a bundle map $\varepsilon^1 \oplus TW \to \bar{\theta}^*\gamma$. If θ is weakly once-stable we may deduce the converse, that a bundle map $\varepsilon^1 \oplus TW \to \bar{\theta}^*\gamma$ is homotopic to one that arises from a θ-structure. More precisely, we have the following useful lemma.

Lemma 5.4. *Let $\theta : B \to BO(d)$ be weakly once-stable. Let W be a d-manifold and let $\ell : TW|_A \to \theta^*\gamma$ be a θ-structure defined on a closed submanifold $A \subset W$. Then ℓ extends to a θ-structure $TW \to \theta^*\gamma$ if and only if the stabilised bundle map $\varepsilon^1 \oplus \ell : \varepsilon^1 \oplus TW|_A \to \varepsilon^1 \oplus \theta^*\gamma$ extends to a bundle map over all of W.*

Proof. Without loss of generality, we may assume that θ and $\bar{\theta}$ are Serre fibrations. Let us write $s : BO(d) \to BO(d+1)$ for the stabilisation map, and let us pick a classifying map $t : W \to BO(d)$ for the tangent bundle. Tangential structures on TW then correspond to lifts of t along some fibration, and tangential structures on $\varepsilon^1 \oplus TW$ correspond to lifts of $s \circ t$ along some fibration.

We write $\widetilde{\theta} : \widetilde{B} \to BO(d)$ for the pullback of $\bar{\theta}$, so the commutative diagram in Definition 5.3 gives a map $i : B \to \widetilde{B}$ over $BO(d)$. A $\bar{\theta}$-structure on $\varepsilon^1 \oplus TW$ is then nothing but a $\widetilde{\theta}$-structure on TW.

The long exact sequence on homotopy for the various fibrations combine to give

$$\cdots \longrightarrow \pi_i(\widetilde{B}, B) \longrightarrow \pi_i(BO(d), B) \longrightarrow \pi_i(BO(d+1), \bar{B}) \longrightarrow \pi_{i-1}(\widetilde{B}, B) \longrightarrow \cdots$$

from which we deduce that $(\widetilde{B}, B)$ is d-connected. Now, the situation described in the statement is a lifting problem

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ W \arrow[r] \arrow[ur, dotted] & \tilde{B}, \end{tikzcd}$$

which has a solution as (W, A) has cells of dimension at most d, and $(\widetilde{B}, B)$ is d-connected. $\square$

Lemma 5.5. *The tangential structure $\theta : B \to BO(d)$ is weakly once-stable if and only if any θ-structure on a disc $D^d \subset S^d$ extends to one on S^d.*

Proof. Given any bundle map $\ell : TS^d|_{D^d} \to \theta^*\gamma$ we can of course extend the stabilised map to $\varepsilon^1 \oplus TS^d \to \varepsilon^1 \oplus \theta^*\gamma$, and if θ is weakly once stable, the above lemma implies that the θ structure extends.

Conversely, given $\theta : B \to BO(d)$ we may pick θ-structures $\ell_i : TS^d \to \theta^*\gamma$, one for each path component of B, and form $\bar{B}$ by attaching an $(n+1)$-cell to B along each map. The compositions $S^d \to B \to BO(d) \to BO(d+1)$ are null-homotopic, so we obtain an extension $\bar{\theta} : \bar{B} \to BO(d+1)$.

It follows that $H_i(\bar{B}, B) \to H_i(BO(d+1), BO(d))$ is surjective for $i = d+1$ and an isomorphism for $i \leq d$, even with local coefficients. By the Hurewicz theorem, $\pi_i(\bar{B}, B) \to \pi_i(BO(d+1), BO(d))$ is surjective for $i = d+1$ and bijective for $i \leq d$, for all basepoints. It follows that $\pi_i(BO(d), B) \to \pi_i(BO(d+1), \bar{B})$ is surjective for $i = d+1$ and bijective for $i \leq d$. $\square$

Thus a tangential structure is weakly once-stable if and only if every structure on a disc extends to one on the sphere. We now show that these conditions are also equivalent to reversibility.

Proposition 5.6. *The tangential structure θ is reversible if and only if every θ-structure on a disc $D^d \subset S^d$ extends to one on S^d.*

Proof. If θ is reversible and a structure on D^d is given, we think of D^d as a morphism from the empty set to S^{d-1}. By assumption, a compatible structure exists on the disc, thought of as a morphism from S^{d-1} to the empty set.

For the reverse direction we use Lemma 5.4. Suppose given a cobordism $C : M \rightsquigarrow N$ with θ-structure $\ell : TC \to \theta^*\gamma$. Let $\overline{C} : N \rightsquigarrow M$ be the cobordism whose underlying manifold is C, but regarded as a morphism in the other direction. Since $TC|_{\partial C} = \varepsilon^1 \oplus T(\partial C)$, we may reflect in the ε^1-direction to get a reversed θ-structure near $\partial\bar{C} = N \amalg M$, and our task is to extend the reversed structure to $\bar{C}$. By the lemma, it suffices to extend the stabilised bundle map, but that is easy: Pick a non-zero section of the vector bundle $\varepsilon^1 \oplus TC$ which over ∂C is the inwards pointing normal to $T(\partial C) \subset TC|_{\partial C}$, and reflect the stabilised bundle map in that field. $\square$

One key property of reversible tangential structures is that they allow us to connect-sum θ-manifolds, which of course is not possible in general: the connect-sum of framed manifolds is not typically framable. In fact, more is true. We can perform arbitrary surgeries on a θ-manifold and find a θ-structure on the new manifold.

Proposition 5.7. *Let (M, ℓ_M) be a d-dimensional θ-manifold, and suppose that*

$$e : S^{n-1} \times D^{d-n+1} \hookrightarrow M$$

is a piece of surgery data such that the map $S^{n-1} \to B$ induced by $e \circ \ell_M$ is null-homotopic. Then if θ is reversible, the surgered manifold

$$\overline{M} = (M - \mathrm{int}(S^{n-1} \times D^{d-n+1})) \cup_{S^{n-1}\times S^{d-n}} (D^n \times S^{d-n})$$

admits a θ-structure which agrees with ℓ_M on $(M - \mathrm{int}(S^{n-1} \times D^{d-n+1}))$.

Proof. If we let V denote the trace of the surgery, then the θ-structure on M and a choice of null-homotopy of $e \circ \ell_M$ induces a bundle map $TV \to \varepsilon^1 \oplus \theta^*\gamma$, and by restriction a bundle map $\varepsilon^1 \oplus T\overline{M} \to \varepsilon^1 \oplus \theta^*\gamma$, which we can assume agrees with the stabilisation of ℓ_M on $(M - \mathrm{int}(S^{n-1} \times D^{d-n+1})) \subset \overline{M}$. But when θ is weakly once-stable, Lemma 5.4 says that this bundle map can be replaced with one induced from a θ-structure. $\square$

For tangential structures that are once-stable (not just weakly), we can say that for a d-manifold W with a fixed θ-structure $\ell_0 : TW|_{\partial W} \to \theta^*\gamma$, the stabilisation map

$$\mathrm{Bun}^{\partial}(TW, \theta^*\gamma; \ell_0) \longrightarrow \mathrm{Bun}^{\partial}(\varepsilon^1 \oplus TW, \bar{\theta}^*\gamma; \varepsilon^1 \oplus \ell_0)$$

is a weak homotopy equivalence. (Weakly once-stable only implies that this map is 0-connected.) We shall not make explicit use of the stronger condition in this paper, but point out that most of the naturally occuring tangential structures *are* once-stable. In particular, the following construction will be our main source of once-stable tangential structures. Let W be a connected d-dimensional manifold with basepoint, and $\tau : W \to BO(d)$ be its Gauss map, which we may assume to be pointed. For each k there are Moore–Postnikov factorisations of τ

$$W \xrightarrow{j_k} B_W(k) \xrightarrow{p_k} BO(d)$$

where $\pi_*(j_k)$ is an isomorphism for $* < k$ and an epimorphism for $* = k$, and $\pi_*(p_k)$ is an isomorphism for $* > k$ and a monomorphism for $* = k$. These connectivity properties characterise $B_W(k)$, by obstruction theory. Then $\theta_W(k) = p_k$ is a tangential structure.

Lemma 5.8. *The tangential structure $\theta_W(k) : B_W(k) \to BO(d)$ is once-stable for any $k \le d$.*

Proof. We let $\bar{B}_W(k)$ denote the same Moore–Postnikov construction applied to the composition $W \to BO(d) \to BO(d+1)$. The claim then follows as $BO(d) \to BO(d+1)$ is d-connected. $\square$

Remark 5.9. There do exist tangential structures which are reversible but not once-stable, which justifies our emphasis on reversibility. An interesting example is $BU(3) \to BO(6)$, which is reversible as S^6 admits an almost complex structure, but is not once-stable: if it were pulled back from a fibration $f : \bar{B} \to BO(7)$, one can easily use the Serre spectral sequence to check that the kernel of the map f^* on $\mathbb{F}_2$-cohomology would be the ideal $I = (w_1, w_3, w_5) \subset H^*(BO(7); \mathbb{F}_2)$, but this is not closed under the action of the Steenrod algebra as $Sq^4(w_5) = w_4 \cdot w_5 + w_3 \cdot w_6 + w_2 \cdot w_7 \notin I$.

5.2. **The standard family.** We will prove Theorem 5.2 by performing $(n-1)$-surgery on objects until we reach an object in $\mathcal{A}$, just as in Section 4 we performed l-surgery on objects to make them l-connected (relative to L). As in that section, the surgery shall be performed by gluing in a suitable family of manifolds along certain families of embeddings, whose existence we shall prove in Section 6. The standard family to be glued in is very similar to that in Section 4, where we started with a certain submanifold $K \subset \mathbb{R}^{d-l} \times \mathbb{R}^{l+1}$. In this section, $l = n-1$, so we have a submanifold $K \subset \mathbb{R}^{n+1} \times D^n$ defined as follows. We first chose a smooth function $\rho : \mathbb{R} \to \mathbb{R}$ which is the identity on $(-\infty, \frac{1}{2})$, has nowhere negative derivative, and has $\rho(t) = 1$ for all $t \geq 1$, and we let

$$K = \{(x, y) \in \mathbb{R}^{n+1} \times \mathbb{R}^n \mid |y|^2 = \rho(|x|^2 - 1)\}.$$

The first coordinate restricts to a Morse function $h = x_1 : K \to \mathbb{R}$ with exactly two critical points: $(-1, 0, ..., 0; 0)$ and $(+1, 0, ..., 0; 0)$ both of index n.

In Section 4, we constructed from K a one-parameter family of manifolds $\mathcal{P}_t \subset (-6, -2) \times \mathbb{R}^{d+1}$, obtained from K by moving the lowest critical point down as $t \in [0, 1]$ increases, as in Figure 4. In this section we shall need a two-parameter family $\mathcal{P}_{t,w} \subset \mathbb{R} \times (-6, -2) \times \mathbb{R}^n \times \mathbb{R}^n$ which is constructed from $\{0\} \times K$ by moving *both* critical points down as $t \in [0, 1]$ increases, as in Figure 5. As $w \in [0, 1]$ decreases, we shrink the width of the handle so that the distance between the two critical values is $2w$. In order for the manifold to stay embedded in the limit $w = 0$, we need an extra ambient dimension.

Let us first construct a 1-parameter family of submanifolds $K_w \subset \mathbb{R} \times \mathbb{R}^{n+1} \times D^n$ such that $K_1 = \{0\} \times K$. Let $\rho : \mathbb{R} \to [0, 1]$ be a smooth function which is zero on $(2, \infty)$ and identically 1 on $(-\infty, \sqrt{2})$, and define a 1-parameter family of embeddings

$$\begin{aligned} \varphi_w : \mathbb{R}^{n+1} \times D^n &\longrightarrow \mathbb{R} \times \mathbb{R}^{n+1} \times D^n \\ (x, y) &\longmapsto (x_1(1 - (1-w)\rho(|x|)), x_1(1-w)\rho(|x|), x_2, ..., x_{n+1}, y). \end{aligned}$$

We now let

$$K_w = \varphi_w(K) \subset \mathbb{R} \times \mathbb{R}^{n+1} \times D^n$$

for $w \in [0, 1]$. A calculation shows that (for $w > 0$) the critical points of the projection to the first coordinate of $K_w \subset \mathbb{R}^{n+2}$ are $\varphi_w(\pm 1, 0, ..., 0)$ and so lie at heights $\pm w$. They remain Morse of index n.

We now define a 2-parameter family of d-dimensional submanifolds $\mathcal{P}_{t,w}$ inside $(-6, -2) \times \mathbb{R}^{n+1} \times D^n$ in much the same way as $\mathcal{P}_t$ was constructed from K in Section 4.1. Apart from the extra width parameter, the main difference is that we in this section will use a larger part of K, including *both* critical points. Pick a smooth one-parameter family of embeddings $\lambda_s : (-6, -2) \to (-6, 2)$, such that $\lambda_0 = \mathrm{id}$, that $\lambda_s|_{(-6,-5)} = \mathrm{id}$ for all s, and that $\lambda_1(-4) = -1$ and $\lambda_1(-3) = 1$. Then we get embeddings $\lambda_t \times 1 : (-6, -2) \times \mathbb{R}^{2n+1} \to (-6, 2) \times \mathbb{R}^{2n+1}$ and define

$$\mathcal{P}_{t,w} = (\lambda_t \times 1)^{-1}(K_w) \in \Psi_d((-6, -2) \times \mathbb{R}^{n+1} \times \mathbb{R}^n).$$

It is easy to verify that $\mathcal{P}_{t,w}$ agrees with $(-6,-2) \times \{0\} \times \mathbb{R}^n \times S^{n-1}$ outside $(-5,-2) \times \{0\} \times B_2^n(0) \times D^n$, independently of t and w.

We shall also need a tangentially structured version of this construction, given a structure $\ell : TK|_{(-6,2)} \to \theta^*\gamma$. For this purpose, let $\omega = \rho : \mathbb{R} \to [0,1]$ be the function defined above and define a 1-parameter family of embeddings by

$$\begin{array}{rcl}\psi_t : (-6,-2) \times \mathbb{R}^{n+1} \times \mathbb{R}^n & \longrightarrow & (-6,2) \times \mathbb{R}^{n+1} \times \mathbb{R}^n \\ (s,x,y) & \longmapsto & (\lambda_{t\omega(|x|)}(s), x, y),\end{array}$$

It is easy to see that we also have $\psi_t^{-1}(K_w) = (\lambda_t \times 1)^{-1}(K_w) = \mathcal{P}_{t,w}$, and we define a θ-structure on $\mathcal{P}_{t,w}$ by pullback along ψ_t. This gives a two-parameter family

$$\mathcal{P}_{t,w}(\ell) \in \Psi_\theta((-6,-2) \times \mathbb{R}^{n+1} \times \mathbb{R}^n).$$

We will omit ℓ from the notation when it is unimportant. We record some important properties of this family in Proposition 5.11 below, using the following definition.

Definition 5.10. Let $\ell : TK \to \theta^*\gamma$ be a θ-structure on K. Recall that outside of $\mathbb{R} \times B_2^n(0) \times D^n$ the manifold K agrees with $\mathbb{R} \times \mathbb{R}^n \times S^{n-1}$. We say that ℓ is *extendible* if the θ-structure $\ell|_{\mathbb{R}\times(\mathbb{R}^n - B_2^n(0))\times S^{n-1}}$ extends to a θ-structure on the whole of $\mathbb{R} \times \mathbb{R}^n \times S^{n-1}$.

Proposition 5.11. *The elements $\mathcal{P}_{t,w}(\ell) \in \Psi_\theta((-6,-2) \times \mathbb{R}^{n+1} \times \mathbb{R}^n)$ are θ-submanifolds of $(-6,-2) \times \mathbb{R}^{n+1} \times D^n$ satisfying*

(i) $\mathcal{P}_{0,w}(\ell) = K_w|_{(-6,-2)} = (-6,-2) \times \{0\} \times \mathbb{R}^n \times S^{n-1}$ as θ-manifolds, for all $w \in [0,1]$.

(ii) For all t and w, $\mathcal{P}_{t,w}(\ell)$ agrees with $K_1|_{(-6,-2)}$ as a θ-manifold, outside of $(-5,-2) \times B_2^{n+1}(0) \times D^n$.

(iii) For all t and w and each pair of regular values $-6 < a < b < -2$ of the height function $h : \mathcal{P}_{t,w} \to \mathbb{R}$, the pair

$$(\mathcal{P}_{t,w}|_{[a,b]}, \mathcal{P}_{t,w}|_b)$$

is $(n-1)$-connected.

(iv) For $w = 0$, the height function $h : \mathcal{P}_{t,0} \to (-6,-2)$ (which is not Morse) has at most one critical value. If ℓ is extendible then for each regular value a, then $\mathcal{P}_{t,0}|_a$ is isomorphic as a θ-manifold to $\mathcal{P}_{0,0}|_a = \{a\} \times \{0\} \times \mathbb{R}^n \times S^{n-1}$.

(v) For $w = 1$, the critical values of $h : \mathcal{P}_{1,1} \to (-6,-2)$ are -4 and -3. For $a \in (-4,-3)$, $\mathcal{P}_{1,1}|_a$ is obtained by $(n-1)$-surgery from $\mathcal{P}_{0,1}|_a = \{0\} \times \mathbb{R}^n \times S^{n-1}$ along the standard embedding. If ℓ is extendible then for a outside that interval, $\mathcal{P}_{1,1}|_a$ is isomorphic as a θ-manifold to $\mathcal{P}_{0,1}|_a = \{0\} \times \mathbb{R}^n \times S^{n-1}$.

In (v), the θ-structure on the surgered manifold is determined (up to homotopy) by the θ-structure ℓ on $K|_{(-6,2)}$.

The precise meaning of the word isomorphic in (iv) and (v) above is the following: By (ii) we know that the manifolds are *equal* outside $(-5,-2)\times B_2^{n+1}(0)\times D^n$. Being isomorphic means that the identity extends to a diffeomorphism which preserves θ-structures up to a homotopy of bundle maps which is constant outside $(-5,-2) \times B_2^{n+1}(0) \times D^n$.

Proof. (i) and (ii) follow easily from the properties of λ_t and ψ_t, and the fact that K agrees with $\mathbb{R}^{n+1} \times S^{n-1}$ outside $B_2^{d-l} \times \mathbb{R}^{l+1}$. For (iii), the Morse function $\mathcal{P}_{t,w} \to (-6,-2)$ has at most two critical point, both of index n. If a critical value is in (a,b), then the pair is $(n-1)$-connected, otherwise $\mathcal{P}_{t,w}|_{[a,b]}$ deformation retracts to $\mathcal{P}_{t,w}|_b$. The fact that the function $K_w \to \mathbb{R}$ has exactly two critical points with value $\pm w$ implies that $K_w|_a$ is diffeomorphic to $\{a\} \times \mathbb{R}^n \times S^{n-1}$ for regular values $a \in \mathbb{R} - [-w,w]$ and extendibility implies that they are also isomorphic as

θ-manifolds. In the limit $w = 0$ we see that $K_0|_a$ is isomorphic to $\{a\} \times \mathbb{R}^n \times S^{n-1}$ for all $a \neq 0$, and this implies (iv). (v) is similar. $\square$

5.3. **Surgery data.** We can now describe the semi-simplicial space of surgery data in the middle dimension. It is similar to the space of surgery data below the middle dimension, but taking into account the slightly different range of definition of the standard family in this case.

Before doing so, we choose once and for all, smoothly in the data $(a_i, \varepsilon_i, a_p, \varepsilon_p)$ increasing diffeomorphisms

$$\psi = \psi(a_i, \varepsilon_i, a_p, \varepsilon_p) : (-6, -2) \cong (a_i - \varepsilon_i, a_p + \varepsilon_p) \tag{5.1}$$

sending $[-4, -3]$ linearly onto $[a_i - \frac{1}{2}\varepsilon_i, a_i + \frac{1}{2}\varepsilon_i]$.

Definition 5.12. Let $(a, \varepsilon, (W, \ell)) \in D^{n-1,n-2}_{\theta,L}(\mathbb{R}^N)_p$, and write $M_i = W \cap x_1^{-1}(a_i)$. Define the set $Y_0(a, \varepsilon, (W, \ell))$ to consist of $(p+1)$-tuples of pairs (e_i, ℓ_i) defined for $i \in \{0, ..., p\}$. Each e_i is an embedding

$$e_i : \Lambda_i \times (a_i - \varepsilon_i, a_p + \varepsilon_p) \times \mathbb{R}^{n+1} \times D^n \hookrightarrow \mathbb{R} \times (0, 1) \times (-1, 1)^{N-1}$$

where $\Lambda_i \subset \Omega$ is a finite set, and the e_i's are required to satisfy

(i) The e_i have disjoint images.
(ii) $e_i^{-1}(W) = \Lambda_i \times (a_i - \varepsilon_i, a_p + \varepsilon_p) \times \{0\} \times \mathbb{R}^n \times \partial D^n$ for each i. We let

$$\partial e_i : \Lambda_i \times (a_i - \varepsilon_i, a_p + \varepsilon_p) \times \mathbb{R}^n \times \partial D^n \hookrightarrow W$$

denote the embedding restricted to the boundary.
(iii) For $t \in \cup_i (a_i - \varepsilon_i, a_i + \varepsilon_i)$, we have $(x_1 \circ e_i)^{-1}(t) = \Lambda_i \times \{t\} \times \mathbb{R}^{n+1} \times D^n$.

