9 Fiber bundles

9/1. First steps toward fiber bundles. Covering spaces may be considered as a perfect tool to study the fundamental group. Fiber bundles provide the same kind of tool to study the higher homotopy groups, as we shall see soon.

A *fiber bundle* (or locally trivial fiber bundle) is a four-tuple (E, B, F, p), where E, B, F are spaces, and $p : E \longrightarrow B$ is a map with the following property:

- *For each point $x \in B$ there exists a neighborhood U of x such that $p^{-1}(U)$ is homeomorphic to $U \times F$, moreover the homeomorphism $\varphi_U : p^{-1}(U) \longrightarrow U \times F$ should make the diagram*

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "\varphi_U"] \arrow[dr, "p"'] & & U \times F \arrow[dl, "pr"] \\ & U & \end{tikzcd}$$

  commute. Here $pr : U \times F \longrightarrow U$ is a projection on the first factor.

The spaces E, B, F have their special names: E is a *total space*, B is a *base*, and F is a *fiber*. The inverse image $F_x \cong p^{-1}(x)$ is clearly homeomorphic to the fiber F for each point $x \in B$. However, these homemorphisms depend on x. As in the case of covering spaces, the following commutative diagram

$$\begin{tikzcd} E_1 \arrow[rr, "f"] \arrow[dr, "p_1"'] & & E_2 \arrow[dl, "p_2"] \\ & B & \end{tikzcd}$$

gives a *morphism* of fiber bundles (E_1, B, F_1, p_1) to (E_2, B, F_2, p_2). Two fiber bundles (E_1, B, F_1, p_1) and (E_2, B, F_2, p_2) are *equivalent* if there exist morphisms

$$f : (E_1, B, F_1, p_1) \longrightarrow (E_2, B, F_2, p_2), \quad g : (E_2, B, F_2, p_2) \longrightarrow (E_1, B, F_1, p_1)$$

such that $f \circ g = Id$ and $g \circ f = Id$. In particular, a fiber bundle $p : E \longrightarrow B$ is *trivial* if it is equivalent to the bundle $B \times F \longrightarrow B$:

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "p"'] & & B \times F \arrow[dl, "pr"] \\ & B & \end{tikzcd}$$

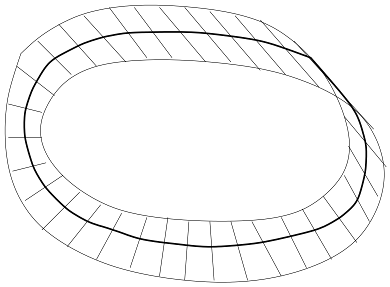


Fig. 9.1.

Examples. 1. Trivial bundle $B \times F \longrightarrow B$.

2. Covering spaces.

3. A projection $M \longrightarrow S^1$ of the Mebius band on the middle circle, see Fig. 9.1. The fiber is the interval I.

4. The Hopf bundle $h : S^{2n+1} \longrightarrow \mathbf{CP}^n$ with a fiber S^1.

5. Let G be a Lie group and H be its compact subgroup. A space of cosets G/H is a base space of the bundle $G \longrightarrow G/H$ with a fiber H.

6. Let a group Lie G acts freely on the manifold M, M/G be a space of orbits, then the projection $M \longrightarrow M/G$ is a fiber bundle with the fiber G

It is not so difficult to verify that the examples above are indeed locally trivial fiber bundles. To give a sample of such verification, we consider the Example 4 in more detail:

Lemma 9.1. *The Hopf map $h : S^{2n+1} \longrightarrow \mathbf{CP}^n$ is a locally trivial fiber with a fiber S^1.*

Proof. We use the construction given in the proof of Lemma 2.1 (Section 2). Again, we take a close look at the Hopf map $h : S^{2n+1} \longrightarrow \mathbf{CP}^n$: we take a point $(z_1, \cdots, z_{n+1}) \in S^{2n+1}$, (where $|z_1|^2 + \cdots + |z_{n+1}|^2 = 1$), then h maps the point $(z_1, \cdots, z_{n+1})$ to the line $(z_1 : \cdots : z_{n+1}) \in \mathbf{CP}^n$. Moreover $h(z_1, \cdots, z_{n+1}) = h(z_1', \cdots, z_{n+1}')$ if and only if $z_j' = e^{i\varphi} z_j$. Thus we can identify $\mathbf{CP}^n$ with the following quotient space:

$$\mathbf{CP}^n = S^{2n+1}/\sim, \quad \text{where } (z_1, \cdots, z_{n+1}) \sim (e^{i\varphi} z_1, \cdots, e^{i\varphi} z_{n+1}). \tag{24}$$

Now consider the following $(n+1)$ open subsets in $\mathbf{CP}^n$

$$U_j = \left\{ (z_1, \cdots, z_{n+1}) \in S^n \mid z_j \neq 0 \ \text{ and } \ (z_1, \cdots, z_{n+1}) \sim (e^{i\varphi} z_1, \cdots, e^{i\varphi} z_{n+1}) \right\}.$$

Since $z_j \neq 0$, we may write $z_j = re^{i\alpha}$, where $0 < r \leq 1$. Then the map $g : U_j \longrightarrow \overset{o}{D}{}^{2n}$ is given by

$$(z_1, \cdots, z_{j-1}, re^{i\alpha}, z_{j+1}, \cdots, z_{n+1}) \mapsto (e^{-i\alpha} z_1, \cdots, e^{-i\alpha} z_{j-1}, r, e^{-i\alpha} z_{j+1}, \cdots, e^{-i\alpha} z_{n+1})$$

is a homeomorphism. Indeed, we have:

$$|z_1|^2 + \cdots + |z_{j-1}|^2 + |z_{j+1}|^2 + \cdots + |z_{n+1}|^2 = 1 - r^2,$$

and for a given r, $0 < r \leq 1$, a point $(e^{-i\alpha} z_1, \cdots, e^{-i\alpha} z_{j-1}, e^{-i\alpha} z_{j+1}, \cdots, e^{-i\alpha} z_{n+1}) \in \mathbf{C}^n$ belongs to the sphere S^{2n-1} of radius $\sqrt{1-r^2}$. Since $0 < r \leq 1$, this gives parametrization of open disk $\overset{o}{D}{}^{2n}$ of radius 1.

Now let $\ell \in \mathbf{CP}^n$. In order to prove that the Hopf map $h : S^{2n+1} \longrightarrow \mathbf{CP}^n$ is a locally trivial fiber bundle, we have to find a neighborhood U of ℓ such that $h^{-1}(U)$ is homemorphic to the product $U \times S^1$. Notice that there exists j, $1 \leq j \leq n+1$, such that $\ell \in U_j$. Hence it is enough to show that $h^{-1}(U_j)$ is homeomorphic to $U_j \times S^1$, and that a projection on the first factor $pr : U_j \times S^1 \longrightarrow U_j$ coincides with the Hopf map.

Now we see that $h^{-1}(U_j) \subset S^{2n+1}$ is given as

$$\begin{aligned} h^{-1}(U_j) &= \{(z_1, \cdots, z_{n+1}) \in S^n \mid z_j = re^{i\alpha} \neq 0\} \\ &\cong \{((e^{-i\alpha}z_1, \cdots, e^{-i\alpha}z_{j-1}, r, e^{-i\alpha}z_{j+1}, \cdots, e^{-i\alpha}z_{n+1}), e^{-i\alpha})\} = U_j \times S^1. \end{aligned}$$

Clearly the projection on the first factor coincides with the Hopf map. □

Exercise 9.1. *Prove that the Hopf map $S^{4n+3} \longrightarrow \mathbf{HP}^n$ is locally trivial fiber bundle.*

Exercise 9.2. *Here we specify the example 5. Note that $Sp(1) = S^3$, and S^1 is a subgroup of $Sp(1)$. Prove that the fiber bundle $Sp(1) \longrightarrow Sp(1)/S^1$ is equivalent to the Hopf bundle $S^3 \longrightarrow \mathbf{CP}^1$.*[††]

Exercise 9.3. *Here we specify the example 6. Let S^{2n+1} be a unit sphere in $\mathbf{C}^{n+1}$, $S^{2n+1} = \{|z_1|^2 + \ldots + |z_{n+1}| = 1|\}$. The group $S^1 = \{e^{i\varphi}\}$ acts on S^{2n+1} by the formula $(z_1, \ldots, z_{n+1}) \longrightarrow (e^{i\varphi}z_1, \ldots, e^{i\varphi}z_{n+1})$. Prove that this action is free, and that a fiber bundle $S^{2n+1} \longrightarrow S^{2n+1}/S^1$ is equivalent to the Hopf bundle $S^{2n+1} \longrightarrow \mathbf{CP}^n$.*

Exercise 9.4. *Let $f : M \longrightarrow N$ be a smooth map, where M, N are smooth manifolds. Assume that the map f is a submersion, i.e. f is onto and the differential $df_x : TM_x \longrightarrow TM_{f(x)}$ is an epimorphism for any $x \in M$. Prove that $(M, N, f^{-1}(x), f)$ is a locally-trivial fiber bundle.* **Hint:** *Probably you have to use the Implicit Function Theorem from the course "Analysis on Manifolds".*

Exercise 9.5. *Prove that the fiber bundles from the examples 3–6 are nontrivial fiber bundles.*

9/2. Constructions of new fiber bundles. There are two important ways to construct new fiber bundles.