Each ℓ_i is an extendible θ-structure

$$\ell_i : T(\Lambda_i \times K) \longrightarrow \theta^*\gamma.$$

Under the chosen diffeomorphism

$$K|_{(-6,-2)} = (-6, -2) \times \mathbb{R}^n \times S^{n-1} \cong_\psi (a_i - \varepsilon_i, a_p + \varepsilon_p) \times \mathbb{R}^n \times S^{n-1}$$

we insist that ℓ_i and $\ell \circ D(\partial e_i)$ are equal. The data $(e_i, \ell_i|_{[-4,0]})$ is enough to perform θ-surgery on M_i (as $K|_{[-4,0]}$ is the trace of an $(n-1)$-surgery), and we further insist that

(iv) The resulting θ-manifold $\overline{M}_i$ lies in $\mathcal{A}$.

We let $Y_q(a, \varepsilon, (W, \ell))$ be the set of $(q+1)$-tuples of elements in $Y_0(a, \varepsilon, (W, \ell))$ which are all disjoint. Then $Y_\bullet(a, \varepsilon, (W, \ell))$ forms a semi-simplicial set.

Define a bi-semi-simplicial space $D^{n-1,\mathcal{A}}_{\theta,L}(\mathbb{R}^N)_{\bullet,\bullet}$ (augmented in the second semi-simplicial direction) from this, as in Definition 4.4. The main result about this bi-semi-simplicial space of manifolds equipped with surgery data is the following, whose proof we defer until Section 6.

Theorem 5.13. *The augmentation map*

$$D^{n-1,\mathcal{A}}_{\theta,L}(\mathbb{R}^N)_{\bullet,\bullet} \longrightarrow D^{n-1,n-2}_{\theta,L}(\mathbb{R}^N)_\bullet$$

induces a weak homotopy equivalence after geometric realisation, as long as the conditions of Theorem 5.2 are satisfied.

The inclusion

$$|D^{n-1,\mathcal{A}}_{\theta,L}(\mathbb{R}^N)_{\bullet,0}| \longrightarrow |D^{n-1,\mathcal{A}}_{\theta,L}(\mathbb{R}^N)_{\bullet,\bullet}|$$

is a weak homotopy equivalence under the same conditions.

5.4. **Proof of Theorem 5.2.** The proof of this theorem will be almost identical with that of Theorem 4.1, apart from some small differences which control the width of the standard family. Thus, suppose that the conditions in the statement of Theorem 5.2 are satisfied, and let $(a, \varepsilon, (W, \ell_W), \{(e_i, \ell_i)\}_{i=0}^p) \in D_{\theta,L}^{n-1,\mathcal{A}}(\mathbb{R}^N)_{p,0}$. We construct a 2-parameter family

$$\mathcal{K}_{e_i,\ell_i}^{t,w}(W, \ell_W) \in \Psi_\theta((a_0 - \varepsilon_0, a_p + \varepsilon_p) \times \mathbb{R}^N) \tag{5.2}$$

for $t, w \in [0, 1]$, as follows. Changing the first coordinate of the manifolds $\mathcal{P}_{t,w}(\ell_i)$ by composing with the reparametrisation functions of (5.1), we get manifolds

$$\overline{\mathcal{P}}_{t,w}(\ell_i) \in \Psi_\theta((a_i - \varepsilon_i, a_p + \varepsilon_p) \times \mathbb{R}^{n+1} \times \mathbb{R}^n),$$

having all the properties in Proposition 5.11 (where all subintervals of $(-6, -2)$ are replaced appropriately). Then for $t, w \in [0, 1]$ let

$$\mathcal{K}_{e_i,\ell_i}^{t,w}(W, \ell_W) \in \Psi_\theta((a_i - \varepsilon_i, a_p + \varepsilon_p) \times \mathbb{R}^N)$$

be equal to $W|_{(a_0-\varepsilon_0,a_p+\varepsilon_p)}$ outside of the image of e_i, and on $e_i(\Lambda_i \times (a_i - \varepsilon_i, a_p + \varepsilon_p) \times \mathbb{R}^{n+1} \times D^n$ be given by $e_i(\Lambda_i \times \overline{\mathcal{P}}_{t,w}(\ell_i))$. This gives a θ-manifold, because $\Lambda_i \times \overline{\mathcal{P}}_{t,w}(\ell_i)$ and $\Lambda_i \times \overline{\mathcal{P}}_{0,1}(\ell_i)$ agree as θ-manifolds outside of $(a_i - \frac{1}{2}\varepsilon_i, a_p + \varepsilon_p) \times B_2^{n+1}(0) \times D^n$.

As the embeddings e_i are all disjoint, this procedure can be iterated, and for tuples $t = (t_0, \ldots, t_p) \in [0, 1]^{p+1}$ and $w = (w_0, \ldots, w_p) \in [0, 1]^{p+1}$ we let

$$\mathcal{K}_{e,\ell}^{t,w}(W, \ell_W) = \mathcal{K}_{e_p,\ell_p}^{t_p,w_p} \circ \cdots \circ \mathcal{K}_{e_0,\ell_0}^{t_0,w_0}(W, \ell_W) \in \Psi_\theta((a_0 - \varepsilon_0, a_p + \varepsilon_p) \times \mathbb{R}^N).$$

To apply the same proof as that of Theorem 4.1, we need an analogue of Lemma 4.6 to tell us how the manifold improves when we apply the various surgery operations.

Lemma 5.14. *Firstly, the tuple $(a, \frac{1}{2}\varepsilon, \mathcal{K}_{e,\ell}^{t,w}(W, \ell_W))$ is an element of $X_p^{n-1,n-2}$. Secondly, if t_i and w_i are 1—so the surgery for the regular value a_i is fully done—and for each remaining j, either $t_j = 1$ or $w_j = 0$, then for each regular value $b \in (a_i - \frac{1}{2}\varepsilon_i, a_i + \frac{1}{2}\varepsilon_i)$ of $x_1 : \mathcal{K}_{e,\ell}^{t,w}(W, \ell_W) \to \mathbb{R}$, the θ-manifold $\mathcal{K}_{e,\ell}^{t,w}(W, \ell_W)|_b$ lies in $\mathcal{A}$.*

Proof. For the first part we must verify the conditions of Definition 2.18. This part of the argument of Lemma 4.6 applies equally well when $\kappa = n - 1$, $l = n - 1$.

For the second part, we suppose $t_i = w_i = 1$ and that for each remaining j either $t_j = 1$ or $w_j = 0$. Let $b \in (a_i - \frac{1}{2}\varepsilon_i, a_i + \frac{1}{2}\varepsilon_i)$ be a regular value of the height function on $\mathcal{K}_{e,\ell}^{t,w}(W, \ell_W)$ and define θ-manifolds

$$\begin{aligned} \overline{M} &= \big(\mathcal{K}_{e,\ell}^{t,w}(W, \ell_W)\big)|_b \\ \widetilde{M} &= \big(\mathcal{K}_{e_i,\ell_i}^{t_i,w_i}(W, \ell_W)\big)|_b \\ M &= W|_b. \end{aligned}$$

By property (iv) of the semi-simplicial space of surgery data, performing surgery on M using the data (e_i, ℓ_i) gives a θ-manifold in $\mathcal{A}$. By property (v) of the standard family, $\mathcal{K}_{e_i,\ell_i}^{t_i,w_i}(W, \ell_W)$ has this surgery done, so $\widetilde{M}$ lies in $\mathcal{A}$. Now $\overline{M}$ is obtained from $\widetilde{M}$ by applying the remaining $\mathcal{K}_{e_j,\ell_j}^{t_j,w_j}$ for $j \neq i$. If $t_j = 1$ then the effect of applying $\mathcal{K}_{e_j,\ell_j}^{1,w_j}$ is independent of w_j outside of $x_1^{-1}(a_j - \varepsilon_j, a_j + \varepsilon_j)$. Thus we may suppose that $w_j = 0$ in either case. But then by property (iv) of the standard family, the level sets of regular values are isomorphic as θ-manifolds to what they were before, so still lie in $\mathcal{A}$. $\square$

We define a map

$$\begin{array}{rcl}\mathscr{S}_p : [0,1]^{p+1} \times [0,1]^{p+1} \times D^{n-1,\mathcal{A}}_{\theta,L}(\mathbb{R}^N)_{p,0} & \longrightarrow & X^{n-1,n-2}_p \\ \big(t, w, (a, \varepsilon, (W, \ell_W), \{(e_i, \ell_i)\}_{i=0}^p)\big) & \longmapsto & (a, \tfrac{1}{2}\varepsilon, \mathcal{K}^{t,w}_{e,\ell}(W, \ell_W))\end{array}$$

which has the desired range by the first part of Lemma 5.14. Let $P : [0,1] \to [0,1]^2$ be the piecewise linear path with $P(0) = (0,0)$, $P(\frac{1}{2}) = (1,0)$ and $P(1) = (1,1)$, and define

$$\begin{array}{rcl}H : [0,1] \times \Delta^p & \longrightarrow & ([0,1] \times [0,1])^{p+1} \times \Delta^p \cong_{\text{shuffle}} [0,1]^{p+1} \times [0,1]^{p+1} \times \Delta^p \\ (s,t) & \longmapsto & \left(P(s \cdot \lambda_0(t)), ..., P(s \cdot \lambda_p(t)), \dfrac{\psi(t)}{\sum_j \psi_j(t)}\right)\end{array}$$

where the functions $\lambda, \psi : \Delta^p \to [0,1]^{p+1}$ are as in the proof of Theorem 4.1. Using these maps we form the composition

$$\begin{array}{rcl}F_p : [0,1] \times D^{n-1,\mathcal{A}}_{\theta,L}(\mathbb{R}^N)_{p,0} \times \Delta^p & \xrightarrow{H} & [0,1]^{p+1} \times [0,1]^{p+1} \times D^{n-1,\mathcal{A}}_{\theta,L}(\mathbb{R}^N)_{p,0} \times \Delta^p \\ & \xrightarrow{\mathscr{S}_p} & X^{n-1,n-2}_p \times \Delta^p.\end{array}$$

Lemma 5.15. *The maps F_p glue to a homotopy $\mathscr{S} : [0,1] \times |D^{n-1,\mathcal{A}}_{\theta,L}(\mathbb{R}^N)_{\bullet,0}| \to |X^{n-1,n-2}_\bullet|$.*

Proof. We must see that the points $F_p(s, x, d^i t)$ and $F_{p-1}(s, d_i(x), t)$ are identified under the face maps among the $X^{\kappa,l-1}_p \times \Delta^p$.

Note that $\psi_i(d^i t) = 0$, and so $F_p(s, x, d^i t)$ lies on the ith boundary of a simplex, so is identified with the point in $X^{n-1,n-2}_{p-1} \times \Delta^{p-1}$ obtained by applying the ith face map. Furthermore, as the ith entry of $d^i t$ is zero, so is $\lambda_i(t)$ and so the ith surgery is not performed at all: hence after applying the ith face map, we obtain $F_{p-1}(s, d_i(x), t)$. □

This homotopy is constant on the subspace $|D^{n-1,\mathcal{A}}_{\theta,L}(\mathbb{R}^N)_\bullet| \hookrightarrow |D^{n-1,\mathcal{A}}_{\theta,L}(\mathbb{R}^N)_{\bullet,0}|$ of manifolds equipped with no surgery data, and so all that is left to show is that the homotopy ends up in the subspace $|X^{n-1,\mathcal{A}}_\bullet|$. To do this it is enough to show that $F_p(1, -)$ has image in $X^{n-1,\mathcal{A}}_p \times \Delta^p$, so consider the element

$$F_p\big(1, (a, \varepsilon, (W, \ell_W), \{e_i, \ell_i\}), t\big) = \left(a, \tfrac{1}{2}\varepsilon, \mathcal{K}^{P^{p+1}(\lambda(t))}_{e,\ell}(W, \ell_W), \frac{\psi(t)}{\sum_j \psi_j(t)}\right),$$

where we implicitly shuffle the coordinates of $P^{p+1}(\lambda(t)) \in ([0,1]^2)^{p+1}$. We need to check that if the ith simplicial coordinate is non-zero, then for every regular value in $(a_i - \frac{1}{2}\varepsilon_i, a_i + \frac{1}{2}\varepsilon_i)$ of the height function on the surgered manifold $\mathcal{K}^{P^{p+1}(\lambda(t))}_{e,\ell}(W, \ell_W)$, the level set lies in $\mathcal{A}$.

Let $b \in (a_i - \frac{1}{2}\varepsilon_i, a_i + \frac{1}{2}\varepsilon_i)$ be such a regular value, then $\mathcal{K}^{P^{p+1}(\lambda(t))}_{e,\ell}(W, \ell_W)|_b$ is obtained from $\mathcal{K}^{P(\lambda_i(t))}_{e_i,\ell_i}(W, \ell_W)|_b$ by doing further surgery. We have assumed that the ith simplicial coordinate is non-zero, so $\psi_i(t) > 0$, but then $\lambda_i(t) = 1$ and so this manifold is $\mathcal{K}^{1,1}_{e_i,\ell_i}(W, \ell_W)|_b$. By property (iv) of the semi-simplicial space of surgery data, and property (v) of the standard family, this manifold lies in $\mathcal{A}$. It remains to see that the further surgeries required to obtain $\mathcal{K}^{P^{p+1}(\lambda(t))}_{e,\ell}(W, \ell)|_b$ from $\mathcal{K}^{P(\lambda_i(t))}_{e_i,\ell_i}(W, \ell_W)|_b$ do not change this property. However, for each path $P(\lambda_j(t))$, either the width is zero or the surgery is fully done, so by the second part of Lemma 5.14 the manifold $\mathcal{K}^{P^{p+1}(\lambda(t))}_{e,\ell}(W, \ell_W)|_b$ lies in $\mathcal{A}$.

This establishes that $|X_\bullet^{n-1,\mathcal{A}}| \to | X_\bullet^{n-1,n-2}|$ is a weak homotopy equivalence, and Theorem 5.2 follows from this, the equivalence $B\mathcal{C}_{\theta,L}^{n-1,n-2}(\mathbb{R}^N) \simeq |X_\bullet^{n-1,n-2}|$, and its analogue for $\mathcal{A}$.

6. Contractibility of spaces of surgery data

In order to finish the proofs of the results of the last three sections, we must supply proofs of Theorems 3.4, 4.5 and 5.13 concerning the bi-semi-simplicial spaces of manifolds equipped with surgery data. For convenience we assume that the domain B of the map θ defining the tangential structure is path connected. In the category $\mathcal{C}_\theta^{\kappa,l}$, this implies that objects are path connected as long as $l > -1$, and morphisms are path connected if in addition $\kappa > -1$.

6.1. **The second part of Theorems 4.5 and 5.13.** Theorems 4.5 and 5.13 consist of two parts, the second of which asserts that the inclusions

$$|D_{\theta,L}^{\kappa,l}(\mathbb{R}^N)_{\bullet,0}| \longrightarrow |D_{\theta,L}^{\kappa,l}(\mathbb{R}^N)_{\bullet,\bullet}| \qquad |D_{\theta,L}^{n-1,\mathcal{A}}(\mathbb{R}^N)_{\bullet,0}| \longrightarrow |D_{\theta,L}^{n-1,\mathcal{A}}(\mathbb{R}^N)_{\bullet,\bullet}|$$

are weak homotopy equivalences. This is easier than the first part of these theorems, but requires slightly different methods.

Proof (assuming the first part). The proof in both cases is the same, so let us write $D_{\bullet,\bullet}$ for either $D_{\theta,L}^{\kappa,l}(\mathbb{R}^N)_{\bullet,\bullet}$ or $D_{\theta,L}^{n-1,\mathcal{A}}(\mathbb{R}^N)_{\bullet,\bullet}$. We define, for this proof only, a bi-semi-simplicial space $D'_{\bullet,\bullet}$ in the same way as $D_{\bullet,\bullet}$ except that the usual inequalities $a_i < a_{i+1}$ and $\varepsilon_i > 0$ are replaced by $a_i \leq a_{i+1}$ and $\varepsilon_i \geq 0$, and the intervals $[a_i - \varepsilon_i, a_i + \varepsilon_i]$ are allowed to overlap.

The inclusion $D_{\bullet,\bullet} \hookrightarrow D'_{\bullet,\bullet}$ is easily seen to be a levelwise weak homotopy equivalence, by spreading the a_i out and making the ε_i positive but small, so it is enough to work with $D'_{\bullet,\bullet}$ throughout and show that $|D'_{\bullet,0}| \to | D'_{\bullet,\bullet}|$ is a weak homotopy equivalence.

To do so, we describe a retraction $r : |D'_{\bullet,\bullet}| \to |D'_{\bullet,0}|$ which will be a weak homotopy inverse to the inclusion. The map r does not change the underlying manifold $W \in \psi_\theta(N+1,1)$, but only modifies the a_i and barycentric coordinates. There is a map

$$D'_{p,q} \longrightarrow D'_{(p+1)(q+1)-1,0}$$

given by considering $(p+1)$ regular values, each equipped with $(q+1)$ pieces of surgery data, as $(p+1)(q+1)$ not-necessarily distinct regular values, each with a *single* piece of surgery data. There is also a map $\Delta^p \times \Delta^q \to \Delta^{(p+1)(q+1)-1} \subset \mathbb{R}^{(p+1)(q+1)}$ with $(j+(q+1)i)$th coordinate given by $(t,s) \mapsto s_i t_j$. Taking the product of these maps gives

$$r_{p,q} : D'_{p,q} \times \Delta^p \times \Delta^q \longrightarrow D'_{(p+1)(q+1)-1,0} \times \Delta^{(p+1)(q+1)-1}$$

which glue together to give the map $r : |D'_{\bullet,\bullet}| \to |D'_{\bullet,0}|$. It is clear that r is a retraction (i.e. left inverse to the inclusion), so the induced map on homotopy groups is surjective. To see that it is injective, we use the map $|D'_{\bullet,\bullet}| \to | D'_\bullet|$ induced by the augmentation in the second bi-semi-simplicial direction (induced by forgetting all surgery data). This is a weak equivalence by the first part of Theorem 4.5 or 5.13 respectively, but it clearly factors as

$$|D'_{\bullet,\bullet}| \xrightarrow{r} |D'_{\bullet,0}| \longrightarrow |D'_\bullet|,$$

where the second map is again induced by the augmentation in the second bi-semi-simplicial direction. Therefore r is also injective on homotopy groups, and hence a weak homotopy equivalence. $\square$

6.2. **A simplicial technique.** In order to give the proofs of Theorems 3.4, 4.5 and 5.13, we need a technique for showing that for certain augmented semi-simplicial spaces $X_\bullet \to X_{-1}$, the map $|X_\bullet| \to X_{-1}$ is a weak homotopy equivalence. The semi-simplicial spaces occurring in those theorems are all of the following special type.

Definition 6.1. Let $X_\bullet \to X_{-1}$ be an augmented semi-simplicial space. We say it is an *augmented topological flag complex* if

(i) The map $X_n \to X_0 \times_{X_{-1}} X_0 \times_{X_{-1}} \cdots \times_{X_{-1}} X_0$ to the $(n+1)$-fold product—which takes an n-simplex to its $(n+1)$ vertices—is a homeomorphism onto its image, which is an open subset.

(ii) A collection $(v_0, ..., v_n) \in X_0 \times_{X_{-1}} X_0 \times_{X_{-1}} \cdots \times_{X_{-1}} X_0$ lies in X_n if and only if each pair (v_i, v_j) lies in X_1.

If elements v, $w \in X_0$ lie in the same fibre over X_{-1} and $(v, w) \in X_1$, we say they are *orthogonal*. If $X_{-1} = *$ we omit the adjective augmented.

The semi-simplicial space $Z_\bullet(a, \varepsilon, W) \to *$ from Definition 3.2 and the semi-simplicial spaces $Y_\bullet(a_\bullet, \varepsilon_\bullet, (W, \ell_W)) \to *$ from Definitions 4.3 and 5.12 are topological flag complexes. Furthermore, $D^{\kappa}_{\theta,L}(\mathbb{R}^N)_{p,\bullet} \to D^{\kappa-1}_{\theta,L}(\mathbb{R}^N)_p$ from Definition 3.3, $D^{\kappa,l}_{\theta,L}(\mathbb{R}^N)_{p,\bullet} \to D^{\kappa,l-1}_{\theta,L}(\mathbb{R}^N)_p$ from Definition 4.4 and $D^{n-1,\mathcal{A}}_{\theta,L}(\mathbb{R}^N)_{p,\bullet} \to D^{n-1,n-2}_{\theta,L}(\mathbb{R}^N)_p$ from Section 5.3 are all augmented topological flag complexes. In all cases this is immediate from the definition: firstly, a p-simplex of these semi-simplicial spaces consists of $(p+1)$-tuples of surgery data, which are each 0-simplices; secondly, the pieces of surgery data are subject to the requirement that they are all disjoint, but disjointness is a property that can be verified pairwise.

Theorem 6.2. *Let $X_\bullet \to X_{-1}$ be an augmented topological flag complex. Suppose that*

(i) The map $\varepsilon : X_0 \to X_{-1}$ has local sections (in the strong sense that given any $x \in X_0$, there is a neighbourhood $U \subset X_{-1}$ of $\varepsilon(x)$ and a section $s : U \to X_0$ with $s(\varepsilon(x)) = x$).

(ii) $\varepsilon : X_0 \to X_{-1}$ is surjective.

(iii) For any $p \in X_{-1}$ and any (non-empty) finite set $\{v_1, ..., v_n\} \subset \varepsilon^{-1}(p)$ there exists a $v \in \varepsilon^{-1}(p)$ which is orthogonal to all v_i.

Then $|X_\bullet| \to X_{-1}$ is a weak homotopy equivalence.

Condition (ii) can be viewed as the special case $n = 0$ of condition (iii), but we prefer to keep the cases $n = 0$ and $n > 0$ separate.

To give this theorem some motivation, suppose that $X_{-1} = *$ and that each X_i is discrete, so $|X_\bullet|$ has the structure of a Δ-complex. Then any map $f : S^n \to |X_\bullet|$ may be homotoped to be simplicial, for some triangulation of S^n, and so hits finitely many vertices $v_0, ..., v_k$. By (iii) there exists a $v \in X_0$ such that (v, v_i) is a 1-simplex for all i. But then the map f extends to the join

$$\{v\} * f : \{v\} * S^n \longrightarrow |X_\bullet|$$

and so f is null-homotopic. The proof we give below follows this in spirit, although is necessarily more complicated when the X_i carry a topology. To deal with the topology, we require the following technical lemma.

Lemma 6.3. *Let Ω be some uncountable set (regarded as a discrete topological space). Let X be a topological space and let*

$$P \subset \mathbb{N} \times \Omega \times X$$

be a subset such that if we write $P \cap (\{n\} \times \Omega \times X) = \{n\} \times P_n$, then each $P_n \subset \Omega \times X$ is open in the product topology and surjects to X. We give $\mathbb{N} \times \Omega$ the partial order *in which* $(n,\alpha) < (m,\beta)$ *if and only if* $n < m$*, give* $\mathbb{N}\times\Omega\times X$ *the product order (where* X *has no non-trivial order relations), and give* P *the induced ordering. In other words,*

$$(n,\alpha,x) < (m,\beta,y) \textit{ iff } n < m \textit{ and } x = y.$$

Then the natural map $\pi : |N_\bullet P| \to X$ *is a Serre fibration with weakly contractible fibres.*

Proof. We will prove that π is a Serre microfibration, i.e. for any homotopy $D^k \times [0,1] \to X$ with prescribed lift over $D^k \times \{0\}$, there exists $\varepsilon > 0$ and a lift over $D^k \times [0,\varepsilon]$ extending the prescribed lift. It is clear that π has weakly contractible fibers (every finite set is bounded above), so [Wei05, Lemma 2.2] implies that π is a Serre fibration.

First let us describe $|N_\bullet P|$ a little. Let us write $P \cap (\{n\}\times\{\alpha\}\times X) = \{n\}\times\{\alpha\}\times U_{n,\alpha}$. A point in $|N_\bullet P|$ may then be described as follows

(i) a p-tuple $n = (0 \leq n_0 < \cdots < n_p)$ of integers and $\alpha = (\alpha_0,\dots,\alpha_p) \in \Omega^{p+1}$
(ii) a point $x \in U_{n,\alpha} = \cap_i U_{n_i,\alpha_i} \subset X$,
(iii) $t = (t_0,\dots,t_p) \in \Delta^p$,

with the relation that if $t_i = 0$ we may forget n_i and α_i. Each point is uniquely representable by a tuple (n,α,x,t) where no t_i is zero and we shall do so in what follows. Without the requirement that $x \in U_{n,\alpha}$, we would get a description of the space $|N_\bullet(\mathbb{N}\times\Omega\times X)|$ which contains $|N_\bullet P|$ as a subset, but the topology is finer than the subspace topology. For a sequence (n^k,α^k,x^k,t^k) to converge to $(\overline{n},\overline{\alpha},\overline{x},\overline{t})$, it is necessary that it only involve finitely many elements of $\mathbb{N}\times\Omega$. (The necessity of this can be seen by projecting to the simplicial complex defined by the above data without the x: a convergent sequence in a simplicial complex takes place in a finite subcomplex.) Thus some pair (n,α) occurs infinitely often, so there is a (reindexed) subsequence of the form (n,α,x^k,t^k). We deduce that $\bar{x}$ lies in the closure of $U_{n,\alpha}$, but we claim it actually lies in $U_{n,\alpha}$. Otherwise, the collection of points in the sequence is easily seen to be a closed subset of $|N_\bullet P|$: The inverse image in $\{\alpha\}\times\{n\}\times U_{n,\alpha}\times\Delta^n$ is closed because the sequence of x^k's has no accumulation point in $U_{n,\alpha}$ and the preimage in other simplices is treated similarly.

Let us return to the proof that π is a Serre microfibration, for which we must show that we may ε-lift homotopies parametrised by discs. A prerequisite for this is that we may ε-lift paths, and we first show that we may do this in a canonical way. Let

$$\begin{tikzcd} \{0\} \arrow[r, "\hat{f}|_0"] \arrow[d, hook] & {|N_\bullet P|} \arrow[d, "\pi"] \\ {[0,1]} \arrow[r, "f"] & X \end{tikzcd}$$

be a lifting problem, so $\hat{f}|_0(0) \in |N_\bullet P|$ which may be described by the data (n,α,x,t) as above, with $x_0 \in U_{n,\alpha}$. By continuity of f, there is an $\varepsilon > 0$ such that $f|_{[0,\varepsilon]}$ has image in the open set $U_{n,\alpha}$ and so restricted to this interval the map f lifts *canonically* to

$$\hat{f}|_{[0,\varepsilon]} : s \longmapsto (n,\alpha,f(s),t).$$

Thus we have a sort of "ε parallel transport" on π. Given a general lifting problem

$$\begin{tikzcd} D^k \times \{0\} \arrow[r, "\hat{f}|_0"] \arrow[d, hook] & {|N_\bullet P|} \arrow[d, "\pi"] \\ D^k \times [0,1] \arrow[r, "f"] & X \end{tikzcd}$$

we know that for each $a \in D^k$ we can canonically lift $f|_{\{a\}\times[0,\varepsilon_a]}$ for some $\varepsilon_a > 0$. We must show that there is some uniform choice of ε that works over the whole of D^k. If not, we can find a sequence of points $x^k \in D^k$ whose canonical lift is not defined on $[0, 1/k]$, i.e. that there exists an $s_k < 1/k$ such that $\hat{f}|_0(x^k) = (n^k, \alpha^k, x^k, t^k)$ but $f(x^k, s_k) \notin U_{n^k,\alpha^k}$. By compactness of D^k we can assume convergence $x^k \to \bar{x}$ and the above discussion of sequential convergence in $|N_\bullet P|$ shows that after passing to a subsequence we may further assume that (n^k, α^k) is constantly (n, α), and that $f(\bar{x}, 0) \in U_{n,\alpha}$. Therefore we may pick $\varepsilon > 0$ so that $f(\{\bar{x}\} \times [0, \varepsilon]) \subset U_{n,\alpha}$, but then continuity and compactness implies that $f(\{x^k\} \times [0, \varepsilon]) \subset U_{n,\alpha}$ for sufficiently large k, contradicting that $f(x^k, s_k) \notin U_{n^k,s^k}$ when $k > 1/\varepsilon$. □

Proof of Theorem 6.2. We begin with an element of the relative homotopy group of the pair of spaces $(X_{-1}, |X_\bullet|)$,

$$\begin{tikzcd} \partial D^k \arrow[r, "\hat{f}"] \arrow[d] & {|X_\bullet|} \arrow[d] \\ D^k \arrow[r, "f"] & X_{-1} \end{tikzcd}$$

and we will show it is trivial, i.e. after changing $(f, \hat{f})$ by a homotopy, there is a diagonal map $D^k \to |X_\bullet|$ making the diagram commutative.

Using the maps $f : D^k \to X_{-1}$ and $\varepsilon : X_0 \to X_{-1}$, we can form the fiber product $D^k \times_{X_{-1}} X_0$. The diagonal map in the diagram will be constructed using certain auxilliary spaces. First choose an uncountable set Ω. We will construct compact subsets $B_n \subset D^k \times_{X_{-1}} X_0$, $n \geq -1$ and open sets $P_n \subset \Omega \times D^k$, $n \geq 0$ surjecting to D^k, together with a map $g_n : P_n \to X_0$ with the properties

(i) $\varepsilon \circ g_n = \pi_n$, where $\pi_n : P_n \to D^k$ denotes the projection,
(ii) $B_n \cap \{x\} \times X_0$ is finite for all n,
(iii) for $-1 \leq n < m$, any $p \in P_m$ is orthogonal to any $y \in X_0$ for which $(\pi_m(p), y) \in B_n \subset D^k \times X_0$.

For the construction of B_n and P_n, we replace X_p by the fiber product $D^k \times_{X_{-1}} X_p$, reducing to the case where $X_{-1} = D^k$ and f is the identity map. Thus we are looking for subsets $B_n \subset X_0$ and maps $g_n : P_n \to X_0$ such that $\varepsilon \circ g_n = \pi_n : P_n \to D^k$.

We first construct B_{-1}. To this end we shall, for the rest of this proof, replace the usual coordinates $(t_0, \ldots, t_p) \in \Delta^k$ (which are non-negative numbers with $\sum t_i = 1$) by the coordinates $s_i = t_i / \max t_i$ (which are non-negative numbers with $\max s_i = 1$). For each $x \in \partial D^k$, we have $\hat{f}(x) = (s, y)$, where $s = (s_0, \ldots, s_p) \in \Delta^p$ and $y = (y_0, \ldots, y_p) \in X_p \subset X_0^{p+1}$. We identify this point (s, y) with the function $w(x) : X_0 \to [0, 1]$ which takes $y_i \mapsto s_i$ and is otherwise 0, and define

$$B_{-1} = \{x \in \varepsilon^{-1}(\partial D^k) | w(\varepsilon(x))(x) \geq 1/2\}.$$

We now change $\hat{f} : \partial D^k \to |X_\bullet|$ by deforming all barycentric coordinates: Change $s \in \Delta^p$ by replacing s_i by $\max(0, 2s_i - 1)$ (in fact this gives a self-map of $|X_\bullet|$ which commutes with the augmentation and is homotopic to the identity through such maps). After this change of $\hat{f}$, we have achieved that only elements of B_{-1} ever appear as vertices of any $\hat{f}(x) \in |X_\bullet|$.

To define P_n for $n \geq 0$, we first pick for each $x \in D^k$ an element $g_x(x) \in \varepsilon^{-1}(x)$ which is orthogonal to all elements in the set $(\cup_{i<n} B_i) \cap \varepsilon^{-1}(x)$. Since that set is finite, this is possible by assumption. Then, since ε has local sections, we can extend to a map $g_x : V_x \to X_0$ which is a section of $\varepsilon : X_0 \to D^k$, defined on a compact neighbourhood $V_x \ni x$. Since orthogonality is an open relation and $B_i \subset X_0$ is compact, we can assume that $g_x(y)$ is orthogonal to $(\cup_{i<n} B_i) \cap \varepsilon^{-1}(y)$ for all $y \in V_x$. The disc is then covered by the open sets $U_x = \mathrm{int}(V_x)$, and we can pick a finite subcover $D^k = \cup U_i$, and the corresponding sections $V_i \to X_0$ assemble to a map

$$\coprod V_i \xrightarrow{\overline{g}_n} X_.$$

We then define $B_n = \overline{g}_n(\amalg V_i)$, and let g_n be the restriction of $\overline{g}_n$ to $P_n = \amalg U_i$. Chosing distinct elements of Ω for each i we can regard P_n as an open subset of $\Omega \times D^k$, such that π_n is the projection to D^k.

The outcome of this procedure is a sequence of open subspaces $P_n \subset \Omega \times D^k$, surjecting to D^k, and maps $g_n : P_n \to X_0$, such that if $n < n'$ and $x \in P_n$ and $x' \in P_{n'}$ lie over the same point in D^k, then $g_n(x)$ and $g_{n'}(x')$ are orthogonal. If we let $P = \cup\{n\} \times P_n \subset \mathbb{N} \times \Omega \times X$ and give it a partial order as in Lemma 6.3, then this says that the g_n glue to a map $g : N_\bullet P \to X_\bullet$. The lemma says that the augmentation map $|N_\bullet P| \to D^k$ is a Serre fibration with contractible fibers, and hence it admits a section $s : D^k \to |N_\bullet P|$, and we let the composition $g \circ s : D^k \to |X_\bullet|$ be the diagonal map in the diagram

$$\begin{tikzcd}
\partial D^k \arrow[r, "\hat{f}"] \arrow[d] & {|X_\bullet|} \arrow[d] \\
D^k \arrow[r, "f"] \arrow[ur] & X_{-1}
\end{tikzcd}$$

Then the lower triangle commutes strictly, and we claim the upper triangle is homotopy commutative. To see this, we note that for $x \in \partial D^k$, all vertices of any $g \circ s(x) \in |X_\bullet|$ are in $B_0 \cup B_1 \cup \cdots \subset X_0$, and all vertices of $\hat{f}(x) \in |X_\bullet|$ are in $B_{-1} \subset X_0$. Since $g \circ s(x)$ and $\hat{f}(x)$ both lie in $\varepsilon^{-1}(x)$, their vertices must be orthogonal, so there is a straight line in $\varepsilon^{-1}(x) \subset |X_\bullet|$ between the two points $g \circ s(x)$ and $\hat{f}(x)$ inside the join of the simplices that contain them. These straight lines then gives the homotopy between $g \circ s(x)$ and $\hat{f}(x)$. □

6.3. **Proof of Theorem 3.4.** Recall that this theorem states that the augmentation

$$D^\kappa_{\theta,L}(\mathbb{R}^N)_{\bullet,\bullet} \longrightarrow D^{\kappa-1}_{\theta,L}(\mathbb{R}^N)_\bullet$$

induces a weak homotopy equivalence after geometric realisation, as long as certain conditions are satisfied. We remind the reader that these conditions are

(i) $2\kappa \leq d-1$,
(ii) $\kappa + 1 + d < N$,
(iii) L admits a handle decomposition only using handles of index $< d - \kappa - 1$.

We will use Theorem 6.2 to prove that in fact for each p the augmentation map induces a weak equivalence

$$|D^\kappa_{\theta,L}(\mathbb{R}^N)_{p,\bullet}| \longrightarrow |D^{\kappa-1}_{\theta,L}(\mathbb{R}^N)_p|.$$

Theorem 6.2 does not apply directly to the augmentation $D^\kappa_{\theta,L}(\mathbb{R}^N)_{p,\bullet} \to D^{\kappa-1}_{\theta,L}(\mathbb{R}^N)_p$, but we will show that it does after replacing with weakly equivalent spaces.

Recall that an element of $D^{\kappa}_{\theta,L}(\mathbb{R}^N)_{p,q}$ consists of an element $(a,\varepsilon,W) \in D^{\kappa-1}_{\theta,L}(\mathbb{R}^N)_p$, together with an element $e \in Z_q(a,\varepsilon,W)$, given by an embedding $e : \Lambda \times \overline{V} \to \mathbb{R} \times (-1,1)^N$, where $\Lambda \subset \Omega$ is a finite set equipped with a map $\delta : \Lambda \to [p]^\vee \times [q] = \{0,\dots,p+1\} \times \{0,\dots,q\}$.

Definition 6.4. The *core* of $\overline{V}$ is the submanifold $C = [-2,0] \times D^\kappa \times \{0\} \subset \overline{V} = [-2,0] \times \mathbb{R}^\kappa \times \mathbb{R}^{d-\kappa}$. Let $\widetilde{Z}_\bullet(a,\varepsilon,W)$ be the semi-simplicial space defined as in Definition 3.2 except that instead of demanding that $e : \Lambda \times \overline{V} \to \mathbb{R} \times (-1,1)^N$ be an embedding, we demand only it be a smooth map which restricts to an embedding of a neighbourhood of $\Lambda \times C$. We still require that e satisfy the numbered conditions listed in Definition 3.2. Let $\widetilde{D}^{\kappa}_{\theta,L}(\mathbb{R}^N)_{\bullet,\bullet} \to D^{\kappa-1}_{\theta,L}(\mathbb{R}^N)_\bullet$ be the augmented bi-semi-simplicial space defined as in Definition 3.3, but using $\widetilde{Z}_\bullet(a,\varepsilon,W)$ instead of $Z_\bullet(a,\varepsilon,W)$.

Proposition 6.5. *The inclusion $D^{\kappa}_{\theta,L}(\mathbb{R}^N)_{\bullet,\bullet} \hookrightarrow \widetilde{D}^{\kappa}_{\theta,L}(\mathbb{R}^N)_{\bullet,\bullet}$ induces a weak homotopy equivalence in each bidegree, and so on geometric realisation.*

Proof. It is easy to see that there is an isotopy of embeddings $j_t : \bar{V} \to \bar{V}$, $t \in [1,\infty)$, such that $j_1 = \mathrm{id}$, $j_t|_C = \mathrm{id}$ for all t and $j_t(\bar{V})$ is contained in the $(1/t)$-neighbourhood of C for large t, and also such that every j_t preserves the submanifold $\mathrm{int}(\partial_- D^{\kappa+1}) \times \mathbb{R}^{d-\kappa}$ and preserves the height function $h : \bar{V} \to [-2,0]$.

Precomposing the embeddings $e_{i,j} : \Lambda_i \times \bar{V} \to \mathbb{R} \times (0,1) \times (-1,1)^{N-1}$ with the maps $j_t \times \mathrm{id}$ induces a deformation $[1,\infty) \times \widetilde{Z}_q(a,\varepsilon,W) \to \widetilde{Z}_q(a,\varepsilon,W)$ and in turn $[1,\infty) \times \widetilde{D}^{\kappa}_{\theta,L}(\mathbb{R}^N)_{p,q} \to \widetilde{D}^{\kappa}_{\theta,L}(\mathbb{R}^N)_{p,q}$. Elements of $\widetilde{Z}_q(a,\varepsilon,W)$ have disjoint cores, so in a compact family $K \to \widetilde{D}^{\kappa}_{\theta,L}(\mathbb{R}^N)_{p,q}$, there exists an $\varepsilon > 0$ such that the ε-neighbourhoods of all cores are also disjoint. Composing with the deformation of $\widetilde{D}^{\kappa}_{\theta,L}(\mathbb{R}^N)_{p,q}$, the map from K will eventually deform into $D^{\kappa}_{\theta,L}(\mathbb{R}^N)_{p,q}$. It follows easily from this that the relative homotopy groups vanish. □

In order to prove Theorem 3.4, we will show that for each p the map

$$\widetilde{D}^{\kappa}_{\theta,L}(\mathbb{R}^N)_{p,\bullet} \longrightarrow D^{\kappa-1}_{\theta,L}(\mathbb{R}^N)_p$$

is a weak homotopy equivalence after geometric realisation, by applying Theorem 6.2. Hence we must verify the conditions of that theorem. First we establish condition (i).

Proposition 6.6. *The map $\widetilde{D}^{\kappa}_{\theta,L}(\mathbb{R}^N)_{p,0} \to D^{\kappa-1}_{\theta,L}(\mathbb{R}^N)_p$ has local sections.*

Proof. Let's consider a point $x \in \widetilde{D}^{\kappa}_{\theta,L}(\mathbb{R}^N)_{p,0}$, given by elements $(a,\varepsilon,W) \in D^{\kappa-1}_{\theta,L}(\mathbb{R}^N)_p$ and $\{e_{i,j}\} \in \widetilde{Z}_0(a,\varepsilon,W)$. Choose $t_0 < a_0 - \varepsilon_0$ and $t_1 > a_p + \varepsilon_p$ which are regular values for $x_1 : W \to \mathbb{R}$, and such that $x_1 \circ e_{i,j}(\Lambda_{i,j} \times \bar{V}) \subset (t_0,t_1)$. There is a (connected) open neighbourhood $U \subset D^{\kappa-1}_{\theta,L}(\mathbb{R}^N)_p$ of (a,ε,W) on which the t_i remain regular values.