1. Restriction. Let $E \xrightarrow{p} B$ be a fiber bundle with a fiber F, and let $B' \subset B$ be a subset. Let $E' = p^{-1}(B')$. The bundle $E' \xrightarrow{p'} B'$, where $p' = p|_{E'}$ is a *restriction of the bundle $E \xrightarrow{p} B$ on the subspace $B' \subset B$.*

[††] It may be helpfull to remember that $Sp(1)$ is the set of all unit quaternions $\alpha = \alpha_1 + i\alpha_2 + j\alpha_3 + k\alpha_4 \in Sp(1)$, and $S^1 = \{\alpha_1 + i\alpha_2\}$.

2. Induced fiber bundle. Let $E \xrightarrow{p} B$ be a fiber bundle with a fiber F, and $X \xrightarrow{f} B$ be a map. Let $f^*(E) \subset X \times E$ be the following subspace:

$$f^*(E) = \{(x, e) \in X \times E \mid f(x) = p(e)\ \}.$$

There are two natural maps: $f^*(E) \xrightarrow{f^*} E$ ($f^*(x, e) = e$) and $f^*(E) \xrightarrow{p'} X$ ($p'(x, e) = x$). It is easy to check that the map $f^*(E) \xrightarrow{p'} X$ is a locally–trivial bundle over X with the same fiber F and that the diagram

$$\begin{tikzcd}
f^*(E) \arrow[r, "f^*"] \arrow[d, "p'"'] & E \arrow[d, "p"] \\
X \arrow[r, "f"] & B
\end{tikzcd}$$

commutes. The bundle $f^*(E) \xrightarrow{p'} X$ is the *induced fiber bundle.*

Now we prove the following fact.

Lemma 9.2. *Any locally–trivial fiber bundle over the cube I^q is trivial.*

Proof. Let $E \xrightarrow{p} I^q$ be a locally–trivial fiber bundle. We prove the statement in two steps.

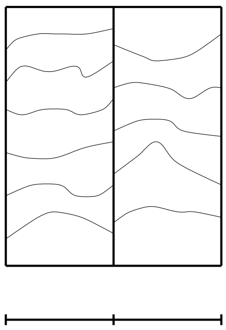


Fig. 9.2.

STEP 1. First we assume that the restriction of the bundle $E \xrightarrow{p} I^q$ on each of the cubes

$$I_1^q = \{(x_1, \ldots, x_q) \in I^q \mid x_q \leq 1/2\ \},$$

$$I_2^q = \{(x_1, \ldots, x_q) \in I^q \mid x_q \geq 1/2\ \}.$$

is a trivial fiber bundle. Let $p_1 : E_1 \longrightarrow I_1^q$, $p_2 : E_2 \longrightarrow I_2^q$ be these restrictions. Since these bundles are trivial, we can assume that $E_1 = I_1^q \times F$, $E_1 = I_1^q \times F$, so a point of E_1 has coordinates (x, y), $x \in I_1^q$, $y \in F$, and, analogously, a point of E_2 has coordinates (x, y'), $x \in I_2^q$, $y' \in F$. In particular, if $x \in I_1^q \cap I_2^q$, then the map $f_x : y \mapsto y'$ is well-defined, and is a homeomorphism of the fiber F. We define a projection $\pi : I_1^q \longrightarrow I_1^q \cap I_2^q$ by the formula:

$$\pi(x_1, \ldots, x_q) = (x_1, \ldots, x_{q-1}, 1/2).$$

Define new map $\varphi : E_1 \longrightarrow I_1^q \times F$ by the formula $\varphi(x, y) = (x, f_{\pi(x)}(y))$. It gives a homeomorphism $E_1 \cong I_1^q \times F$ which coincide with the chosen trivialization over I_2^q, i.e. we obtain a homeomorphism $E \longrightarrow I^q \times F$.

Step 2. Now we prove the general case. Since the bundle $p : E \longrightarrow I^q$ is a locally–trivial bundle, we may cut the cube I^q into finite number of small cubes I_i^q, $i = 1, 2, \ldots$, such that a restriction of the bundle $p : E \longrightarrow I^q$ on each of these small cubes is trivial, and each space $J_k = \bigcup_{i=1}^{k} I_i^q$ is homeomorphic to a cube I_1^q. Assume that we have constructed a trivialization of the bundle over J_k, then $J_{k+1} = J_k \cup I_{k+1}^q$ homeomorphic to a cube. Choose homeomorphisms $J_k \cong I_1^q$ and I_2^q and then Step 1 completes the proof. □

We will say that a *covering homotopy property* (CHP) holds for a map $p : E \longrightarrow B$, if for any CW-complex Z and commutative diagram

$$\begin{tikzcd} & E \arrow[d, "p"] \\ Z \arrow[ur, "\widetilde{g}"] \arrow[r, "g"] & B \end{tikzcd}$$

and a homotopy $G : Z \times I \longrightarrow B$, such that $G|_{Z\times\{0\}} = g$ there exists a homotopy $\widetilde{G} : Z \times I \longrightarrow E$ such that $\widetilde{G}|_{Z\times\{0\}} = \widetilde{g}$ and the diagram

$$\begin{tikzcd} & E \arrow[d, "p"] \\ Z \times I \arrow[ur, "\widetilde{G}"] \arrow[r, "G"] & B \end{tikzcd}$$

commutes.