The inclusion $U \hookrightarrow D^{\kappa-1}_{\theta,L}(\mathbb{R}^N)_p$ has graph $\Gamma \subset U \times \mathbb{R}^N$. All fibres of the projection $\pi : \Gamma|_{[t_0,t_1]} \to U$ are diffeomorphic to the same manifold $M = W|_{[t_0,t_1]}$. Sending a point in U to its fibre defines a function

$$\begin{aligned} F : U &\longrightarrow \mathrm{Emb}_\partial(M, [t_0,t_1] \times (-1,1)^N)/\mathrm{Diff}(M) \\ u &\longmapsto \pi^{-1}(u) \end{aligned}$$

where Emb_∂ denotes embeddings which send the boundary to the boundary, and the definition of the topology on $\Psi_\theta(\mathbb{R} \times \mathbb{R}^N)$ makes this continuous (manifolds near to a point $W \in \psi_\theta(N+1,1) \subset \Psi_\theta(\mathbb{R} \times \mathbb{R}^N)$ look like a section of the normal bundle of W inside a compact set, e.g. inside $[t_0,t_1] \times [-1,1]^N$).

We now require two results on spaces of embeddings. Firstly, the map

$$\mathrm{Emb}_\partial(M, [t_0, t_1] \times (-1,1)^N) \longrightarrow \mathrm{Emb}_\partial(M, [t_0, t_1] \times (-1,1)^N)/\mathrm{Diff}(M)$$

is well-known to be a principal $\mathrm{Diff}(M)$-bundle, and has local sections (see e.g. [BF81]). Thus, after perhaps passing to a smaller open neighbourhood, which we will still call U, F has a lift $\widetilde{F} : U \to \mathrm{Emb}_\partial(M, [t_0, t_1] \times (-1,1)^N)$, and we call $f = \widetilde{F}(a, \varepsilon, W)$.

Secondly, we need the following generalisation of a technical theorem of Cerf [Cer61, 2.2.1 Théorème 5] (the "first isotopy and extension theorem"), an especially elementary proof of which was given by Lima [Lim63]. We follow Lima's proof.

Lemma 6.7. *Let $C \subset [t_0, t_1]$ be a closed subset and let $S \subset \mathrm{Emb}_\partial(M, [t_0, t_1] \times (-1,1)^N)$ be the open subset of those embeddings e for which $\pi_1 \circ e : M \to [t_0, t_1]$ has no critical values inside C.*

Given an $f \in S$, there is a neighbourhood U of f in S and a continuous map $\varphi : U \to \mathrm{Diff}([t_0, t_1] \times (-1,1)^N)$ such that $\varphi(g) \circ f$ and g have the same image, and $\varphi(g)$ is height-preserving over C.

Proof. Consider M to be a submanifold of $[t_0, t_1] \times (-1,1)^N$ via f. We choose a tubular neighbourhood $\pi : T \to M$ of radius ε which over the boundary and $x_1^{-1}(C)$ has fibres contained in level sets of x_1 (this is possible as C is closed and consists of regular values). If $g \in S$ is sufficiently close to f, it will have image in T and we may define an element $\bar{\varphi}(g) \in C^\infty(M, M)$ by

$$\bar{\varphi}(g)(x) = \pi(g(x)).$$

This is a diffeomorphism for $g = f$, and so there is a neighbourhood U' of f in S where this remains true. We get a function $\bar{\varphi} : U' \to \mathrm{Diff}(M)$ and for each $g \in U'$ we define a new embedding $G = G(g) : M \to [t_0, t_1] \times (-1,1)^N$ by $G = g \circ (\bar{\varphi}(g)^{-1})$. It has the same image as g and has $\pi(G(x)) = x$. Therefore x and $G(x)$ have the same height when $x \in x_1^{-1}(C)$.

Let λ be a bump function which is 1 on $[0, \varepsilon/4)$ and 0 on $[\varepsilon/2, \infty)$. Now let

$$\varphi(g)(x) = x + \lambda(|x - \pi(x)|) \cdot (G(\pi(x)) - \pi(x))$$

define a compactly-supported smooth map $\varphi(g) \in \mathcal{C}_c^\infty([t_0, t_1] \times (-1,1)^N)$. For $g = f$ it is a diffeomorphism, and so there is a smaller neighbourhood U of f in S where this remains true. We get a function $\varphi : U \to \mathrm{Diff}_c([t_0, t_1] \times (-1,1)^N)$.

By construction $\varphi(g) \circ f(x) = \varphi(g)(x) = x + (G(x) - x) = G(x)$, so $\varphi(g) \circ f$ has the same image as g. Also, if $x \in x_1^{-1}(C)$ then the vector $G(\pi(x)) - \pi(x)$ has no component in the x_1 direction, so $x_1(\varphi(g)(x)) = x_1(x)$ and $\varphi(g)$ is height function preserving over C. □

It now follows that π also has this local structure: after posibly shrinking U, there is a map

$$\varphi : U \longrightarrow \mathrm{Diff}([t_0, t_1] \times (-1,1)^N)$$

with the properties described in the lemma, such that $\Gamma|_{[t_0,t_1]} \subset U \times [t_0, t_1] \times (-1,1)^N$ is obtained from $W|_{[t_0,t_1]}$ by applying the family of diffeomorphisms φ.

The element $e \in \widetilde{Z}_0(a, \varepsilon, W)$ is given by embedded surgery data

$$e : \Lambda \times \bar{V} \hookrightarrow [t_0, t_1] \times (0,1) \times (-1,1)^{N-1},$$

so we attempt to define a section $U \to \widetilde{D}^\kappa_{\theta,L}(\mathbb{R}^N)_{p,0}$ by sending u to the embedding $\varphi(u) \circ e$. We must verify that this is indeed an element of $\widetilde{Z}_0(u)$ by checking the conditions of Definition 3.2. Conditions (i)–(iv) hold as φ is height preserving over each $[a_i - \varepsilon_i, a_i + \varepsilon_i]$. Condition (v) holds by construction, as $\varphi(x)$ is a diffeomorphism which carries W into $\pi^{-1}(x)$. Condition (vi) need not hold in general, but it does hold at the point (a, ε, W), and is an open condition. Thus, after possibly replacing U with a smaller open set, this does define a continuous section as required. $\square$

Next, we establish condition (iii) in Theorem 6.2.

Proposition 6.8. *Fix a point $(a, \varepsilon, W) \in D^{\kappa-1}_{\theta,L}(\mathbb{R}^N)_p$, and let $v_1, \ldots, v_k \in \widetilde{Z}_0(a, \varepsilon, W)$ be a non-empty collection of pieces of surgery data (not necessarily forming a $(k-1)$-simplex). Then, if $2\kappa < d$ and $\kappa + 1 + d < N$, there exists a piece of surgery data $v \in \widetilde{Z}_0(a, \varepsilon, W)$ such that each (v_i, v) is a 1-simplex.*

Proof. Each v_j is given by a set Λ^j (which is a subset of the uncountable set Ω) and a map $e^j : \Lambda^j \times \bar{V} \to \mathbb{R} \times (0, 1) \times (-1, 1)^{N-1}$, satisfying certain properties. We first pick a set Λ which is disjoint from all Λ^j and a bijection $\varphi : \Lambda \to \Lambda^1$, and then set

$$\tilde{e} = e^1 \circ (\varphi \times \mathrm{id}) : \Lambda \times \bar{V} \longrightarrow \mathbb{R} \times (0, 1) \times (-1, 1)^{N-1}.$$

This gives a new element of $\widetilde{Z}_0(a, \varepsilon, W)$, but it of course not orthogonal to v_1 (and not necessarily orthogonal to the other v_j). We then perturb $\tilde{e}$ inside the class of functions satisfying the requirements of Definition 3.2, to a new function $e : \Lambda \times \bar{V} \to \mathbb{R} \times (0, 1) \times (-1, 1)^{N-1}$ whose core is in general position with respect to the cores of the v_j. More explicitly, $\tilde{e}$ restricts to a map

$$\Lambda \times \partial_- D^{\kappa+1} \times \mathbb{R}^{d-\kappa} \longrightarrow W,$$

and we first perturb this so that $\Lambda \times \partial_- D^{\kappa+1} \times \{0\}$ is transverse in W to the corresponding part of the other embeddings, and remains disjoint from L, then we extend this perturbation to a map $e : \Lambda \times \bar{V} \to \mathbb{R} \times (0, 1) \times (-1, 1)^{N-1}$ whose restriction to the interior of C is transverse to the corresponding part of the other embeddings. In the first step we make κ-dimensional manifolds transverse in a d-dimensional manifold, and in the second we make $(\kappa + 1)$-dimensional manifolds disjoint in an $(N+1)$-dimensional manifold. As $2\kappa < d$ and $2(\kappa+1) \leq \kappa+d+2 < N + 1$, the new core will actually be disjoint from all other cores, producing the required element $v \in \widetilde{Z}_0(a, \varepsilon, W)$. $\square$

Finally, we establish condition (ii) of Theorem 6.2.

Proposition 6.9. *$\widetilde{Z}_0(a, \varepsilon, W)$ is non-empty as long as $2\kappa < d$, $\kappa + 1 + d < N$, and L admits a handle decomposition only using handles of index $< d - \kappa - 1$.*

Proof. For each $i = 1, \ldots, p$ we consider the pair $(W|_{[a_{i-1},a_i]}, W|_{a_i})$. Since it is $(\kappa - 1)$-connected, it is homotopy equivalent to a finite relative CW complex $(X, W|_{a_i})$ with cells of dimension $\geq \kappa$ only. Since $2\kappa < d$, the homotopy equivalence $(X, W|_{a_i}) \to (W|_{[a_{i-1},a_i]}, W|_{a_i})$ may be assumed to restrict to a smooth embedding of the relative κ-cells. If we pick a set $\Lambda_{i,0}$ with one element for each relative κ-cell, we may therefore pick an embedding

$$\hat{e}_{i,0} : \Lambda_{i,0} \times (D^\kappa, \partial D^\kappa) \longrightarrow (W|_{[a_{i-1}+\varepsilon_{i+1}, a_i+\varepsilon_i]}, W|_{a_i+\varepsilon_i}),$$

which we may assume collared on $[a_i - \varepsilon_i, a_i + \varepsilon_i]$, such that the pair

$$(W|_{[a_{i-1},a_i]}, W|_{a_i} \cup \mathrm{Im}(\hat{e}_{i,0})|_{[a_{i-1},a_i]})$$

is κ-connected. Furthermore, $\mathbb{R} \times L \subset W$ has a core of dimension $< d - \kappa$, by our assumption on the indices of handles of L, and so we may suppose that the embedding $\hat{e}_{i,0}$ is disjoint from $\mathbb{R} \times L$.

The embedding

$$\hat{e}_{i,0}|_{\Lambda_{i,0} \times \partial D^\kappa} : \Lambda_{i,0} \times \partial D^\kappa \times \{0\} \longrightarrow W|_{a_i+\varepsilon_i} \subset W|_{[a_i+\varepsilon_i, a_{i+1}+\varepsilon_{i+1}]}$$

extends to an embedding of $\Lambda_{i,0} \times \partial D^\kappa \times [0,1]$, where $\Lambda_{i,0} \times \partial D^\kappa \times \{1\}$ is sent into $W|_{a_{i+1}+\varepsilon_{i+1}}$ and is collared on the ε-neighbourhoods of both boundaries. This may be seen as follows: to extend $\hat{e}_{i,0}|_{\Lambda_{i,0}\times\partial D^\kappa}$ to a continuous map having this property is possible as $\pi_{\kappa-1}(W|_{[a_{i-1},a_i]}, W|_{a_i}) = 0$, but this may then be perturbed to be an embedding as $2\kappa < d$. As above, this may be made disjoint from $\mathbb{R} \times L$.

We may glue the two embeddings together. Using a suitable diffeomorphism $D^\kappa \approx D^\kappa \cup \big(\partial D^\kappa \times [0,1]\big)$, this gives a new embedding of $\Lambda_{i,0} \times D^\kappa$. Continuing in this way, we obtain an extension of $\hat{e}_{i,0}$ to an embedding

$$\tilde{e}_{i,0} : \Lambda_{i,0} \times (D^\kappa, \partial D^\kappa) \longrightarrow (W|_{[a_{i-1}+\varepsilon_{i-1}, a_p+\varepsilon_p]}, W|_{a_p+\varepsilon_p})$$

which is disjoint from $\mathbb{R}\times L$. Identifying D^κ with the disc $\partial_- D^{\kappa+1} \subset [-1,0]\times\mathbb{R}^{\kappa+1}$ gives a height function $D^\kappa \to [-1,0]$ and if we pick the diffeomorphisms $D^\kappa \approx D^\kappa \cup \big(\partial D^\kappa \times [0,1]\big)$ carefully, we can arrange that on each $\tilde{e}_{i,0}^{-1}(W|_{(a_k-\varepsilon_k, a_k+\varepsilon_k)})$, the embedding $\tilde{e}_{i,0}$ is height function preserving up to an affine transformation.

We now want to extend the $\tilde{e}_{i,0}$ from $\Lambda_{i,0} \times (\partial_- D^{\kappa+1} \times \{0\}) \subset \Lambda_{i,0} \times \overline{V}$ to the whole of $\Lambda_{i,0}\times\overline{V}$ so that it satisfies the conditions of Definition 3.2. As $\kappa+1+d < N$, there is no trouble with extending the maps $\tilde{e}_{i,0}$ to maps $e_{i,0}$ from $\Lambda_{i,0} \times \bar{V}$ to $\mathbb{R}\times(0,1)\times(-1,1)^{N-1}$ satisfying conditions (i)–(v) of Definition 3.2: we first extend each $\tilde{e}_{i,0}$ to an embedding of $[-2,0] \times \mathbb{R}^\kappa \times \{0\}$ (which is possible as $2(\kappa+1) \leq d+\kappa+1 < N$), then make this intersect W only in $\partial_- D^{\kappa+1}$ (which is possible as $\kappa+1+d < N$), and finally thicken it up by $\mathbb{R}^{d-\kappa}$. Property (vi) is ensured by the way we chose $\hat{e}_{i,0}$. □

6.4. **Proof of Theorem 4.5.** Recall that the first part of this theorem states that the augmentation map

$$D^{\kappa,l}_{\theta,L}(\mathbb{R}^N)_{\bullet,\bullet} \longrightarrow D^{\kappa,l-1}_{\theta,L}(\mathbb{R}^N)_\bullet$$

induces a weak homotopy equivalence after geometric realisation, as long as certain conditions are satisfied. We remind the reader that these conditions are

(i) $2(l+1) < d$,
(ii) $l \leq \kappa$,
(iii) $l+2+d < N$,
(iv) L admits a handle decomposition only using handles of index $< d-l-1$,
(v) the map $\ell_L : L \to B$ is $(l+1)$-connected.

We have already proved the second part of Theorem 4.5 in Section 6.1.

We will proceed much as in the previous section. Recall that each point of $D^{\kappa,l}_{\theta,L}(\mathbb{R}^N)_{p,0}$ lying over $(a,\varepsilon,(W,\ell_W)) \in D^{\kappa,l-1}_{\theta,L}(\mathbb{R}^N)_p$ is a $(p+1)$-tuple of pairs (e_i,ℓ_i), where

$$e_i : \Lambda_i \times (a_i-\varepsilon_i, a_p+\varepsilon_p) \times \mathbb{R}^{d-l-1} \times D^{l+1} \hookrightarrow \mathbb{R}\times(0,1)\times(-1,1)^{N-1}$$

is an embedding, and ℓ_i is a θ-structure on $\Lambda_i \times K|_{(-6,0)}$ where K is defined in Section 4.2. Let us define

$$C = (a_i-\varepsilon_i, a_p+\varepsilon_p) \times \{0\} \times D^{l+1} \subset (a_i-\varepsilon_i, a_p+\varepsilon_p) \times \mathbb{R}^{d-l-1} \times D^{l+1}$$

and call it the *core*. Shrinking in the $\mathbb{R}^{d-l-1}$-direction gives an isotopy from the identity map of $(a_i-\varepsilon_i, a_p+\varepsilon_p) \times \mathbb{R}^{d-l-1} \times D^{l+1}$ into any neighbourhood of its core.

Definition 6.10. Let $\widetilde{Y}_\bullet(a,\varepsilon,(W,\ell_W))$ be the semi-simplicial space defined as in Definition 4.3 except that we replace condition (i) by asking that the embeddings $e_{i,j}$ have disjoint cores, instead of being completely disjoint. Note that condition (iv) still makes sense: although the surgery data is no longer disjoint, it is still disjoint when restricted to a small enough neighbourhood of each core.

Let $\widetilde{D}^{\kappa,l}_{\theta,L}(\mathbb{R}^N)_{\bullet,\bullet} \to D^{\kappa,l-1}_{\theta,L}(\mathbb{R}^N)_\bullet$ be the augmented bi-semi-simplicial space defined as in Definition 4.4, but using $\widetilde{Y}_\bullet(a,\varepsilon,(W,\ell_W))$ instead of $Y_\bullet(a,\varepsilon,(W,\ell_W))$

We have the following analogue of Proposition 6.5, although the proof is slightly more complicated in this case, due to the tangential structures on the surgery data.

Proposition 6.11. *The inclusion $D^{\kappa,l}_{\theta,L}(\mathbb{R}^N)_{\bullet,\bullet} \hookrightarrow \widetilde{D}^{\kappa,l}_{\theta,L}(\mathbb{R}^N)_{\bullet,\bullet}$ induces a weak homotopy equivalence in each bidegree, and so on geometric realisation.*

Proof. This is very similar to Proposition 6.5. We pick an isotopy of maps $\psi_t : \mathbb{R}^{d-l-1} \to \mathbb{R}^{d-l-1}$, $t \in [0,\infty)$ which starts at the identity, has $\psi_t(0) = 0$ for all t, and has image in the ball of radius $1/t$ for all t. Applying ψ_t in the $\mathbb{R}^{d-l-1}$ direction gives an isotopy of self-embeddings of $\Lambda_{i,j} \times (a_i - \varepsilon_i, a_p + \varepsilon_p) \times \mathbb{R}^{d-l-1} \times D^{l+1}$. Similarly, we can get an isotopy of self-embeddings of the manifold $K|_{(-6,0)}$ from Section 4.2, which applies ψ_t in the $\mathbb{R}^{d-l-1}$ direction on $h^{-1}((-6,-2])$, is the identity on $h^{-1}((-\sqrt{2},0))$, and interpolates inbetween. Precomposing with these isotopies gives a homotopy of self-maps of $\widetilde{D}^{\kappa,l}_{\theta,L}(\mathbb{R}^N)_{\bullet,\bullet}$, which eventually deforms any compact space into $D^{\kappa,l}_{\theta,L}(\mathbb{R}^N)_{\bullet,\bullet}$. □

Therefore it is enough to show that for each p, the augmentation map

$$\widetilde{D}^{\kappa,l}_{\theta,L}(\mathbb{R}^N)_{p,\bullet} \longrightarrow D^{\kappa,l-1}_{\theta,L}(\mathbb{R}^N)_p$$

which forgets all surgery data induces a weak homotopy equivalence after geometric realisation, which we do by establishing the conditions of Theorem 6.2. The proofs that conditions (i) and (iii) hold are very similar to the analogous case in Section 6.3, so we consider those first.

Proposition 6.12. *The map $\widetilde{D}^{\kappa,l}_{\theta,L}(\mathbb{R}^N)_{p,0} \to D^{\kappa,l-1}_{\theta,L}(\mathbb{R}^N)_p$ has local sections.*

Proof. Exactly as in the proof of Proposition 6.6. □

Proposition 6.13. *Fix a point $(a,\varepsilon,(W,\ell_W)) \in D^{\kappa,l-1}_{\theta,L}(\mathbb{R}^N)_p$, and let v_1, ..., $v_k \in \widetilde{Y}_0(a,\varepsilon,(W,\ell_W))$ be a collection of pieces of surgery data, $k \geq 1$. Then, if $2(l+1) < d$ and $l+2+d < N$, there exists a piece of surgery data $v \in \widetilde{Y}_0(a,\varepsilon,(W,\ell_W))$ such that each (v_i, v) is a 1-simplex.*

Proof. This is essentially the same as Proposition 6.8: first we let $v = v_1$, then we perturb it to have its cores transverse to the cores of all the v_j. We first do the perturbation on the part of the cores inside W. On the boundary the cores are $(l+1)$-dimensional, so disjoint when they are transverse as $2(l+1) < d$. We now make sure the cores intersect W only on their boundary, which is possible as $l+2+d < N$. We finally make sure that the cores are also disjoint on their interiors, which is possible as $(l+2)+(l+2) \leq (l+2)+d < N$. □

Finally, we establish condition (ii).

Proposition 6.14. *$\widetilde{Y}_0(a,\varepsilon,(W,\ell_W))$ is non-empty as long as $2(l+1) < d$, $l \leq \kappa$, $l+2+d < N$, L admits a handle decomposition only using handles of index $< d-l-1$, and the map $\ell_L : L \to B$ is $(l+1)$-connected.*

Proof. For each i, we consider the map $\pi_*(W|_{a_i}) \to \pi_*(B)$, induced by the tangential structure. By assumption, this map is injective for $* \leq l-1$. Since $\{a_i\} \times L \subset W|_{a_i}$, and $L \to B$ is assumed $(l+1)$-connected, we deduce that the map $L \to W|_{a_i}$ is $(l-1)$-connected, $W|_{a_i} \to B$ is l-connected and that

$$\pi_l(W|_{a_i}) \longrightarrow \pi_l(B) \approx \pi_l(L) \tag{6.1}$$

is split surjective and $\pi_l(L) \to \pi_l(W|_{a_i})$ is split injective. We first claim that the kernel of (6.1) is finitely generated as a module over $\pi_1(L)$ (interpreted appropriately when $l = 0$ and $l = 1$; we shall leave the necessary modifications of the following argument in those two cases to the reader). Since the kernel is isomorphic to the cokernel of the splitting, we deduce the exact sequence

$$\pi_l(W|_{a_i}, L) \longrightarrow \pi_l(W|_{a_i}) \longrightarrow \pi_l(B) \longrightarrow 0. \tag{6.2}$$

As $(W|_{a_i}, L)$ is $(l-1)$-connected, we can find a relative CW-complex (K, L), where K is built from L by attaching only cells of dimension $\geq l$, and a weak homotopy equivalence $p : (K, L) \to (W|_{a_i}, L)$. Since $(W|_{a_i}, L)$ has the homotopy type of a CW pair, this map has a homotopy inverse $q : (W|_{a_i}, L) \to (K, L)$, and since $W|_{a_i}$ is compact, its image in K is contained in a finite subcomplex $K' \subset K$. Then $\pi_l(K') \to \pi_l(W|_{a_i})$ is surjective. Since $\pi_l(W|_{a_i}) \to \pi_l(W|_{a_i}, L)$ is also surjective (as $\pi_{l-1}(L) \to \pi_{l-1}(W|_{a_i})$ is split injective), we conclude that $\pi_l(K', L) \to \pi_l(W|_{a_i}, L)$ is surjective, and hence that $\pi_l(W|_{a_i}, L)$ is a finitely generated module over $\pi_1(L)$, as claimed.

Let $\{\hat{f}_\alpha : S^l \to W|_{a_i}\}_{\alpha \in \Lambda_i}$ be a finite collection of elements which generate the kernel of $\pi_l(W|_{a_i}) \to \pi_l(B)$. As the vector bundle $\varepsilon^1 \oplus TW|_{a_i}$ is pulled back from B, it becomes trivial when pulled back via $\hat{f}_\alpha$ so we can pick an isomorphism $\varepsilon^1 \oplus \hat{f}_\alpha^*(TW|_{a_i}) \cong \varepsilon^d$. As $l + 1 < d$ this isomorphism can be destabilised to an isomorphism $\hat{f}_\alpha^*(TW|_{a_i}) \cong \varepsilon^{d-1} \cong \varepsilon^{d-l-1} \oplus TS^l$ and by Smale–Hirsch theory $\hat{f}_\alpha$ is then homotopic to an immersion with trivial normal bundle. We can make this immersion self-transverse, and as $2l < d - 1$ it is then an embedding with trivial normal bundle. Thus each $\hat{f}_\alpha$ gives rise to an embedding $f_\alpha : \mathbb{R}^{d-l-1} \times S^l \hookrightarrow W|_{a_i}$ representing the same homotopy class. As $2l < d - 1$ we may also assume that the f_α are disjoint, so we obtain an embedding

$$f_i|_{a_i} : \Lambda_i \times \{a_i\} \times \mathbb{R}^{d-l-1} \times S^l \hookrightarrow W|_{a_i},$$

and as L has handles of index $< d-l-1$, we may suppose the embedding is disjoint from L. As $l + 1 + d < N$, this extends to an embedding

$$e_i|_{a_i} : \Lambda_i \times \{a_i\} \times \mathbb{R}^{d-l-1} \times D^{l+1} \hookrightarrow \{a_i\} \times (0, 1) \times (-1, 1)^{N-1}$$

which intersects $W|_{a_i}$ precisely on the boundary. Furthermore, as each $S^l \xrightarrow{\hat{f}_\alpha} W|_{a_i} \to B$ is null-homotopic, the θ-structure $\ell|_{a_i} \circ Df_\alpha$ extends to a θ-structure on $\mathbb{R}^{d-l-1} \times D^{l+1}$ and so gives $f_i|_{a_i}$ the data of a θ-surgery, c.f. Section 4.1.

We can extend the map $e_i|_{a_i}$ to an embedding $\Lambda_i \times (a_i - \varepsilon_i, a_i + \varepsilon_i) \times \mathbb{R}^{d-l-1} \times D^{l+1} \hookrightarrow (a_i - \varepsilon_i, a_i + \varepsilon_i) \times (0, 1) \times (-1, 1)^{N-1}$ using just the cylindrical structure of W over $(a_i - \varepsilon_i, a_i + \varepsilon_i)$, but we wish to extend it to an embedding of $\Lambda_i \times (a_i - \varepsilon_i, a_p + \varepsilon_p) \times \mathbb{R}^{d-l-1} \times D^{l+1}$, which is cylindrical over each $(a_j - \varepsilon_j, a_j + \varepsilon_j)$ and intersects W precisely on the boundary. We will do this by extending it step-by-step over each interval $[a_j, a_{j+1}]$: if it is defined up to a_j we have an embedding

$$e_i|_{a_j} : \Lambda_i \times \{a_j\} \times \mathbb{R}^{d-l-1} \times D^{l+1} \hookrightarrow \{a_j\} \times (0, 1) \times (-1, 1)^{N-1},$$

and as the pair $(W|_{[a_j,a_{j+1}]}, W|_{a_{j+1}})$ is κ-connected, and $l \leq \kappa$, on the boundary this extends to a continuous map

$$f_i|_{[a_j,a_{j+1}]} : \Lambda_i \times [a_j, a_{j+1}] \times \mathbb{R}^{d-l-1} \times S^l \longrightarrow W|_{[a_j,a_{j+1}]}.$$

By the Smale–Hirsch argument above, we may perturb this to be a self-transverse immersion of the core, and hence an embedding of the core as $2(l + 1) < d$, while keeping it as it was near a_j. Shrinking in the $\mathbb{R}^{d-l-1}$-direction, we can ensure that it is an embedding of the whole manifold, and then make the embedding cylindrical over the necessary ε-neighbourhood of the ends and disjoint from $[a_j, a_{j+1}] \times L$. Finally, as $l+2+d < N$ we may extend this to an embedding

$$e_i|_{[a_j,a_{j+1}]} : \Lambda_i \times [a_j, a_{j+1}] \times \mathbb{R}^{d-l-1} \times D^{l+1} \hookrightarrow [a_j, a_{j+1}] \times (0,1) \times (-1,1)^{N-1}$$

which is cylindrical over each $(a_j - \varepsilon_j, a_j + \varepsilon_j)$ and intersects W precisely on the boundary. In total we obtain an embedding

$$e_i : \Lambda_i \times (a_i - \varepsilon_i, a_p + \varepsilon_p) \times \mathbb{R}^{d-l-1} \times D^{l+1} \hookrightarrow (a_i - \varepsilon_i, a_p + \varepsilon_p) \times (0,1) \times (-1,1)^{N-1}$$

which is cylindrical over each $(a_j - \varepsilon_j, a_j + \varepsilon_j)$ and intersects W precisely on the boundary. Furthermore, by doing the above in increasing order of i, we can further ensure that the different e_i have disjoint cores: while constructing e_i make sure that its core stays disjoint from those of the e_j for all $j < i$, which is possible as $2(l+1) < d$ and $2(l+2) < N$.

The collection $\{e_i\}$ gives the embedding part of an element of $\widetilde{Y}_0(a, \varepsilon, (W, \ell_W))$, and we must now provide the bundle part. Under the chosen diffeomorphisms

$$K|_{(-6,-2)} = (-6,-2) \times \mathbb{R}^{d-l-1} \times S^l \cong_\varphi (a_i - \varepsilon_i, a_p + \varepsilon_p) \times \mathbb{R}^{d-l-1} \times S^l,$$

the embedding $f_i : \Lambda_i \times (a_i - \varepsilon_i, a_p + \varepsilon_p) \times \mathbb{R}^{d-l-1} \times S^l \hookrightarrow W$ gives a θ-structure $\ell_i|_{(-6,-2)} = \ell \circ Df_i$ on $\Lambda_i \times K|_{(-6,-2)}$. For ℓ_i we may take any extension of this θ-structure to $\Lambda_i \times K$, and so only need to know that such an extension exists. This is a purely homotopical problem, and homotopically $K|_{(-6,0)}$ is obtained from $K|_{(-6,-2)}$ by attaching a D^{l+1}, so the extension problem can be solved if and only if $\ell \circ Df_i|_{a_i} : T(\Lambda_i \times \{a_i\} \times \mathbb{R}^{d-l-1} \times S^l) \to \theta^*\gamma$ extends over $\Lambda_i \times \{a_i\} \times \mathbb{R}^{d-l-1} \times D^{l+1}$, but we have seen above that it does, because $\Lambda_i \times \{a_i\} \times \mathbb{R}^{d-l-1} \times S^l \to W|_{a_i} \to B$ is null-homotopic. □

6.5. **Proof of Theorem 5.13.** Recall that the statement of the theorem is as follows. We work in dimension $2n$, and fix a tangential structure θ which is reversible (c.f. Definition 5.1), a $(2n-1)$-manifold with boundary L equipped with θ-structure, and a collection $\mathcal{A} \subset \pi_0(\mathrm{ob}(\mathcal{C}_{\theta,L}^{n-1,n-2}(\mathbb{R}^N)))$ of objects which hit every path component of the category $\mathcal{C}_{\theta,L}^{n-1,n-2}(\mathbb{R}^N)$. This allows us to define the augmented bi-semi-simplicial space

$$D_{\theta,L}^{n-1,\mathcal{A}}(\mathbb{R}^N)_{\bullet,\bullet} \longrightarrow D_{\theta,L}^{n-1,n-2}(\mathbb{R}^N)_\bullet$$

of surgery data, and Theorem 5.13 states that if the conditions of Theorem 5.2 are satisfied, then the induced map on geometric realisation is a weak homotopy equivalence. We recall that these conditions are

(i) $2n \geq 6$,
(ii) $3n+1 < N$,
(iii) θ is reversible,
(iv) L admits a handle decomposition only using handles of index $< n$,
(v) $\ell_L : L \to B$ is $(n-1)$-connected.

Note that the last condition implies that for any object, the map $M \to B$ induced by the tangential structure induces a surjection on π_* for $* < n$.

In many respects the proof of this theorem is very similar to what we did in Section 6.4, but in that section we often used the inequality $2(l+1) < d$ so that pairs of transverse $(l+1)$-dimensional submanifolds of a d-manifold are automatically disjoint. In Theorem 5.13, $d = 2n$ and the analogue of l is $(n-1)$ so this observation fails. Instead, we will use a version of the Whitney trick to separate n-dimensional submanifolds of our $2n$-manifolds; this accounts for the restriction $2n \geq 6$ in the statement of the theorem.

We proceed precisely as in Definition 6.10 by for $(a, \varepsilon, (W, \ell_W)) \in D_{\theta,L}^{n-1,n-2}(\mathbb{R}^N)_p$ letting $\widetilde{Y}_\bullet(a, \varepsilon, (W, \ell_W))$ be the analogue of $Y_\bullet(a, \varepsilon, (W, \ell_W))$ from Definition 5.12

where we only require the surgery data to have disjoint cores, and use this to define the bi-semi-simplicial space $\widetilde{D}_{\theta,L}^{n-1,\mathcal{A}}(\mathbb{R}^N)_{\bullet,\bullet}$. By the same argument as Proposition 6.11, the inclusion

$$D_{\theta,L}^{n-1,\mathcal{A}}(\mathbb{R}^N)_{\bullet,\bullet} \hookrightarrow \widetilde{D}_{\theta,L}^{n-1,\mathcal{A}}(\mathbb{R}^N)_{\bullet,\bullet}$$

is a weak homotopy equivalence in each bidegree. We are now left to verify the conditions of Theorem 6.2 for the augmented semi-simplicial spaces

$$\widetilde{D}_{\theta,L}^{n-1,\mathcal{A}}(\mathbb{R}^N)_{p,\bullet} \longrightarrow D_{\theta,L}^{n-1,n-2}(\mathbb{R}^N)_p.$$

That the map on 0-simplices has local sections is proved as in the previous two sections.

Proposition 6.15. *Fix a point $(a,\varepsilon,(W,\ell_W)) \in D_{\theta,L}^{n-1,n-2}(\mathbb{R}^N)_p$, and let v_1, ..., $v_k \in \widetilde{Y}_0(a,\varepsilon,(W,\ell_W))$ be a collection of pieces of surgery data, $k \geq 1$. Then if $2n \geq 6$ and $3n+1 < N$ there exists a piece of surgery data $v \in \widetilde{Y}_0(a,\varepsilon,(W,\ell_W))$ such that each (v_i, v) is a 1-simplex.*

Proof. Let us write

$$v_j = \{(e_i^j : \Lambda_i^j \times (a_i - \varepsilon_i, a_p + \varepsilon_p) \times \mathbb{R}^n \times D^n \hookrightarrow \mathbb{R} \times (0,1) \times (-1,1)^{N-1}, \ell_i^j)\}.$$

First we let $v = v_1$, then we perturb it to have its cores transverse to the cores of all the v_j. We first do the perturbation on the part of the cores inside W. On the boundary the cores are n-dimensional, so when they are transverse, they intersect in a finite set of points. We now make sure the cores intersect W only on their boundary, which is possible as $(n+1)+2n < N$. We finally make sure that the cores are also disjoint on their interiors, which is possible as $2(n+1) < N$.

We are left with surgery data v whose core is disjoint from the cores of v_j away from W, and on W intersects the other cores transversely. It has a finite number of transverse intersections with all the other cores in W, so it is enough to give a procedure which reduces the number of intersections by 1. Let x be such an intersection point, between v and some v_j. More precisely, suppose it is a point of intersection of the cylinders

$$e_i(\Lambda_i \times (a_i - \varepsilon_i, a_p + \varepsilon_p) \times \{0\} \times S^{n-1}) \qquad e_k^j(\Lambda_k^j \times (a_k - \varepsilon_k, a_p + \varepsilon_p) \times \{0\} \times S^{n-1}).$$

Claim 6.16. Let $T \subset \mathbb{R}^2$ denote the triangle $\{(x,y) | y \leq 0, y+1 \geq |x|\}$ and U a small open neighbourhood of it, e.g. defined by $y < \varepsilon$, $y+1 < |x| + \varepsilon$. There is a *Whitney disc* $w : U \hookrightarrow W$ such that

(i) w is disjoint from $\mathbb{R} \times L$.
(ii) $w|_{[-1,1]\times\{0\}}$ is a path in $W|_{a_p}$ which on its interior is disjoint from all the cores.
(iii) The inverse image of the first cylinder is the line on ∂T from $(0,-1)$ to $(-1,0)$. The inverse image of the second cylinder is the line from $(0,-1)$ to $(1,0)$.
(iv) The height functions $x_1 \circ w$ and $y : T \to \mathbb{R}$ agree up to an additive constant inside each $w^{-1} \circ x_1^{-1}(a_j - \varepsilon_j, a_j + \varepsilon_j)$.

Given such a disc, we can extend it to a standard neighbourhood $w(U) \times \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \subset W$ as in the proof of [Mil65, Theorem 6.6]. Note the argument is easier in this case as we are canceling intersection points against the boundary instead of against each other, and so no framing problems arise. We can further extend this to a neighbourhood

$$w(U) \times \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \mathbb{R}^{N+1-2n} \subset \mathbb{R} \times (0,1) \times \mathbb{R}^{N-1}.$$

There is a compactly supported vector field on U which is $\partial/\partial x$ on D^2_-, and we extend it using bump functions in the euclidean directions to this open subset of $\mathbb{R} \times (0,1) \times \mathbb{R}^{N-1}$. The flow associated to this vector field gives a 1-parameter family of diffeomorphisms φ_t, and flowing e_i along using φ_t will eventually lead to a new e_i whose core has one fewer intersection point with other cores (at least inside $W|_{(a_0-\varepsilon_0,a_p]}$, but we can then use the cylindrical structure of $W|_{(a_p-\varepsilon_p,a_p+\varepsilon_p)}$ to remove any intersections above a_p). It will still satisfy condition (iii) of Definition 5.12 by property (iv) above, and the other conditions are clear.

It remains to prove the claim. If it were not for property (iv) the argument is clear: choose an embedded path from x in each cylinder up to $W|_{a_p}$. Together these give an element of $\pi_1(W|_{[a_i,a_p]}, W|_{a_p})$ which is 0 as $\kappa = n-1 \geq 2$, and so this extends to a continuous map $w|_T : T \to W$ which gives these two paths along the lower part of its boundary and lies in $W|_{a_p}$ in the top part of its boundary. As $2 \cdot 2 < 2n$, this map may be perturbed to be an embedding into W, still enjoying these two properties. Finally, as $2 + n < 2n$, $w|_{D^2_-}$ can be made disjoint from the other cores on its interior. This may now be extended to a map on U, enjoying properties (ii) and (iii).

To obtain property (iv) as well, we instead build up the embedding $w|_T$ in pieces inside each $W|_{[a_j,a_{j+1}]}$, which is possible as each $\pi_1(W|_{[a_j,a_{j+1}]}, W|_{a_{j+1}})$ is 0. □

In the proof of Proposition 6.14, it was easy to see that for an object $M \in \mathcal{C}^{\kappa,l-1}_{\theta,L}(\mathbb{R}^N)$ there is a piece of θ-surgery data $e : \Lambda \times \mathbb{R}^{d-l-1} \times S^l \hookrightarrow M$ such that the resulting manifold $\overline{M}$ has $\pi_l(\overline{M}) \to \pi_l(B)$ injective, so satisfies condition (iv) of Definition 4.3. In the present situation we have $M \in \mathcal{C}^{n-1,n-2}_{\theta,L}(\mathbb{R}^N)$ and require surgery data so that $\overline{M} \in \mathcal{A}$, to satisfy condition (iv) of Definition 5.12. This is rather more difficult, and we first describe how to accomplish this step.

Lemma 6.17. *Let $M \in \mathcal{C}^{n-1,n-2}_{\theta,L}(\mathbb{R}^N)$ be an object, and suppose θ is reversible, $2n \geq 6$, L has a handle structure with only handles of index $< n$, and $\ell_L : L \to B$ is $(n-1)$-connected. Then there is a piece of θ-surgery data, given by an embedding $e : \Lambda \times \mathbb{R}^{d-l-1} \times S^l \hookrightarrow M$ disjoint from L and a compatible bundle map $T(\Lambda \times \mathbb{R}^{d-l-1} \times D^{l+1}) \to \theta^*\gamma$, such that the resulting surgered manifold $\overline{M}$ lies in $\mathcal{A}$.*

Proof. Part of this proof is very similar to [Kre99, pp. 722–724].

We first claim that if there is a morphism $W : M_0 \rightsquigarrow M_1 \in \mathcal{C}^{n-1,n-2}_{\theta,L}(\mathbb{R}^N)$, then there is another, W' say, with the property that (W', M_0) is also $(n-1)$-connected. (By definition, (W', M_1) is $(n-1)$-connected.) In fact, we claim that it is possible to do surgery along a finite set of embeddings of $S^{n-1} \times D^{n+1}$ into the interior of W (and disjoint from L), such that the resulting cobordism W' is $(n-1)$-connected with respect to both boundaries. Let us first point out that doing any such $(n-1)$-surgery does not change $\pi_k(W, M_1) = 0$ for $k \leq (n-1)$: up to homotopy it amounts to cutting out a manifold of codimension $(n+1)$ and then attaching a cells of dimension n and $2n$. We have assumed that $L \to B$ is $(n-1)$-connected so $\pi_k(M_0) \to \pi_k(B)$ is surjective for $k \leq n-1$. Since $M_0 \in \mathcal{C}^{n-1,n-2}_{\theta,L}$, it is an isomorphism for $k \leq n-2$ and similarly for M_1. Since $\pi_k(M_1) \to \pi_k(W)$ is an isomorphism for $k \leq n-2$, we conclude $\pi_k(M_0) \to \pi_k(W)$ is an isomorphism for $k \leq n-2$, but it need not be surjective for $k = n-1$. In fact, the long exact sequence in homotopy groups identifies the cokernel with $\pi_{n-1}(W, M_0) \cong H_{n-1}(\widetilde{W}, \widetilde{M_0})$. The fact that $\pi_{n-1}(M_0) \to \pi_{n-1}(B)$ is surjective implies that the map

$$\mathrm{Ker}\big(\pi_{n-1}(W) \to \pi_{n-1}(B)\big) \to \pi_{n-1}(W) \to \pi_{n-1}(W, M_0)$$

is still surjective. By the Hurewicz theorem, $\pi_{n-1}(W, M_0) \cong H_{n-1}(\widetilde{W}, \widetilde{M_0})$ is finitely generated as a module over π_1, and we have proved that there exist finitely many elements $\alpha_i \in \mathrm{Ker}(\pi_{n-1}(W) \to \pi_{n-1}(B))$ which generate the cokernel of $\pi_{n-1}(M_0) \to \pi_{n-1}(W)$. These elements may be represented by disjoint embedded framed spheres in the interior of W, and as L has a handle structure with only handles of index $< n$ they can be made disjoint from L, and we let W' denote the result of performing surgery. Both pairs (W', M_0) and (W', M_1) are now $(n-1)$-connected, and by Proposition 5.7, W' again admits a θ-structure.