Theorem 9.3. (Theorem on Covering Homotopy) *The covering homotopy property holds for a locally-trivial fiber bundle $E \longrightarrow B$.*

We will prove a stronger version of Theorem 9.3, namely we assume in addition the following.

- *There is a subcomplex $Z' \subset Z$ and a homotopy $\widetilde{G}' : Z' \times I \longrightarrow E$ covering the homotopy $G|_{Z'\times I}$.*

Proof. Case 1. *Let the fiber bundle $p : E \longrightarrow B$ be trivial, and Z be any CW-complex.* We identify $E \cong B \times F$, and maps to $B \times F$ with the pairs of maps to B and F. Then the map $\widetilde{g} : Z \longrightarrow E = B \times F$ is given by a pair $\widetilde{g} = (g, h)$, where $g : Z \longrightarrow B$ is the above map, and $h : Z \longrightarrow F$ be some continuous map. The homotopy $\widetilde{G}' : Z' \times I \longrightarrow E$ is given by the pair $\widetilde{G}' = (G', H')$, where $G' = G|_{Z'\times I} : Z' \times I \longrightarrow B$ is determined by the homotopy G and and the homotopy $H' : Z' \times I \longrightarrow F$ is such that $H'|_{Z'\times\{0\}} = h|_{Z'}$. Thus the Borsuk Theorem gives us that there exists a homotopy $H : Z \times I \longrightarrow F$ extending the map $h : Z \longrightarrow F$ and the homotopy $H' : Z' \times I \longrightarrow F$. The covering homotopy $\widetilde{G} : Z \times I \longrightarrow B \times F = E$ is defined by $H(z,t) = (G(z,t), H(z,t))$.

Case 2. *The fiber bundle $p : E \longrightarrow B$ is arbitrary, $Z = D^n$, $Z' = S^{n-1}$.* Let $g : D^n \longrightarrow B$, $g' : S^{n-1} \longrightarrow B$, $\widetilde{f} : D^n \longrightarrow E$, $\widetilde{g}' : S^{n-1} \longrightarrow E$, and $G : D^n \times I \longrightarrow B$, $\widetilde{G}' : S^{n-1} \times I \longrightarrow E$ be the corresponding maps and homotopies.

The map $G : D^n \times I \longrightarrow B$ induces the bundle $G^*(E) \longrightarrow D^n \times I$, which is trivial by Lemma 9.2. Recall that the total space

$$E' = G^*(E) = \{((x,t), e) \mid G(x,t) = p(e)\} \subset (D^n \times I) \times E.$$

Let $G^* : G^*(E) \longrightarrow E$ be a natural map (projection). We define a map $\widetilde{h} : D^n \longrightarrow G^*(E)$ by $\widetilde{h}(x) = ((x,0), \widetilde{g}(x))$. This map is well-defined since $G(x,0) = g(x) = p \circ \widetilde{g}(x)$ and $\widetilde{h}$ covers the map $h : D^n \longrightarrow D^n \times I$ given by $x \mapsto (x,0)$. The homotopy $H : D^n \times I \longrightarrow D^n \times I$ (the identity map!), and $\widetilde{H}' : S^{n-1} \times I \longrightarrow G^*(E)$, where $\widetilde{H}'(x,t) = ((x,t), \widetilde{G}'(x,t))$ satisfy the conditions of the theorem. Indeed, we have the commutative diagrams:

$$\begin{tikzcd}
 & & G^*(E) \arrow[d, "p'"] \\
S^{n-1} \arrow[r] & D^n \arrow[ur, "\widetilde{h}"] \arrow[r, "h"] & D^n \times I
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & G^*(E) \arrow[d, "p'"] \\
S^{n-1} \times I \arrow[r] \arrow[urr, "\widetilde{H}'"] & D \times I \arrow[ur, "\widetilde{H}"] \arrow[r, "H=Id"] & D^n \times I
\end{tikzcd}
\tag{25}$$

The map $\widetilde{H}$ from (25) exists be the Case 1. Thus the map $\widetilde{G} = G^* \circ \widetilde{H} : D^n \times I \longrightarrow E$ covers the homotopy $G : D^n \times I \longrightarrow B$ as required.