We now return to the proof of the lemma. There is a zig-zag of morphisms in the category $\mathcal{C}_{\theta,L}^{n-1,n-2}(\mathbb{R}^N)$ from M to an object of $\mathcal{A}$, as $\mathcal{A}$ was chosen to hit every path component of the category. By the above discussion we can suppose that it is a zig-zag of θ-cobordisms which are $(n-1)$-connected relative to both ends. Then, by reversibility, we can reverse the backwards-pointing arrows and obtain a single morphism

$$(C, \ell_C) : (M, \ell_M) \rightsquigarrow (A, \ell_A) \in \mathcal{C}_{\theta,L}^{n-1,n-2}(\mathbb{R}^N),$$

which is $(n-1)$-connected relative to both ends, so $\pi_*(C, A) = \pi_*(C, M) = 0$ for $* \leq n-1$.

If such a cobordism C admits a Morse function with only critical points of index n, then the descending manifolds of the critical points, and ℓ_C restricted to them, gives the required θ-surgery data. It remains to produce such a Morse function.

If $\pi_1(L) = 0$ then all of the manifolds appearing above are also simply-connected, and we deduce by Poincaré duality and the Universal coefficient theorem that $H_*(C, M)$ is concentrated in degree n and is free abelian. We can choose a self-indexing Morse function on C and as in the proof of the h-cobordism theorem we can first modify it to have no critical points of index 0 or 1 [Mil65, Theorem 8.1], do the same to the negative of the Morse function to remove critical points of index $2n$ and $(2n-1)$, and finally by the Basis Theorem [Mil65, Theorem 7.6] we can diagonalise the differentials in the Morse homology complex, and so modify the Morse function to only have critical points of index n.

When $\pi_1(L) \neq 0$ we must go to a little more trouble, and use techniques from the proof of the s-cobordism theorem. As these are less well known, we go into more detail, but recommend [Lüc02] and [Ker65] for details of that argument. As above, pick a self-indexing Morse function on C and let us write

$$\pi = \pi_1(L) = \pi_1(M) = \pi_1(C) = \pi_1(A)$$

for the common fundamental group, and $\mathbb{Z}[\pi]$ for its integral group ring.

When $M \hookrightarrow C$ is 1-connected, [Mil65, Theorem 8.1] is still true: we may modify the Morse function to have no critical points of index 0 or 1, and as above do the same on the opposite Morse function to eliminate critical points of index $2n$ and $(2n-1)$. The cores of the handles given by this Morse function on the universal cover give a cell complex with cellular chain complex $C_*(\widetilde{C}, \widetilde{M})$, and $C_*(\widetilde{C}, \widetilde{A})$ for the opposite Morse function. These are chain complexes of based free $\mathbb{Z}[\pi]$-modules, and geometric Poincaré duality gives an isomorphism

$$C_*(\widetilde{C}, \widetilde{M}) \cong \mathrm{Hom}_{\mathbb{Z}[\pi]}(C_{2n-*}(\widetilde{C}, \widetilde{A}), \mathbb{Z}[\pi])$$

of chain complexes, by sending basis elements to their "dual" basis elements.

The chain complex $C_{2n-*}(\widetilde{C}, \widetilde{A})$ is one of free $\mathbb{Z}[\pi]$-modules and $0 = \pi_*(C, A) = \pi_*(\widetilde{C}, \widetilde{A}) = H_*(\widetilde{C}, \widetilde{A}; \mathbb{Z})$ for $* \leq n-1$, so it is acyclic in degrees $2n - * \leq n-1$. By the Universal coefficient spectral sequence, the same is true for its $\mathbb{Z}[\pi]$-dual and so $C_*(\widetilde{C}, \widetilde{M})$ is acyclic for $* \geq n+1$. Furthermore $0 = \pi_*(C, M) = \pi_*(\widetilde{C}, \widetilde{M}) = H_*(\widetilde{C}, \widetilde{M}; \mathbb{Z})$ for $* \leq n-1$, so the homology of $C_*(\widetilde{C}, \widetilde{M})$ is concentrated in degree n. By the usual modification technique, we can use handle exchanges to modify the Morse function to only have critical points of index n and $(n-1)$. We are left with a short exact sequence of $\mathbb{Z}[\pi]$-modules

$$0 \longrightarrow H_n(\widetilde{C}, \widetilde{M}; \mathbb{Z}) \longrightarrow C_n(\widetilde{C}, \widetilde{M}) \xrightarrow{\partial_n} C_{n-1}(\widetilde{C}, \widetilde{M}) \longrightarrow 0.$$

The rightmost term is a free $\mathbb{Z}[\pi]$-module and so this sequence is split: in particular, $H_n(\widetilde{C}, \widetilde{M}; \mathbb{Z})$ is stably free as a $\mathbb{Z}[\pi]$-module. If $H_n(\widetilde{C}, \widetilde{M}; \mathbb{Z})$ is not actually free as a $\mathbb{Z}[\pi]$-module, there cannot exist a Morse function on C with only critical points of index n. In this case we replace C by $C \#^g S^n \times S^n$ for g sufficiently large (and this manifold admits a θ-structure by Proposition 5.7). This has the effect of adding on a large free $\mathbb{Z}[\pi]$-module to $H_n(\widetilde{C}, \widetilde{M}; \mathbb{Z})$, so we may assume that this homology group is now free, and pick a basis of it.

Choosing a splitting of the short exact sequence above, we obtain an isomorphism

$$C_n(\widetilde{C}, \widetilde{M}) \cong H_n(\widetilde{C}, \widetilde{M}; \mathbb{Z}) \oplus C_{n-1}(\widetilde{C}, \widetilde{M}) \tag{6.3}$$

of based free $\mathbb{Z}[\pi]$-modules, and so an element of $K_1(\mathbb{Z}[\pi])$. However, the basis we chose for $H_n(\widetilde{C}, \widetilde{M}; \mathbb{Z})$ was not geometrically meaningful and we are free to change it. After possibly stabilising C further, it is possible to choose a basis for which (6.3) represents the zero class in $K_1(\mathbb{Z}[\pi])$, and hence in the Whitehead group $\mathrm{Wh}(\pi)$ too. We may then use the modification lemma to rearrange the index n critical points of the Morse function so that $\partial_n : C_n(\widetilde{C}, \widetilde{M}) \to C_{n-1}(\widetilde{C}, \widetilde{M})$ is simply projection onto the first few basis elements: this allows us to cancel all the critical points of index $(n-1)$. $\square$

Proposition 6.18. *$\widetilde{Y}_0(a, \varepsilon, (W, \ell_W))$ is non-empty as long as $3n+1 < N$, $2n \geq 6$, θ is reversible, L admits a handle structure with only handles of index $< n$, and $\ell_L : L \to B$ is $(n-1)$-connected.*

Proof. We follow the proof of Proposition 6.14, with a few changes. Let $d = 2n$ and $l = n-1$. The first step of Proposition 6.14 is to produce for each $W|_{a_i}$ the θ-surgery data $f_i|_{a_i}$. The method described in that proposition no longer works, and we use Lemma 6.17 to produce the necessary data instead. From this up to constructing the maps $e_i|_{(a_i-\varepsilon_i, a_i+\varepsilon_i)}$ there is no difference, and the argument given there works.

It remains to explain how given an embedding $e_i|_{a_j}$ we can extend it to $e_i|_{[a_j, a_{j+1}]}$. We proceed in the same way: we have the embedding

$$f_i|_{a_j} : \Lambda_i \times \{a_j\} \times \mathbb{R}^n \times S^{n-1} \hookrightarrow W|_{a_j}$$

disjoint from L, which extends to a continuous map

$$f_i|_{[a_j, a_{j+1}]} : \Lambda_i \times [a_j, a_{j+1}] \times \mathbb{R}^n \times S^{n-1} \longrightarrow W|_{[a_j, a_{j+1}]}$$

as $(W|_{[a_j, a_{j+1}]}, W|_{a_{j+1}})$ is $(n-1)$-connected by assumption. We can again make this be a self-transverse immersion of the core, but this no longer implies that the core is embedded: it will have isolated points of self-intersection. As $2n \geq 6$ we can remove these using the Whitney trick, as in the proof of Proposition 6.15. The core will still intersect the core of $[a_j, a_{j+1}] \times L$, as they are both of dimension n inside a $2n$-manifold, but we can again use the Whitney trick to separate them. Given $f_i|_{[a_j, a_{j+1}]}$ which is an embedding of the core and whose core is disjoint from that of $[a_j, a_{j+1}] \times L$, we can shrink in the $\mathbb{R}^n$ direction and isotope it to get an embedding disjoint from $[a_j, a_{j+1}] \times L$, and then extend this to $e_i|_{[a_j, a_{j+1}]}$ as in Proposition 6.14.

This gives the required embeddings e_i. The bundle part of the data consists of an extendible (c.f. Definition 5.10) θ-structure ℓ_i on $\Lambda_i \times K$ for each i which agrees with ℓ on $\Lambda_i \times K|_{(-6,-2)}$, and such that the effect of the θ-surgery described by this data (i.e. the restriction of ℓ to $K|_{(-6,0]}$) lies in $\mathcal{A}$. We will describe a construction which for each $\lambda \in \Lambda_i$ produces a θ-structure $\ell = \ell_{i,\lambda}$ on $K \subset \mathbb{R}^{n+1} \times \mathbb{R}^n$.

Firstly, there is a unique θ-structure on the subspace

$$K|_{(-6,-2)} = \big((-6,-2) \times \mathbb{R}^n\big) \times S^{n-1}, \tag{6.4}$$

such that the embedding e_i preserves θ-structures (i.e. satisfies requirement in Definition 5.12). Secondly, the manifold

$$K|_{(-6,0]} \subset \mathbb{R}^{n+1} \times \mathbb{R}^n \tag{6.5}$$

is obtained from (6.4) by attaching an n-handle. To extend the θ-structure requires a null-homotopy of the structure map $S^{n-1} \to B$ from the θ-structure on (6.4), and this is provided as part of the θ-surgery data in Lemma 6.17. Finally, we need to prove that this structure extends to an extendible structure over all of K. To see this, we observe that the restriction of this structure to the subspace

$$K|_{(-6,0]} - (B_2^{n+1}(0) \times \mathbb{R}^n) = \big((-6,0] \times \mathbb{R}^n - B_2^{n+1}(0)\big) \times S^{n-1} \tag{6.6}$$

admits a (homotopically unique) extension to the manifold $\big(\mathbb{R} \times \mathbb{R}^n\big) \times S^{n-1}$, since this deformation retracts to (6.6). Restrict this extension to the manifold $\big(\mathbb{R}\times\mathbb{R}^n - B_2^{n+1}(0)\big) \times S^{n-1} = K - (B_2^{n+1}(0) \times \mathbb{R}^n)$ and glue it with the structure on (6.5), to get a θ-structure on

$$K|_{(-\infty,0]} \cup (K - B_2^{n+1}(0) \times \mathbb{R}^n). \tag{6.7}$$

It remains to prove that this structure can be extended to all of K. It is easy to see that the stabilised structure extends to a bundle map $\varepsilon^1 \oplus TK \to \theta^*\gamma$, but then Lemma 5.4 implies that the unstabilised bundle map also extends. □

7. Applying the group completion theorem

In the following we shall work entirely in even dimension $d = 2n$. Our results are valid whenever $2n \neq 4$, but we shall assume $2n > 4$. The case $2n = 2$ is covered in [GRW10], and the case $2n = 0$ is classical. For convenience, we recall the setup from the introduction. From now on we shall always set $N = \infty$.

Definition 7.1. Let $2n > 4$ and let $\theta : B \to BO(2n)$ be a map such that any θ-structure on D^{2n} extends to one on S^{2n}.

Let P be a closed $(2n-1)$-dimensional manifold with θ-structure $\ell_P : \varepsilon^1 \oplus TP \to \theta^*\gamma$ such that $\ell_P : P \to B$ is $(n-1)$-connected. Let $f : P \to [0, 2n-1]$ be a self-indexing Morse function and let $L = f^{-1}([0, n-.5])$. Then $\ell_P|_L : L \to B$ is also $(n-1)$-connected.

If we pick a collared embedding $L \subset (-1/2, 0] \times (-1,1)^{\infty-1}$, we have defined a category $\mathcal{C}_{\theta,L}^{n-1,n-2}$. Let $\mathcal{A} \subset \pi_0(\mathrm{ob}(\mathcal{C}_{\theta,L}^{n-1,n-1}))$ be the set of objects (M,ℓ) for which $M - \mathrm{int}(L)$ is diffeomorphic to a handlebody with handles of dimension $\leq n-1$ only. This category is denoted $\mathcal{C}_{\theta,L}^{n-1,\mathcal{A}}$. A morphism in $\mathcal{C}_{\theta,L}^{n-1,\mathcal{A}}$ from P_0 to P_1 is a manifold $W \subset [0,t] \times (-1,1)^\infty$ with $W \cap x_2^{-1}((-\infty,0]) = [0,t] \times L$. We shall write

$$W^\circ = W - [0,t] \times \mathrm{int}(L)$$

for morphisms and similarly

$$P^\circ = P - \mathrm{int}(L)$$

for objects. Morphisms or objects $M \in \mathcal{C}_{\theta,L}^{n-1,\mathcal{A}}$ are completely determined by M°.

We denote by $\mathcal{C}$ the category with

$$\mathrm{ob}(\mathcal{C}) = \{M^\circ \mid M \in \mathrm{ob}(\mathcal{C}_{\theta,L}^{n-1,\mathcal{A}})\}$$
$$\mathrm{mor}(\mathcal{C}) = \{W^\circ \mid W \in \mathrm{mor}(\mathcal{C}_{\theta,L}^{n-1,\mathcal{A}})\}$$

made into a topological category by insisting that the functor $\mathcal{C}_{\theta,L}^{n-1,\mathcal{A}} \to \mathcal{C}$ given by $X \mapsto X^\circ$ is a homeomorphism on the spaces of objects and morphisms.

Our work in Sections 3, 4, 5 and 6 determines the homotopy type of the space $\Omega B\mathcal{C}_{\theta,L}^{n-1,\mathcal{A}}$, as follows.

Theorem 7.2. *In the setup of Definition 7.1, there is a weak equivalence*

$$\Omega B\mathcal{C}_{\theta,L}^{n-1,\mathcal{A}} \simeq \Omega^\infty MT\theta,$$

where $MT\theta$ is the Thom spectrum associated to $\theta : B \to BO(2n)$.

Proof. First, since the statement only involves the loop space of $B\mathcal{C}_{\theta,L}^{n-1,\mathcal{A}}$, we see that we may replace $\mathcal{A}$ by the set of all objects for which $M - \mathrm{int}(L)$ is diffeomorphic to a handlebody with handles of dimension $\leq n-1$ only, together with all objects which are *not* bordant to an object of that type (adding those objects only adds some extra path components to $B\mathcal{C}_{\theta,L}^{n-1,\mathcal{A}}$). Theorem 5.2 then applies: by Proposition 5.6 the tangential structure θ is reversible, and we have supposed that $L \to B$ is $(n-1)$-connected. This implies that the inclusion induces a weak equivalence $B\mathcal{C}_{\theta,L}^{n-1,\mathcal{A}} \to B\mathcal{C}_{\theta,L}^{n-1,n-2}$. We compose this with the string of weak equivalences

$$B\mathcal{C}_{\theta,L}^{n-1,n-2} \simeq B\mathcal{C}_{\theta,L}^{n-1} \simeq B\mathcal{C}_{\theta,L} \simeq \psi_{\theta,L}(\infty,1) \simeq \psi_\theta(\infty,1)$$

obtained by applying Theorem 4.1 $(n-1)$ times, Theorem 3.1 n times, Proposition 2.15 and Proposition 2.16, respectively. Finally, we compose this with the weak equivalence

$$\psi_\theta(\infty,1) \simeq \Omega^{\infty-1} MT\theta$$

from [GRW10, Theorem 3.12]. □

This theorem also identifies the homotopy type of $\Omega B\mathcal{C}$. From now on we will work with this category, and need a lemma to translate what the connectivity conditions in $\mathcal{C}_{\theta,L}^{n-1,\mathcal{A}}$ mean when we think of L as being cut out.

Lemma 7.3. *The objects of $\mathcal{C}$ are the compact $(2n-1)$-manifolds obtained from ∂L by attaching n-handles and above. The morphisms in $\mathcal{C}$ are the cobordisms relative to ∂L which are $(n-1)$-connected relative to either of their boundaries.*

Consequently, a morphism W in $\mathcal{C}_{\theta,L}^{n-1,\mathcal{A}}$ is also $(n-1)$-connected relative to its incoming boundary.

Proof. The objects of $\mathcal{C}$ have this description by definition, as $\mathcal{A}$ consists of manifolds M which can be obtained from L by attaching n-handles and above, so $M - \mathrm{int}(L)$ can be obtained from ∂L by attaching n-handles and above.

In $\mathcal{C}_{\theta,L}^{n-1,\mathcal{A}}$ the morphisms are cobordisms which are $(n-1)$-connected relative to their outgoing boundary. Let $W : M \rightsquigarrow N$ be such a cobordism, and observe that W deformation retracts to $W^\circ \cup N$. Thus N is obtained from L by attaching n-cells and above, W is obtained from N by attaching n-cells and above, N° is obtained from ∂L by attaching n-cells and above and W is obtained from W° by attaching n-cells and above. As $n \geq 3$, it follows that all the maps

$$\begin{tikzcd}
\partial L \arrow[r] \arrow[drr] & L \arrow[r] & N \arrow[r] & W \\
& & N^\circ \arrow[r] \arrow[u] & W^\circ \arrow[u]
\end{tikzcd}$$

induce isomorphisms on fundamental groups, and we write π for the common fundamental group. There are isomorphisms

$$H_*(W, N; \mathbb{Z}[\pi]) \cong H_*(W^\circ \cup N, N; \mathbb{Z}[\pi]) \cong H_*(W^\circ, N^\circ; \mathbb{Z}[\pi])$$

given by the equivalence $W^\circ \cup N \simeq W$ and excision of $\mathrm{int}(L)$ respectively, and so $H_*(W^\circ, N^\circ; \mathbb{Z}[\pi]) = 0$ for $* \leq n-1$. Hence $N^\circ \to W^\circ$ is $(n-1)$-connected.

We have that $\partial W^\circ = M^\circ \cup N^\circ$, so the long exact sequence on homotopy for the triple $(W^\circ, \partial W^\circ, N^\circ)$ shows that $(W^\circ, \partial W^\circ)$ is also $(n-1)$-connected, and then the long exact sequence on homotopy for the triple $(W^\circ, \partial W^\circ, M^\circ)$ shows that (W°, M°) is $(n-1)$-connected.

Finally the claim about morphisms W in $\mathcal{C}_{\theta,L}^{n-1,\mathcal{A}}$ follows by noting that W° is $(n-1)$-connected relative to its incoming boundary and that $W \simeq M \cup W^\circ$. $\square$

Definition 7.4. For later use, let us also note two distinguished objects of $\mathcal{C}$ associated to the Morse function $f : P \to [0, 2n-1]$. Firstly, let us extend the embedding $L \to (-1,0] \times (-1,1)^{\infty-1}$ to an embedding $P \to (-1,1)^\infty$ with $P \cap ((-1,0]\times(-1,1)^{\infty-1}) = L$ and denote the resulting objects $P \in \mathcal{C}_{\theta,L}^{n-1,\mathcal{A}}$ and $P^\circ \in \mathcal{C}$.

Secondly, recall from the proof of Proposition 2.16 that we constructed a θ-manifold $D(L)$ which is diffeomorphic to the double of L. This contains $L \subset D(L)$ with its standard θ-structure, whose complement is denoted $\overline{L}$. As L has a handle structure with handles of index at most $(n-1)$, $D(L)$ can be obtained from L by attaching n-handles and above. We extend the embedding of L to an embedding $D(L) \to (-1,1)^\infty$ and denote the resulting objects $D(L) \in \mathcal{C}_{\theta,L}^{n-1,\mathcal{A}}$ and $D(L)^\circ = \overline{L} \in \mathcal{C}$.

The relevance of the category $\mathcal{C}$ to Theorem 1.8 is evident from the following proposition.

Proposition 7.5. *The space of morphisms $\mathcal{C}(\overline{L}, P^\circ) \cong \mathcal{C}_{\theta,L}^{n-1,\mathcal{A}}(D(L), P)$ is homotopy equivalent to $\mathcal{N}^\theta(P, \ell_P)$. Furthermore, if $K : P \rightsquigarrow P'$ is a θ-cobordism containing $L \times [0,1]$, then the maps*

$$\mathcal{C}(\overline{L}, P^\circ) \xrightarrow{-\circ K^\circ} \mathcal{C}(\overline{L}, (P')^\circ) \qquad \mathcal{N}^\theta(P, \ell_P) \xrightarrow{-\circ K} \mathcal{N}^\theta(P', \ell_{P'})$$

correspond under this equivalence.

Proof. Composition with the θ-cobordism $V : \emptyset \rightsquigarrow D(L)$ constructed in the proof of Proposition 2.16 gives a continuous map

$$\varphi_P : \mathcal{C}(\overline{L}, P^\circ) \cong \mathcal{C}_{\theta,L}^{n-1,\mathcal{A}}(D(L), P) \xrightarrow{V\circ -} \mathcal{C}_\theta^{n-1}(\emptyset, P) \cong \mathcal{N}^\theta(P, \ell_P)$$

and it is clear that the square

$$\begin{tikzcd}
\mathcal{C}(\overline{L}, P^\circ) \arrow[r, "-\circ K^\circ"] \arrow[d, "\varphi_P"'] & \mathcal{C}(\overline{L}, (P')^\circ) \arrow[d, "\varphi_{P'}"'] \\
\mathcal{N}^\theta(P, \ell_P) \arrow[r, "-\circ K"] & \mathcal{N}^\theta(P', \ell_{P'})
\end{tikzcd}$$

commutes. Thus we must just show that φ_P is a homotopy equivalence. To do this, consider first the trivial cobordism $P \times [0,1]$. This contains $L \times [1/4, 3/4]$, which is diffeomorphic to V and has a homotopic θ-structure. Cutting this out gives a θ-cobordism $P \sqcup D(L) \rightsquigarrow P$ containing $L \times I$. Composition with the incoming P of this θ-cobordism defines a continuous map

$$\mathcal{N}^\theta(P, \ell_P) \cong \mathcal{C}_\theta^{n-1}(\emptyset, P) \longrightarrow \mathcal{C}_{\theta,L}^{n-1}(D(L), P) = \mathcal{C}_{\theta,L}^{n-1,\mathcal{A}}(D(L), P) \cong \mathcal{C}(\overline{L}, P^\circ)$$

which is homotopy inverse to φ_P. $\square$

To state the following theorem neatly, it is useful to make sense of the notion of composing a countable sequence of composable morphisms in the category $\mathcal{C}$. (Of course, the result is no longer a morphism in this category.) Let $K|_0, K|_1, \ldots$ be a sequence of objects of $\mathcal{C}$, and $K|_{[i-1,i]} : K|_{i-1} \to K|_i$ be a sequence of morphisms. For $0 \leq a \leq b$, let us write

$$K|_{[a,b]} = K|_{[a,a+1]} \circ K|_{[a+1,a+2]} \circ \cdots \circ K|_{[b-1,b]}$$

for the composition of the morphisms from $K|_a$ to $K|_b$. There are natural inclusions $K|_{[0,a]} \subset K|_{[0,a+1]} \subset \cdots$ and we let K denote the union: a non-compact smooth manifold with θ-structure. The symbol $K|_{[a,b]}$ is not ambiguous, and we can also make sense of $K|_{[a,\infty)} = \cup_{b>a} K|_{[a,b]}$.

Theorem 7.6 (Group completion)**.** *Let $2n > 4$ and fix a map $\theta : B \to BO(2n)$ such that any θ-structure on D^{2n} extends to S^{2n}. Fix a $(2n-1)$-manifold with boundary L with θ-structure so that $L \to B$ is $(n-1)$-connected.*

Let

$$K|_0 \xrightarrow{K|_{[0,1]}} K|_1 \xrightarrow{K|_{[1,2]}} K|_2 \xrightarrow{K|_{[2,3]}} K|_3 \xrightarrow{K|_{[3,4]}} \cdots$$

be a sequence of composable morphisms in $\mathcal{C}$ such that

(i) For each $i \geq 0$, $\pi_n(K|_{[i,\infty)}) \to \pi_n(B)$ is surjective.

(ii) For each $i \geq 0$, the subgroup

$$\operatorname{Ker}\left(\pi_{n-1}(K|_i) \to \pi_{n-1}(K|_{[i,\infty)})\right)$$

of $\pi_{n-1}(K|_i)$ contains $\operatorname{Ker}(\pi_{n-1}(K|_i) \to \pi_{n-1}(B))$.

(iii) For each $i \geq 0$, each path-component of the manifold $K|_{[i,\infty)}$ contains a submanifold diffeomorphic to $S^n \times S^n - \operatorname{int}(D^{2n})$, which in addition has null-homotopic structure map to B.

Then there is a map

$$\operatorname*{hocolim}_{i\to\infty} \mathcal{N}^\theta(K|_i \cup L, \ell_{K|_i} \cup \ell_L) \simeq \operatorname*{hocolim}_{i\to\infty} \mathcal{C}(\overline{L}, K|_i) \longrightarrow \Omega B\mathcal{C}$$

which is a homology equivalence.

The proof of this theorem will be based on Proposition 7.7 below. Let $F(i) : \mathcal{C} \to \mathbf{Top}$ denote the representable functor $\mathcal{C}(-, K|_i)$ and $\eta(i+1) : F(i) \to F(i+1)$ be the natural transformation given by right composition with $K|_{[i,i+1]}$. Let $F_\infty : \mathcal{C} \to \mathbf{Top}$ denote the (objectwise) homotopy colimit of these functors.

Proposition 7.7. *Given the assumptions of Theorem 7.6, the functor F_∞ sends each morphism in $\mathcal{C}$ to a homology equivalence.*

Given this proposition, by [GMTW09, Theorem 7.1] and the discussion following it, the pull-back square

$$\begin{tikzcd} F_\infty(\overline{L}) \arrow[r] \arrow[d] & B(\mathcal{C} \wr F_\infty) \simeq \operatorname{hocolim}_i B(\mathcal{C} \wr F(i)) \simeq * \arrow[d] \\ \{\overline{L}\} \arrow[r] & B\mathcal{C} \end{tikzcd}$$

is homology cartesian and so $F_\infty(\overline{L}) \to \Omega B\mathcal{C}$ is a homology equivalence, which establishes Theorem 7.6. To prove Proposition 7.7, we first make some preparations.

Lemma 7.8. *Let $W : N \rightsquigarrow M$ be a morphism in $\mathcal{C}$. After composing with an endomorphism of M or N in $\mathcal{C}$, the cobordism W admits a handle structure with only handles of index n.*

Proof. The pairs (W, M) and (W, N) are both $(n-1)$-connected, so we proceed as in the proof of Lemma 6.17 to try find a Morse function on W with only critical points of index n. As we saw in that lemma this is not always possible if $\pi_1(L) \neq 0$, but becomes possible after connect-sum with sufficiently many copies of $S^n \times S^n$. In this situation we realise this by composing with a multiple of $(N \times [0,1]) \# S^n \times S^n$ or $(M \times [0,1]) \# S^n \times S^n$ (using as in Lemma 6.17 that $S^n \times S^n$ admits a θ-structure and that we can connect-sum θ-manifolds.) $\square$

If F_∞ sends both $W \circ (M \overset{F}{\rightsquigarrow} M)$ and $(N \overset{G}{\rightsquigarrow} N) \circ W$ to homology isomorphisms, then it sends W to one. Thus it is enough to prove that F_∞ sends each cobordism W which has a handle structure with only n-handles to an isomorphism, and by factoring this cobordism into single handle attachments it is enough to prove it in the case that W has a handle structure with a single n-handle. The cobordism W gives a map of direct systems

$$\begin{tikzcd}
\mathcal{C}(M, K|_0) \arrow[r] \arrow[d, "W\circ-"'] & \mathcal{C}(M, K|_1) \arrow[r] \arrow[d, "W\circ-"'] & \mathcal{C}(M, K|_2) \arrow[r] \arrow[d, "W\circ-"'] & \mathcal{C}(M, K|_3) \arrow[r] \arrow[d, "W\circ-"'] & \cdots \\
\mathcal{C}(N, K|_0) \arrow[r] & \mathcal{C}(N, K|_1) \arrow[r] & \mathcal{C}(N, K|_2) \arrow[r] & \mathcal{C}(N, K|_3) \arrow[r] & \cdots .
\end{tikzcd}$$

Lemmas 7.9 and 7.10 show that this map induces an injection and surjection respectively on the homology of the homotopy colimits.

Lemma 7.9. *For each i there is a $k \geq i$ and a factorisation*

$$\begin{tikzcd}
\mathcal{C}(M, K|_i) \arrow[r, "-\circ K|_{[i,k]}"] \arrow[d, "W\circ-"'] & \mathcal{C}(M, K|_k) \\
\mathcal{C}(N, K|_i) \arrow[ur, dotted] &
\end{tikzcd}$$

up to homotopy.

Proof. By Proposition 7.5, we can replace the solid part of the diagram by

$$\mathcal{N}^\theta(\overline{N} \cup_{\partial L} K|_i) \overset{W\cup \mathrm{Id}}{\longleftarrow} \mathcal{N}^\theta(\overline{M} \cup_{\partial L} K|_i) \overset{\mathrm{Id}\cup K|_{[i,k]}}{\longrightarrow} \mathcal{N}^\theta(\overline{M} \cup_{\partial L} K|_k),$$

where we have omitted the θ-structure from the notation, and the notation $\overline{X}$ denotes the θ-manifold $D(X) - X$ for a $(2n-1)$-dimensional θ-manifold X. Note a θ-cobordism $U : X \rightsquigarrow Y$ can be interpreted as a θ-cobordism $U : \overline{Y} \rightsquigarrow \overline{X}$, using the canonical null-bordisms of $D(X)$ and $D(Y)$.

It is enough to show that there is an embedding of the cobordism $W \cup \mathrm{Id}_{K|_i}$ into $\mathrm{Id}_{\overline{M}} \cup K|_{[i,k]}$, where the restriction of the θ-structure is isomorphic to the standard one, all relative to $\overline{M} \cup_{\partial L} K|_i$. We may choose k as large as we like, so may as well assume that $K|_{[i,k]}$ is the non-compact manifold $K|_{[i,\infty)}$.

The manifold $W \cup \mathrm{Id}_{K|_i}$ is obtained by attaching an n-handle along $S^{n-1} \times D^n \hookrightarrow \overline{M} \subset \partial K|_{[i,\infty)}$, so we need to find an extension of this embedding into $K|_{[i,\infty)}$ (with the correct θ-structure). The map

$$S^{n-1} \times D^n \longrightarrow \overline{M} \subset \partial K|_{[i,\infty)} \longrightarrow K|_{[i,\infty)}$$

is null-homotopic by assumption (ii): it is certainly null-homotopic when composed with $K|_{[i,\infty)} \to B$, because then it factors as $S^{n-1} \times D^n \to D^n \times D^n \to W \to B$. Thus there is a continuous map $f : W \to K|_{[i,\infty)}$ relative to $\overline{M}$. Furthermore, as $\pi_n(K|_{[i,\infty)}) \to \pi_n(B)$ is surjective by assumption (i), we can change f by adding on elements of $\pi_n(K|_{[i,\infty)})$ so that

$$W \overset{f}{\longrightarrow} K|_{[i,\infty)} \overset{\ell_{K|_{[i,\infty)}}}{\longrightarrow} B$$

is homotopic relative to $\overline{M}$ to ℓ_W. Let us write $\hat{f}$ for the map $(D^n, \partial D^n) \to (K|_{[i,\infty)}, \partial K|_{[i,\infty)})$ given by f restricted to the core of the n-handle.

There is an isomorphism $TW \cong f^* TK|_{[i,\infty)}$ relative to $\overline{M}$ coming from

$$TW \cong \ell_W^* \theta^* \gamma \cong f^* \ell_{K|_{[i,\infty)}}^* \theta^* \gamma \quad \text{and} \quad TK \cong \ell_{K|_{[i,\infty)}}^* \theta^* \gamma,$$

so $\hat{f}$ has the data of a formal immersion relative to its boundary. By Smale–Hirsch theory we may homotope $\hat{f}$ so that it is an actual immersion, keeping it fixed on the boundary, and further perturb it so that it intersects itself transversely. We must now explain how to change $\hat{f}$ to remove self-intersections, and will do so by actually changing the map, and not keeping it within its homotopy class.

Around a self-intersection point, choose a coordinate $\mathbb{R}^n \times \mathbb{R}^n \hookrightarrow K|_{[i,\infty)}$ so that $\mathbb{R}^n \times \{0\}$ and $\{0\} \times \mathbb{R}^n$ give local coordinates around the two preimages of the double point. By assumption we can find a $S^n \times S^n - \mathrm{int}(D^{2n}) \subset K|_{[i,\infty)}$ with null-homotopic map to B. We choose an embedded path from this $S^n \times S^n - \mathrm{int}(D^{2n})$ to the patch $\mathbb{R}^n \times \mathbb{R}^n$, and thicken it up: inside this we have a subset diffeomorphic to the boundary connect sum

$$(D^n \times D^n) \natural (S^n \times S^n - \mathrm{int}(D^{2n})),$$

which the image of $\hat{f}$ intersects in $D^n \times \{0\} \cup \{0\} \times D^n$. Inside this subset there are embedded disjoint discs which give the same embedding on the boundary, and we can modify $\hat{f}$ by redefining it to have these discs as image instead. This reduces by 1 the number of geometric self-intersections of $\hat{f}$, and up to homotopy we have added an element of $\pi_n(S^n \times S^n - \mathrm{int}(D^{2n})$ to the homotopy class of $\hat{f}$. As $S^n \times S^n - \mathrm{int}(D^{2n}) \to K|_{[i,\infty)} \to B$ was null-homotopic, we have not changed the homotopy class of $\hat{f}$ in B.

After finitely-many steps, we have changed $\hat{f}$ to an embedding. The corresponding embedding $f : W \to K|_{[i,\infty)}$ (obtained by thickening $\hat{f}$ up again) is homotopic to the original one after composing with $K|_{[i,\infty)} \to B$, so has $\ell_{K|_{[i,\infty)}} \circ f \simeq \ell_W$ relative to $\overline{M}$. Hence it is (isotopic to) an embedding of the θ-manifold W in $K|_{[i,\infty)}$ relative to $\overline{M}$. $\square$

Lemma 7.10. *For each i there is a $k \geq i$ and a factorisation*

$$\begin{tikzcd} & \mathcal{C}(M, K|_k) \arrow[d, "W\circ -"] \\ \mathcal{C}(N, K|_i) \arrow[ur, dotted] \arrow[r, "-\circ K|_{[i,k]}"'] & \mathcal{C}(N, K|_k) \end{tikzcd}$$

up to homotopy.

Proof. This proposition is essentially the same as the last, but a little complicated by the fact that when we make k larger we are changing the vertical map also. By Proposition 7.5, we can replace the solid part of the diagram by

$$\mathcal{N}^\theta(\overline{N} \cup_{\partial L} K|_i) \overset{\mathrm{Id}\cup K|_{[i,k]}}{\longrightarrow} \mathcal{N}^\theta(\overline{N} \cup_{\partial L} K|_k) \overset{W \cup \mathrm{Id}}{\longleftarrow} \mathcal{N}^\theta(\overline{M} \cup_{\partial L} K|_k),$$

where we have omitted the θ-structure from the notation. It is enough to show that there is an embedding of the cobordism $W \cup \mathrm{Id}_{K|_k}$ into $\mathrm{Id}_M \cup K|_{[i,k]}$, where the restriction of the θ-structure is isomorphic to the standard one, all relative to $\overline{M} \cup_{\partial L} K|_k$. As we may choose k as large as we like, this is the same as finding an embedding of W into

$$(K|_{[i,i+1]} \cup \overline{N} \times [0,1]) \circ K|_{[i+1,\infty)}$$

sending the incoming boundary $\overline{N}$ to $\overline{N} \times \{1\}$. As W is obtained from $\overline{N}$ by attaching a single n-handle, this problem is identical with that of the previous proposition, and can be solved by the same technique. $\square$

These lemmas do *not* prove that F_∞ sends each morphism in $\mathcal{C}$ to a weak homotopy equivalence, since the dotted maps we constructed in no sense preserve basepoints. The case $n = 0$ gives rise to the following example from [MS76]: we have $F_\infty(\emptyset) \simeq \mathbb{Z} \times B\Sigma_\infty$ and the morphism $1 : \emptyset \rightsquigarrow \emptyset$ given by a single point induces the shift map on Σ_∞, that is, the map induced hy the self-embedding given by $\{1, 2, ...\} \cong \{2, 3, ...\} \hookrightarrow \{1, 2, ...\}$. This is not surjective, so the map is not a homotopy equivalence; it is however a homology equivalence, by the argument we have presented.

7.1. **Proof of Theorem 1.8.** In the situation of Theorem 1.8 we have a θ-manifold (K, ℓ_K) and a proper map $x_1 : K \to [0, \infty)$ with the integers as regular values. The θ-manifold $K|_0$ is identified with P. Let $\theta' : B' \to B \xrightarrow{\theta} BO(2n)$ be obtained as the $(n-1)$st stage of the Moore–Postnikov tower of $\ell_P : P \to B$.

Lemma 7.11. *The natural map*

$$\mathcal{N}^{\theta'}(P, \ell'_P) \longrightarrow \mathcal{N}^{\theta}(P, \ell_P)$$

is a homotopy equivalence, and the θ-structure on K lifts to a θ'-structure starting with (P, ℓ'_P).

Proof. Let W be a null-bordism of P which is $(n-1)$-connected relative to P, so up to homotopy is obtained by attaching cells of dimension at least n. The homotopy fibre of $B' \to B$ is an $(n-2)$-type, so the space of lifts

$$\begin{tikzcd} P \arrow[r, "\ell'_P"] \arrow[d, hook] & B' \arrow[d] \\ W \arrow[r] \arrow[ur, dotted] & B \end{tikzcd}$$

contractible. Thus $\mathrm{Bun}_\partial(W, (\theta')^*\gamma) \to \mathrm{Bun}_\partial(W, \theta^*\gamma)$ is a homotopy equivalence, and so the same is true after taking the Borel construction for the $\mathrm{Diff}(W, P)$-action.

For the second part, note that each pair $(K|_{[i,i+1]}, K|_i)$ is $(n-1)$-connected, so up to homotopy $K|_{[i,i+1]}$ is constructed from $K|_i)$ by attaching n-cells and above. The same obstruction-theory argument as above shows that any θ'-structure on $K|_i$ extends uniquely to one on $K|_{[i,i+1]}$. □

By this lemma we may now work exclusively with $\mathcal{N}^{\theta'}(P, \ell'_P)$, and suppose that we have a θ'-manifold (K, ℓ'_K) which starts at (P, ℓ'_P). We choose a self-indexing Morse function $f : P \to [0, 2n-1]$ and let $L = f^{-1}[0, n-.5]$. As in Definition 7.1 we obtain a category $\mathcal{C}^{n-1,\mathcal{A}}_{\theta',L}$ where $\mathcal{A}$ denotes the set of objects which admit a handle structure relative to L using only n-handles and above.

Lemma 7.12. *For each $i \geq 1$ there are isomorphisms of each θ'-manifold $K|_i$ to manifolds $\overline{K}|_i$ containing L as a θ'-submanifold, so that $\overline{K}|_i$ is obtained from L by attaching handles of dimension at least n. Furthermore, there are isomorphisms of each $K|_{[i-1,i]}$ to cobordisms $\overline{K}|_{[i-1,i]}$ containing $L \times [0,1]$ as a θ'-submanifold, restricting to the isomorphism to $\overline{K}|_i$ at the ends.*

Proof. Suppose we have done so for all $j < i$, so in particular $K|_{i-1} \cong \overline{K}|^\circ_{i-1} \cup L$ is given. The cobordism $K_{[i-1,i]} : \overline{K}|_{i-1} \rightsquigarrow K|_i$ is $(n-1)$-connected relative to either end, and so has a handle structure with only handles of index at most n. The spheres in $\overline{K}|_{i-1}$ along which these are attached have dimension at most $(n-1)$, and so can be isotoped into $\overline{K}|^\circ_{i-1}$. This produces an isomorphism $K|_{[i-1,i]} \cong \overline{K}|^\circ_{[i-1,i]} \cup (L \times [i-1, i])$ of θ'-manifolds relative to $\overline{K}|_{i-1}$, as required, which we use to define $K|_i \cong \overline{K}|^\circ_i \cup L$.