Case 3. *Now the fiber bundle $E \xrightarrow{p} B$ is arbitrary, and the CW-complex Z is finite.* By induction, we may assume that the difference $Z \setminus Z'$ is a single cell e^n. Let $\Phi : D^n \longrightarrow X$ be a corresponding characteristic map, and $\varphi = \Phi|_{S^{n-1}}$ be an attaching map. Then the map $\widetilde{h} = \widetilde{g} \circ \Phi : D^n \longrightarrow E$ and the homotopies

$$H = G \circ (\Phi \times Id) : D^n \times I \longrightarrow B,$$

$$H' = \widetilde{G}' \circ (\Phi|_{S^{n-1}} \times Id) : S^{n-1} \times I \longrightarrow E$$

satisfy the conditions of the theorem, so by Case 2 one completes the proof. □

Exercise 9.6. *Prove the general case, i.e. when Z is an arbitrary CW-complex and $E \longrightarrow B$ is any locally trivial bundle.*

9/3. Serre fiber bundles. Serre fiber bundles generalize locally trivial fiber bundles. We start with a definition and examples.

Definition 9.4. A map $p : E \longrightarrow B$ is a *Serre fiber bundle* if the CHP holds for any CW-complex.

Remark. We emphasize that we do not assume uniqueness of the covering homotopy.

A Serre fiber bundle in general is not locally trivial, see Fig. 9.3.

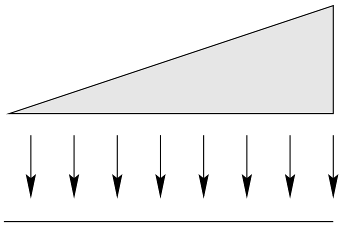


Fig. 9.3.

Examples. 1. Locally-trivial fiber bundles.

2. Let Y be an arbitrary path-connected space, $\mathcal{E}(Y, y_0)$ be the space of paths starting at y_0. The map $p : \mathcal{E}(Y, y_0) \longrightarrow Y$, where $p(s : I \longrightarrow Y) = s(1) \in Y$ is Serre fiber bundle. Note that $p^{-1}(y) = \Omega(Y, y)$. Let $f : X \longrightarrow Y$ be a map, $\widetilde{f} : X \longrightarrow \mathcal{E}(Y, y_0)$ be a covering map, and $F : X \times I \longrightarrow Y$ be a homotopy of f ($F|_{X\times\{0\}} = f$). Then a covering homotopy $\widetilde{F} : X \times I \longrightarrow \mathcal{E}(Y, y_0)$ may be defined by the formula (see Fig. 9.4):

$$(\widetilde{F}(z,t))(\tau) = \begin{cases} (\widetilde{f}(z))(\tau(1+t)) & \text{if } \tau(1+t) \leq 1, \\ F(z, \tau(1+t) - 1) & \text{if } \tau(1+t) \geq 1 \end{cases} \tag{26}$$

Exercise 9.7. *Check that the formula (26) indeed defines a covering homotopy as required.*

3. (A generalization of the previous example.) Let $A \subset Y$, and (X, A) be a Borsuk pair (for example, a CW-pair). Let $E = \mathcal{C}(X, Y)$, $B = \mathcal{C}(A, Y)$, and the map $p : E \longrightarrow B$ be defined as $p(f : X \longrightarrow Y) = (f|_A : A \longrightarrow Y)$.

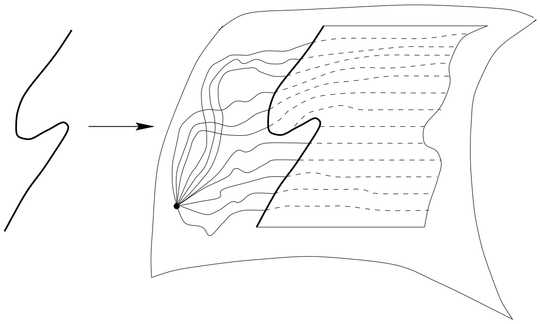


Fig. 9.4.

Exercise 9.8. *Prove that the map*

$$p : \mathcal{C}(X, Y) \longrightarrow \mathcal{C}(A, Y)$$

defined above is indeed a Serre fiber bundle.

As far as the fibers of a Serre fiber bundle $p : E \longrightarrow B$ are concerned, we cannot claim that for any two points $x_0, x_1 \in B$ the fibers $p^{-1}(x_0)$, $p^{-1}(x_1)$ are homeomorphic. However, we will prove here that the fibers are *weak homotopy equivalent.*

Definition 9.5. *Spaces X and Y are weak homotopy equivalent if there is a natural one-to-one correspondence $\varphi_K : [K, X] \xrightarrow{\cong} [K, Y]$ for any CW-complex K. Naturality* *means that for any map $f : K \longrightarrow K'$ the following diagram*

$$\begin{tikzcd}
[K,X] \arrow[r, "\varphi_K"] & [K,Y] \\
[K',X] \arrow[u, "f^*"] \arrow[r, "\varphi_{K'}"] & [K',Y] \arrow[u, "f^*"']
\end{tikzcd}$$

commutes. (Compare with Definitions 3.1, 3.2, 3.3.)