All that is left to show is that $\overline{K}|_i$ is obtained from L by attaching n handles and above, or equivalently that $\overline{K}|^\circ_i$ is obtained from ∂L by attaching n handles and above. The manifold $\overline{K}|^\circ_{i-1}$ is, by assumption, and the cobordism $\overline{K}|^\circ_{[i-1,i]}$ is relative to ∂L and $(n-1)$-connected relative to either end so may be obtained from $\overline{K}|^\circ_{i-1}$ by attaching n-handles and above. Thus the manifold $\overline{K}|^\circ_i$ is obtained by $(n-1)$-surgery and above on the interior of $\overline{K}|^\circ_{i-1}$, so is still obtained from ∂L by attaching n-handles and above. □

As in the above proof, we denote by $\overline{K}|_i^\circ$ the complement of $\mathrm{int}(L)$ in $\overline{K}|_i$, and by $\overline{K}|_{[i-1,i]}^\circ$ the complement of $[i-1,i]\times L$ in $\overline{K}|_{[i-1,i]}$. By Lemma 7.12, the $\overline{K}|_{[i-1,i]}^\circ$ give a sequence of composable morphisms in $\mathcal{C}$, and the assumptions of Theorem 1.8 make this sequence satisfy the assumptions of Theorem 7.6. This identifies the homotopy colimit of the direct system

$$\mathcal{C}(\overline{L},P^\circ)\xrightarrow{\overline{K}|_{[0,1]}^\circ}\mathcal{C}(\overline{L},\overline{K}|_1^\circ)\xrightarrow{\overline{K}|_{[1,2]}^\circ}\mathcal{C}(\overline{L},\overline{K}|_2^\circ)\xrightarrow{\overline{K}|_{[2,3]}^\circ}\mathcal{C}(\overline{L},\overline{K}|_3^\circ)\xrightarrow{\overline{K}|_{[3,4]}^\circ}\cdots$$

as being homology equivalent to $\Omega B\mathcal{C}$, and hence by Theorem 7.2 homology equivalent to $\Omega^\infty MT\theta'$. On the other hand, it is also equivalent to the direct system

$$\mathcal{N}^\theta(P,\ell_P)\xrightarrow{K|_{[0,1]}}\mathcal{N}^\theta(K|_1,\ell_K|_1)\xrightarrow{K|_{[1,2]}}\mathcal{N}^\theta(K|_2,\ell_K|_2)\xrightarrow{K|_{[2,3]}}\cdots,$$

which proves Theorem 1.8.

7.2. **Proof of Theorem 1.6.** We let (W,ℓ_W) be a θ-null-bordism of (P,ℓ_P) which is $(n-1)$-connected relative to P, and consider the sequence of composable θ-cobordisms

$$\emptyset\xrightarrow{(W,\ell_W)}(P,\ell_P)\xrightarrow{(K,\ell_K)}(P,\ell_P)\xrightarrow{(K,\ell_K)}(P,\ell_P)\xrightarrow{(K,\ell_K)}\cdots.$$

Lemma 7.13. *If a cobordism $K : P \rightsquigarrow P$ is $(n-1)$-connected relative to its outgoing boundary, it is also $(n-1)$-connected relative to its incoming boundary. Furthermore, $K\circ K$ is obtained from K by attaching trivial n-cells, as well as $(n+1)$-cells and above.*

Proof. We will show that such a K is isomorphic to a morphism in the category $\mathcal{C}_{\theta,L}^{n-1,\mathcal{A}}$, so the first part of the claim then follows from Lemma 7.3.

Choose a decomposition $P = L\cup P^\circ$ into codimension zero submanifolds each of which has a handle structure with at most $(n-1)$-dimensional handles (as in Definition 7.4). As K is $(n-1)$-connected relative to P_{out}, it can be obtained from that boundary by attaching handles of index at least n, and so it can be obtained from P_{in} by attaching handles of index at most n. As L is constructed from ∂L by attaching n-handles and above, we can isotope the attaching maps of the handles for K to lie in $(P_{in})^\circ$, and so find an isomorphism of θ-cobordisms of K with $(W : P^\circ \rightsquigarrow P_1^\circ)\cup(L\times[0,1] : L\rightsquigarrow L)$.

For the second part of the claim, consider $f : S^{n-1}\to P_{out}\subset K$ the attaching map for a relative n-cell of (K,P_{in}). As $P_{out} = P^\circ\cup L$ is obtained from L by attaching n-handles and above, the map f may be homotoped to lie in L. But the inclusion of L in K also factors as $L\subset P_{in}\subset K$ and so f is null-homotopic in K, as required. ☐

We choose a height function $x_1 : K\to[0,1]$, and let ∞K denote the composition of countably many copies of K, with $x_1 : \infty K\to[0,\infty)$ given by the translation of all the standard height functions. We let $\theta' : B'\to B\xrightarrow{\theta} BO(2n)$ be obtained as the nth stage of the Moore–Postnikov tower for $\ell_K : K\to B$, and call the *standard* θ'-structure ℓ'_P on P the one obtained as the incoming boundary of K.

Note that as a θ'-cobordism, (K,ℓ'_K) is no longer an endomorphism of (P,ℓ'_P), but a morphism from (P,ℓ'_P) to a $(P,\ell'_K|_1)$ (which is isomorphic to P as a θ-manifold, but not necessarily as a θ'-manifold).

Lemma 7.14. *The θ-structure on W lifts to a θ'-structure ℓ'_W which is the standard one on P, and the θ-structure on ∞K lifts to a θ'-structure $\ell'_{\infty K}$ which is the standard one on P. The map*

$$B\mathrm{Diff}^{\theta'}(W\cup gK,\ell'_{\infty K}|_g)_{\ell'_W\cup\ell'_{\infty K}|_{[0,g]}}\longrightarrow B\mathrm{Diff}^{\theta}(W\cup gK,\ell_P)_{\ell_W\cup g\ell_K}$$

is a homotopy equivalence.

Proof. Let the map $B' \to B$ has homotopy fibre F. Then $\pi_*(F) = 0$ when $* \geq n$, as B' is the nth stage of the Moore–Postnikov tower for $\ell_K : K \to B$. Let M be a manifold obtained from P by attaching n-cells and above. Then obstruction theory shows that the map

$$\mathrm{Bun}^\partial(TM, (\theta')^*\gamma; \ell'_P) \longrightarrow \mathrm{Bun}^\partial(TM, \theta^*\gamma; \ell_P)$$

is a homotopy equivalence onto those path-components it hits. Furthermore, the path-component of a map $\ell : M \to B$ is hit if and only if for each attaching map $f : S^{n-1} \to P$ of an n-cell of M, the element $\ell'_P \circ f \in \pi_{n-1}(B')$ is zero: equivalently, if

$$\mathrm{Ker}(\pi_{n-1}(P) \to \pi_{n-1}(M)) \subset \mathrm{Ker}(\ell'_P : \pi_{n-1}(P) \to \pi_{n-1}(B')).$$

For the pair (W, P), we have by the second assumption of Theorem 1.6 that

$$\mathrm{Ker}(\pi_{n-1}(P) \to \pi_{n-1}(W)) \subset \mathrm{Ker}(\pi_{n-1}(P) \to \pi_{n-1}(K)),$$

so this kernel also lies in $\mathrm{Ker}(\ell'_P : \pi_{n-1}(P) \to \pi_{n-1}(B'))$ as $\pi_{n-1}(K) \to \pi_{n-1}(B')$ is an isomorphism. Thus ℓ_W lifts to a θ'-structure.

For the pair $(\infty K, P)$, first note that by the second part of Lemma 7.13, ∞K is obtained from K by attaching trivial n-cells as well as $(n+1)$-cells and above. Hence

$$\mathrm{Ker}(\pi_{n-1}(P) \to \pi_{n-1}(\infty K)) = \mathrm{Ker}(\pi_{n-1}(P) \to \pi_{n-1}(K))$$

which is $\mathrm{Ker}(\pi_{n-1}(P) \to \pi_{n-1}(B'))$ as in the previous case, so there is a θ'-structure on ∞K extending that on P. □

We claim that the data $(\infty K, \ell'_{\infty K}, x_1 : \infty K \to [0, \infty))$ satisfies the conditions of Theorem 1.8, which immediately implies Theorem 1.6 as $B\mathrm{Diff}^{\theta'}(W, \ell'_P)_{\ell'_W}$ is a path-component of $\mathcal{N}^{\theta'}(P, \ell'_P)$.

To verify the first condition of Theorem 1.8, note that (K, P) is $(n-1)$-connected by assumption, and Lemma 7.13 gives the rest. Secondly, the map

$$\ell'_{\infty K}|_{[i-1,i]} : \pi_n(K) \longrightarrow \pi_n(B')$$

is surjective, as $\ell'_{\infty K}|_{[i-1,i]} : K \to B'$ is just another lift of $K \xrightarrow{\ell'_K} B' \to B$, and $\ell_K : \pi_n(K) \twoheadrightarrow \pi_n(B') \hookrightarrow \pi_n(B)$ as we chose B' to be the nth stage of the Moore–Postnikov tower for $\ell_K : K \to B$.

Thirdly, as θ' is obtained as the nth stage of the Moore–Postnikov tower for ℓ_K, all the maps $\ell'_{\infty K}|_{[i-1,i]} : \pi_{n-1}(K) \to \pi_{n-1}(B')$ are isomorphisms. Thus elements in the kernel of $\ell'_{\infty K}|_i : \pi_{n-1}(P) \to \pi_{n-1}(K) \to \pi_{n-1}(B')$ are already in the kernel of $\pi_{n-1}(P) \to \pi_{n-1}(K)$.

Lemma 7.15. *If gK contains a submanifold diffeomorphic to $S^n \times S^n - \mathrm{int}(D^{2n})$, then $3gK$ contains such a submanifold which in addition has null-homotopic structure map to B. We call this the* trivial *θ-structure on $S^n \times S^n - \mathrm{int}(D^{2n})$.*

Proof. Write $\widetilde{K} = gK$. Finding an embedded $S^n \times S^n - \mathrm{int}(D^{2n})$ means finding two embedded n-spheres with trivial normal bundle, which intersect at a single point. We have one in $\widetilde{K}$, so

$$S^n \times S^n - \mathrm{int}(D^{2n}) \hookrightarrow K \xrightarrow{\ell_K} B.$$

This gives $\mathbb{Z} \oplus \mathbb{Z} = \pi_n(S^n \times S^n - \mathrm{int}(D^{2n})) \to \pi_n(B)$ which will not typically be the zero map, instead giving elements $x, y \in \pi_n(B)$ on the basis elements.

In a separate copy of $\widetilde{K}$ we have a framed embedding

$$S^n \times \{*\} \overset{\text{reflection}}{\cong} S^n \times \{*\} \hookrightarrow S^n \times S^n - \mathrm{int}(D^{2n}) \hookrightarrow K$$

which in the homotopy of B gives the element $-x$. Thus in $2\widetilde{K}$ the connect-sum of this embedded framed sphere and the original one gives an embedded framed sphere with null-homotopic map to B. Using another copy of $\widetilde{K}$ we can fix the remaining sphere, without changing that they have geometric intersection 1. $\square$

By this lemma, for any $i \geq 0$ we can find an embedding $S^n \times S^n - \mathrm{int}(D^{2n}) \hookrightarrow 3gK$ such that composing with

$$\ell'_{\infty K}|_{[i,i+3g]} : K|_{[i,i+3g]} \longrightarrow B'$$

and $B' \to B$ is null-homotopic. But $\pi_n(B') \hookrightarrow \pi_n(B)$, and so the composition to B' is already null-homotopic as required.

7.3. **Proof of Theorem 1.3.** We apply Theorem 1.6 with the tangential structure $\theta = \mathrm{id} : BO(2n) \to BO(2n)$. The conditions exactly match up, after noting that $\pi_n(BO(2n)) \to \pi_n(BO)$ is an isomorphism.

7.4. **Proof of Theorem 1.1.** Let W be a closed $(n-1)$-connected $2n$-manifold parallelisable in the complement of a point. We apply Theorem 1.3 to the manifold with boundary $(W - \mathrm{int}(D^{2n}), S^{2n-1})$, stabilising with respect to $K = S^{2n-1} \times I \# S^n \times S^n$. The conditions of Theorem 1.3 are clearly satisfied, and

$$W - \mathrm{int}(D^{2n}) \cup_\partial gK = W \# W_g - \mathrm{int}(D^{2n})$$

in the notation of Theorem 1.1. Furthermore, the nth stage of the Moore–Postnikov tower for $K \to BO(2n)$ is the connective cover $BO(2n)\langle n\rangle$, as K is $(n-1)$-connected and $\pi_n(K) \to \pi_n(BO(2n))$ is the zero map as $S^n \times S^n - \mathrm{int}(D^{2n})$ is parallelisable.

7.5. **The standard stabilisation.** Let us finally discuss the "standard" choice of stabilising manifold K. In the situation of Theorem 1.3, the manifold W determines a subgroup $A = \mathrm{Im}(\pi_n(W) \to \pi_n(BO))$. This group is cyclic, and we pick a generator and represent it by a map $f : S^n \to BSO(n+1)$. The choice of f is unique up to homotopy and up to precomposing with an orientation reversing diffeomorphism of S^n (since the choice of generator is unique only up to a sign). Let $V_f \to S^n$ be the vector bundle represented by f, and let $S(V_f) \to S^n$ be its unit sphere bundle. By obstruction theory, this admits a section $s : S^n \to S(V_f)$, and there is a fiberwise diffeomorphism $S(V_f) \to S(V_f)$, which in the fiber over $x \in S^n$ is induced by the reflection in the vector $s(x)$. This proves that $S(V_f)$ admits an orientation reversing diffeomorphism. Moreover, there is an orientation reversing diffeomorphism $S(V_{-f}) \to S(V_f)$ which is over an orientation reversing diffeomorphism of the base S^n. This proves that the manifold $K = K_A = S(V_f)$ only depends on the cyclic subgroup $A = \langle f\rangle \subset \pi_n(BO)$ and that it supports an orientation reversing diffeomorphism. Therefore the process $W \mapsto W\#K$ depends up to diffeomorphism only on the choice of path component of W at which to perform the connected sum.

7.6. **Proof of Lemma 1.10 and Theorem 1.11.** Let us first show that $\mathcal{K} \subset \mathcal{K}_0$ is a submonoid, and that it is commutative. Recall that $\mathcal{K}_0$ was the set of isomorphism classes of bordisms $K \subset [0,1] \times \mathbb{R}^\infty$ with θ-structure, starting and ending at (P, ℓ_P) and that $\mathcal{K}$ is the subset admitting representatives containing $[0,1] \times (P - A)$ with product θ-structure, where $A \subset P$ is a closed regular neighbourhood of a simplicial complex of dimension at most $(n-1)$ inside P. Let $K_0, K_1 : P \rightsquigarrow P$ be two such cobordisms and let K_i have support in A_i, a regular neighbourhood of a simplicial complex X_i of dimension at most $(n-1)$. As P is $(2n-1)$-dimensional, we can perturb the X_i to be disjoint and then shrink the A_i so they are disjoint. But if W_0 and W_1 have support in the disjoint sets A_0 and A_1, then $W_0 \circ W_1$ has support in $A_0 \sqcup A_1$ which is a regular neighbourhood of $X_0 \sqcup X_1$ which is again a simplicial complex of dimension at most $(n-1)$. Furthermore $K_0 \circ K_1$ is isomorphic to the θ-bordism K_{01} which is supported in $A_0 \sqcup A_1$ and agrees with K_i on $[0,1] \times A_i$, and this in turn is isomorphic to $K_1 \circ K_0$, so $\mathcal{K}$ is commutative.

By the same obstruction theoretic argument as in Lemma 7.14 we see that the map $\mathcal{N}^{\theta'}(P, \ell'_P) \to \mathcal{N}^{\theta}(P, \ell)$ is a weak equivalence, and a similar argument shows that $\mathcal{K}' \to \mathcal{K}$ is an isomorphism. Explicitly, $[0,1] \times P$ has a canonical lift of its θ-structure to a θ'-structure. If an element of $\mathcal{K}$ is represented by a cobordism K supported in $A \subset P$, it contains the subset $(\{0\} \times P) \cup ([0,1] \times (P - A)) \cup (\{1\} \times P)$ which has a canonical θ'-structure. Because A is a regular neighbourhood of a simplicial complex of dimension at most $(n-1)$, the manifold K is obtained up to homotopy from this set by attaching cells of dimension at least n, so up to homotopy there is a unique extension of the lift. This shows that $\mathcal{K}' \to \mathcal{K}$ is a bijection.

Let us now prove the first part of Theorem 1.11 in the case where the monoid $\mathcal{K}$ is countable. We can pick a bijection $\varphi : \mathbb{N} \cong \mathbb{N} \times \mathcal{K}$ and let $K|_{[i,i+1]}$ be a representative for the second coordinate of $\varphi(i)$. This gives a direct system of spaces

$$\mathcal{N}^{\theta}(P, \ell_P) \xrightarrow{K|_{[0,1]}} \mathcal{N}^{\theta}(P, \ell_P) \xrightarrow{K|_{[1,2]}} \dots$$

whose homotopy colimit has homology $H_*(\mathcal{N}^{\theta}(P, \ell_P))[\mathcal{K}^{-1}]$, and we claim Theorem 1.8 applies. Conditions (i) and (ii) of that theorem are satisfied by assumption, and (iii)–(v) are satisfied because $\mathcal{K}$ contains all possible θ-bordisms. Furthermore, the $(n-1)$st Moore–Postnikov stage $K \to B' \to B$ for the non-compact manifold K agrees with that of $P \to B$, so Theorem 1.11 follows from Theorem 1.8 in this case.

The general case follows from this by a direct limit argument, as follows. Recall that the Moore–Postnikov stage $P \to B' \to B$ can be constructed by attaching cells of dimension at least n to P. Let $(C, <)$ be the direct system of finite subcomplexes of B' containing P, and write $B_c \subset B'$ for the subcomplex corresponding to $c \in C$ and $\theta_c : B_c \to BO(2n)$ for the restriction of θ'. Then P has a canonical θ_c-structure for each c which we shall denote ℓ_c, and since each $\pi_i(B_c)$ is countable we see by induction on handles that the corresponding monoid $\mathcal{K}_c$ is also countable. Therefore the previous case identifies $H_*(\Omega^\infty MT\theta_c)$ as the localisation of $H_*(\mathcal{N}^{\theta_c}(P, \ell_c))$ at $\mathcal{K}_c$, and hence the direct limit with the direct limit of the localisations. Since the direct limit of $H_*(\mathcal{N}^{\theta_c}(P, \ell_c))$ is $H_*(\mathcal{N}^{\theta'}(P, \ell'_P)) \cong H_*(\mathcal{N}^{\theta}(P, \ell_P))$ and the direct limit of the monoid $\mathcal{K}_c$ is $\mathcal{K}' \cong \mathcal{K}$, this proves the general case.

Finally, we consider a submonoid $\mathcal{L} \subset \mathcal{K}$ satisfying the conditions in Theorem 1.11. It suffices to prove that the universal localisation $\mathbb{Z}[\mathcal{K}][\mathcal{L}^{-1}] \to \mathbb{Z}[\mathcal{K}][\mathcal{K}^{-1}]$ is an isomorphism, which will follow if we can prove that $k \in \mathbb{Z}[\mathcal{K}][\mathcal{L}^{-1}]$ is a unit for all $k \in \mathcal{K}$. Let $k \in \mathcal{K}$ be represented by a cobordism W with θ-structure ℓ_W. By our assumptions we may find a cobordism L representing an element $l \in \mathcal{L}$ such that

$$\begin{gathered}
\mathrm{Ker}(\pi_{n-1}(P) \to \pi_{n-1}(W)) \subset \mathrm{Ker}(\pi_{n-1}(P) \to \pi_{n-1}(L)),\\
\mathrm{Im}(\pi_n(W) \to \pi_n(B)) \subset \mathrm{Im}(\pi_n(L) \to \pi_n(B)),
\end{gathered}$$

and that L contains a submanifold diffeomorphic to $S^n \times S^n - \mathrm{int}(D^{2n})$. If we let K be the non-compact manifold $K = \infty L = L \cup_P L \cup_P \dots$, the proof of Lemma 7.9 then gives an embedding $W \to K$ and after replacing L with gL for some finite g, we get an embedding $K \to L$. If we let $K' \subset L$ be the (closure of the) complement, we have a diffeomorphism $L \cong K \cup_P K'$ giving a factorisation $l = kk'$. Hence $k \in \mathbb{Z}[\mathcal{K}][\mathcal{L}^{-1}]$ is a unit for all $k \in \mathcal{K}$ as desired.

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