Exercise 9.9. *Prove that finite CW-complexes X, Y are weak homotopy equivalent if and only if they are homotopy equivalent.*

Theorem 9.6. *Let $p : E \longrightarrow B$ be Serre fiber bundle, where B IS a path-connected space. Then the spaces $F_0 = p^{-1}(x_0)$ and $F_1 = p^{-1}(x_1)$ are weak homotopy equivalent for any two points $x_0, x_1 \in B$.*

Proof. Let $s : I \longrightarrow B$ be a path connecting x_0 and x_1. We have to define one-to-one correspondence $\varphi_K : [K, F_0] \longrightarrow [K, F_1]$ for any CW-complex K.

Let $h_0 : K \longrightarrow F_0$ be a map. Denote $i_0 : F_0 \longrightarrow E$ the inclusion map. We have the map:

$$\widetilde{f} : K \xrightarrow{h_0} F_0 \xrightarrow{i_0} E.$$

Consider also the homotopy $F : K \times I \longrightarrow B$, where $G(x,t) = s(t)$. By the CHP there exists a covering homotopy $\widetilde{F} : K \times I \longrightarrow E$ of the map $\widetilde{f}$ such that $p \circ \widetilde{F} = F$, in particular, $\widetilde{F}(K \times \{t\}) \subset p^{-1}(s(t))$, and $\widetilde{F}(K \times \{1\}) \subset F_1$.

We define $\varphi_K(h_0 : K \longrightarrow F_0) = (h_1 : K \longrightarrow F_1)$, where $h_1 = \widetilde{F}|_{K\times\{1\}}$. We should show that the map φ_K is well-defined.

Let s' be a different path connecting x_0 and x_1, and $\widetilde{f}' : K \longrightarrow E$, $F' : K \times I \longrightarrow B$, $h' : K \longrightarrow F_1$ be corresponding maps and homotopies determined by s'. Assume that s and s' are homotopic, and let $S : I \times I \longrightarrow B$ be a corresponding homotopy. Denote by $T : I \times I \longrightarrow B$ a map defined by $T(t_1, t_2) = S(t_1, t_2)$, see Fig. 9.5.

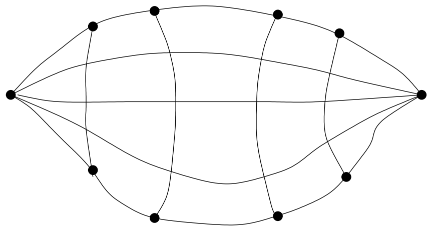


Fig. 9.5.

We define a homotopy $G : (K \times I) \times I \longrightarrow B$ by the formula: $G(k, t_1, t_2) = T(t_1, t_2)$ and a map $\widetilde{g} : K \times I \longrightarrow E$ defined by $\widetilde{g}(k, t_1) = \widetilde{f}(k)$ (see above). The CHP (relative version) implies that there exists $\widetilde{G} : K \times I \longrightarrow E$ covering G and $\widetilde{G}|_{(K\times\{0\})\times I} = \widetilde{F}$ and $\widetilde{G}|_{(K\times\{1\})\times I} = \widetilde{F}'$. The map $(k,t) \longrightarrow \widetilde{G}(k,t,1)$ maps $K \times I$ to F_1: this is the homotopy connecting h_1 and h_1', see Fig. 9.6.

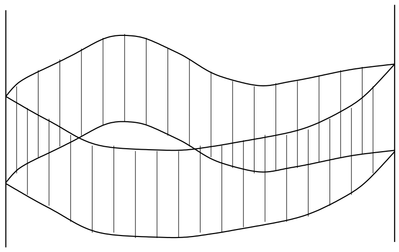


Fig. 9.6.

Thus a path $s : I \longrightarrow B$ defines a map $\varphi_K(s) : [K, F_0] \longrightarrow [K, F_1]$, $F_0 = p^{-1}(s(0))$, $F_1 = p^{-1}(s(1))$, which does depend only of the homotopy class of s. Clearly the map φ_K is natural with respect to K; note also that if s is a constant path, then $\varphi_K = Id_{F_0}$. Moreover, if a composition of paths $s_1 s_2$ (i.e. $s_1(1) = s_2(0)$) gives a map $\varphi_K(s_1 s_2) = \varphi_K(s_2)\varphi_K(s_1)$. In particular, the map $\varphi_K(s^{-1})$ is inverse to $\varphi_K(s)$: it implies that $\varphi_K(s)$ is one-to-one. □

Now let $f : X \longrightarrow Y$ be a map. We say that a map $f_1 : X_1 \longrightarrow Y_1$ is *homotopy equivalent* to f, if there are homotopy equivalences $\varphi : X \longrightarrow X_1$, $\psi : Y \longrightarrow Y_1$ such that the following diagram commutes:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\varphi"] & Y \arrow[d, "\psi"] \\
X_1 \arrow[r, "f_1"] & Y_1
\end{tikzcd}$$

Theorem 9.7. *For any continuous map $f : X \longrightarrow Y$ there exists homotopy equivalent map $f_1 : X_1 \longrightarrow Y_1$, such that $f_1 : X_1 \longrightarrow Y_1$ is Serre fiber bundle.*

Remark. It will be clear from the construction below that the space Y_1 may be chosen to be equal to Y. It is also important that the construction below is natural. It means that the commutative diagram on the left implies a commutativity of the diagram on the right:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\alpha"] & Y \arrow[d, "\beta"] \\
X' \arrow[r, "f'"] & Y'
\end{tikzcd}
\qquad
\begin{tikzcd}
& X \arrow[rr, "f"] \arrow[dd, "\alpha"] & & Y \arrow[dd, "\beta"] \\
X_1 \arrow[ur, "\varphi"] \arrow[rr, "f_1"] \arrow[dd, "\alpha_1"] & & Y_1 \arrow[ur, "\psi"] \arrow[dd, "\beta_1"] & \\
& X' \arrow[rr, "f'"] & & Y' \\
X_1' \arrow[ur, "\varphi'"] \arrow[rr, "f_1'"] & & Y_1' \arrow[ur, "\psi'"] &
\end{tikzcd}$$

Proof of Theorem 9.7. Let $Y_1 = Y$, and

$$X_1 = \{(x, s) \in X \times \mathcal{E}(X) \mid s(0) = f(x)\}.$$

Then $p : X_1 \longrightarrow Y$ is defined by $p(x, s) = s(1)$. Clearly X and X_1 are homotopy equivalent. $\square$

The following statement is "dual" to Theorem 9.7:

Claim 9.1. *Let $f : X \longrightarrow Y$ be a continuous map. Then there exists a homotopy equivalent map $g : X \longrightarrow Y'$, so that g is an inclusion.*

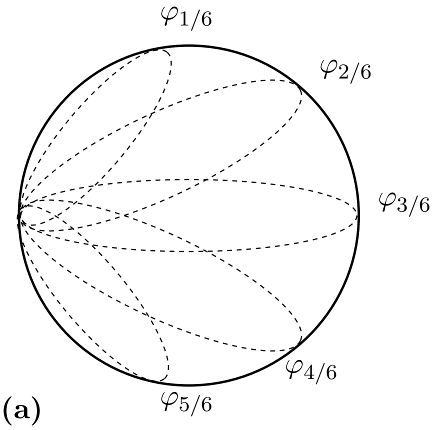
Proof. Let $Y' = (X \times I) \cup_f Y$ be the cylinder of the map f. Clearly $Y' \sim Y$, and $g : X \longrightarrow Y'$ is an embedding of X into the top base of $X \times I$. $\square$

9/4. Homotopy exact sequence of a fiber bundle. First we prove the following important fact:

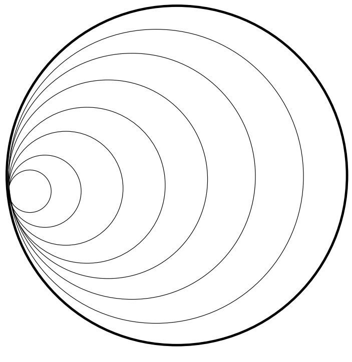
Lemma 9.8. *Let $p : E \longrightarrow B$ be Serre fiber bundle, $y \in E$ be any point, $x = p(y)$, $F = p^{-1}(x)$. The homomorphism*

$$p_* : \pi_n(E, F; y) \longrightarrow \pi_n(B, x)$$

is an isomorphism for all $n \geq 1$.



(a)



(b) **Fig. 9.7.**

Proof. p_* IS A MONOMORPHISM. Let $\alpha \in \pi_n(E, F; y)$ be represented by a map $\tilde{f} : D^n \longrightarrow E$ (where $\tilde{f}|_{S^{n-1}} : S^{n-1} \longrightarrow F$, and $\tilde{f}(s_0) = y$). Then the map

$$f = p \circ \tilde{f} : D^n \longrightarrow B$$

has the property that $f(S^{n-1}) = x$ and $[f] = p_*(\alpha) \in \pi_n(B, x)$. Assume $\alpha \in \text{Ker } p_*$, then there exists a homotopy $f_t : D^n \longrightarrow B$, so that $f_0 = f$ and $f_1(D^n) = x$. The covering homotopy property (the strong version) implies that there exists a homotopy $\tilde{f}_t : D^n \longrightarrow E$ covering the homotopy f_t. In particular $\tilde{f}_1(D^n) \subset F = p^{-1}(x)$ since $p \circ \tilde{f}_1(D^n) = f_1(D^n) = x$.

p_* IS AN EPIMORPHISM. Consider the homotopy $\varphi_t : S^{n-1} \longrightarrow S^n$, so that $\varphi : S^{n-1} \times I \longrightarrow S^n$, $\varphi(S^{n-1} \times I) = S^n$ as it is shown at Fig. 9.7 (a). Let $f : S^n \longrightarrow B$ be a map representing $\beta \in \pi_n(B, x)$. Consider the homotopy $g_t = f \circ \varphi_t : S^{n-1} \longrightarrow B$. Then we lift the homotopy g_t up to a homotopy $\tilde{g}_t : S^{n-1} \longrightarrow E$ by applying the CHP. The homotopy $\tilde{g}_t$ may be considered as a map $\tilde{h} : D^n \longrightarrow E$, where the disk D^n is covered by $(n-1)$-spheres as it is shown, see Fig. 9.7 (b), and the map h on these spheres is given by $\tilde{g}_t$. Clearly the map $h : D^n \longrightarrow E$ gives a representative of an element $\alpha \in \pi_n(E, F)$, so that $p_*(\alpha) = \beta$. $\square$

Now the exact sequence of the pair $(E, F; y)$:

$$\cdots \longrightarrow \pi_n(F, y) \xrightarrow{i_*} \pi_n(E, y) \xrightarrow{j_*} \pi_n(E, F; y) \xrightarrow{\partial} \pi_{n-1}(F, y) \longrightarrow \cdots$$

gives the exact sequence:

$$\cdots \longrightarrow \pi_n(F, y) \xrightarrow{i_*} \pi_n(E, y) \xrightarrow{j_*} \pi_n(B, x) \xrightarrow{\partial} \pi_{n-1}(F, y) \longrightarrow \cdots \tag{27}$$

We call the sequence (27) a *homotopy exact sequence of Serre fibration.*

Exercise 9.10. *Apply the sequence (27) for the Hopf fibration $S^3 \longrightarrow S^2$. Prove that (a) $\pi_2(S^2) = \pi_1(S^1) = \mathbf{Z}$; (b) $\pi_n(S^3) = \pi_n(S^2)$.*

Exercise 9.11. *Let $S^\infty \longrightarrow \mathbf{CP}^\infty$ be the Hopf fibration. Using the fact $S^\infty \sim *$, prove that $\pi_n(\mathbf{CP}^\infty) = 0$ for $n \neq 2$, and $\pi_2(\mathbf{CP}^\infty) = \mathbf{Z}$.*

Exercise 9.12. *Prove that $\pi_n(\Omega(X)) \cong \pi_{n+1}(X)$ for any X and $n \geq 0$.*

Exercise 9.13. *Prove that if the groups $\pi_*(B)$, $\pi_*(F)$ are finite (finitely generated), then the groups $\pi_*(E)$ are finite (finitely generated) as well.*

Exercise 9.14. *Assume that a fiber bundle $p : E \longrightarrow B$ has a section, i.e. a map $s : B \longrightarrow E$, such that $p \circ s = Id_B$. Prove the isomorphism $\pi_n(E) \cong \pi_n(B) \oplus \pi_n(F)$.*

9/5. More on the groups $\pi_n(X, A; x_0)$. Now we construct such a space Y that $\pi_n(X, A; x_0) \cong \pi_{n-1}(Y, y_0)$. First, we construct Serre fiber bundle $A_1 \xrightarrow{p} X$ which is homotopy equivalent to the inclusion $A \xrightarrow{i} X$. Let Y be a fiber of this fiber bundle. By construction above, Y is a space of loops in X which starting in A and ending at the point x_0. We construct a homomorphism $\alpha : \pi_{n-1}(Y, y_0) \longrightarrow \pi_n(X, A; x_0)$ as follows.

A map $S^{n-1} \xrightarrow{g} Y$ gives a map $G : D^n = C(S^{n-1}) \longrightarrow X$ by the formula: $G(s, t) = (g(s)(t)$, $s \in S^{n-1}$, $t \in I$. The map α may be included to the commutative diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_n(A_1) \arrow[r, "(p_1)_*"] \arrow[d] & \pi_n(X) \arrow[r] \arrow[d] & \pi_{n-1}(Y) \arrow[r] \arrow[d, "\alpha"] \arrow[dr, phantom, "\text{(C)}"] & \pi_{n-1}(A_1) \arrow[r] \arrow[d] & \pi_{n-1}(X) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \pi_n(A) \arrow[r, "i_*"] & \pi_n(X) \arrow[r] & \pi_n(X, A) \arrow[r] & \pi_{n-1}(A) \arrow[r] & \pi_{n-1}(X) \arrow[r] & \cdots
\end{tikzcd}$$

where rows are exact. Five–Lemma implies that $\alpha : \pi_{n-1}(Y) \longrightarrow \pi_n(X, A)$ is an isomorphism. In particular, we conclude that $\pi_n(X, A)$ is abelian group for $n \geq 3$.

Exercise 9.15. Prove that the square (C) of the above diagram commutes.