8 Higher homotopy groups

8/1. More about homotopy groups. Let X be a space with a base point $x_0 \in X$. We have defined the homotopy groups $\pi_n(X, x_0)$ for all $n \geq 1$ and even noticed that the groups $\pi_n(X, x_0)$ are commutative for $n \geq 2$ (see the begining of Section 6). Now it is a good time to give more details. First we have defined $\pi_n(X, x_0) = [(S^n, s_0), (X, x_0)]$, where $s_0 \in S^n$ is a based point. Alternatively an element $\alpha \in \pi_n(X, x_0)$ could represented by a map

$$f : (D^n, S^{n-1}) \longrightarrow (X, x_0) \quad \text{or a map}$$

$$f : (I^n, \partial I^n) \longrightarrow (X, x_0).$$

We already defined the group operation in $\pi_n(X, x_0)$, where a unit element is represented my constant map $S^n \longrightarrow \{x_0\} \subset X$. It is convenient to construct a canonical inverse $-\alpha$ for any element $\alpha \in \pi_n(X, x_0)$. Let $f \in \alpha$ be a map

$$f : (D^n, S^{n-1}) \longrightarrow (X, x_0).$$

We construct a map $(-f) : (D^n, S^{n-1}) \longrightarrow (X, x_0)$ as follows. Now consider the sphere $S^n = D^n_+ \cup_{S^{n-1}} D^n_-$, where the hemisphere D^n_+ is identified with the above disk D^n, a domain of the map f, see Fig. 8.1.

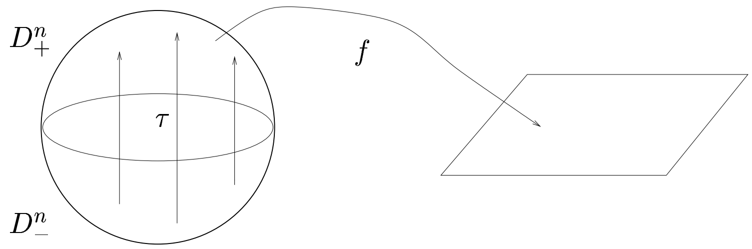

Fig. 8.1.

Let $\tau : S^n \longrightarrow D^n_+$ be a map which is identical on D^n_+ and which maps D^n_- to D^n_+ by the formula $(x_1, \ldots, x_{n+1}) \mapsto (x_1, \ldots, -x_{n+1})$. Then $-f = f \circ \tau : D^n_- \longrightarrow X$.

Exercise 8.1. *Prove that the map $f + (-f) : S^n \longrightarrow X$ is null-homotopic. Hint: It is enough to show that the map $f + (-f) : S^n \longrightarrow X$ exends to a map $g : D^{n+1} \longrightarrow X$.*

Exercise 8.2. *Prove that $\pi_n(X \times Y, x_0 \times y_0) \cong \pi_n(X, x_0) \times \pi_n(Y, y_0)$. Compute $\pi_n(T^k)$ for all n and k.*

8/2. Dependence on the base point. Let X be a path-connected space, and $x_0, x_1 \in X$ be two different points. Choose a path $\gamma : I \longrightarrow X$ so that $\gamma(0) = x_0$ and $\gamma(1) = x_1$. We define a homomorphism

$$\gamma_{\#} : \pi_n(X, x_0) \longrightarrow \pi_n(X, x_1)$$

as follows. Consider the sphere S^n with a base point $s_0 \in S^n$ and the map $\omega : S^n \longrightarrow S^n \vee I$ (see Fig 8.2 how to construct the map ω). Indeed, the map ω takes the base point $s_0 \in S^n$ to the point $\{1\} \in I \subset S^n \vee I$. Then for any map $f : (S^n, s_0) \longrightarrow (X, x_0)$ we define $\gamma_\#(f)$ to be the composition

$$\gamma_\#(f) : S^n \xrightarrow{\omega} S^n \vee I \xrightarrow{f \vee \bar{\gamma}} X,$$

where $\bar{\gamma}(t) = \gamma(1-t)$.

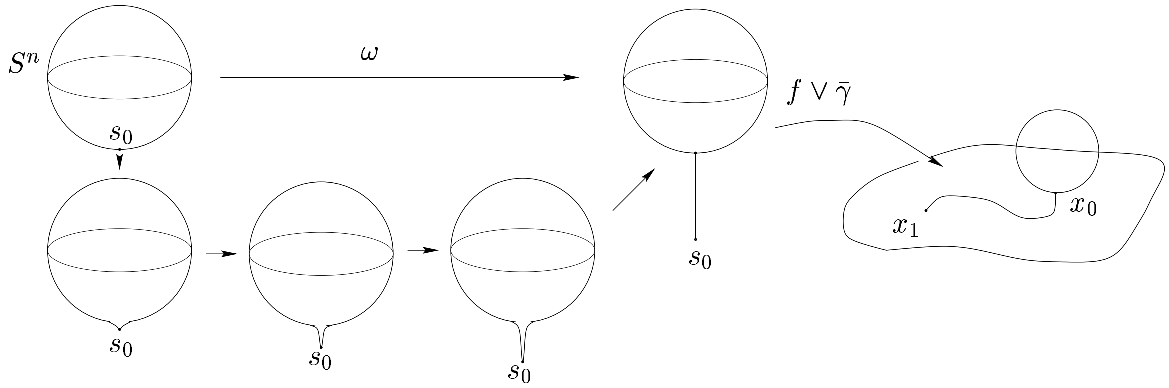

Fig. 8.2.

It is easy to check that $\gamma_\#(f+g) \sim \gamma_\#(f) + \gamma_\#(g)$ and that $(\gamma^{-1})_\# = (\gamma_\#)^{-1}$.

Exercise 8.3. *Prove that $\gamma_\#$ is an isomorphism.*

A path-connected space X is called *n-simple* if the isomorphism $\gamma_\# : \pi_k(X, x_0) \longrightarrow \pi_n(X, x_1)$ does not depend for $k \leq n$ on the choice of a path γ conecting any two points.

Consider the case when $x_0 = x_1$ in more detail. We have that any element $\omega \in \pi_1(X, x_0) = \pi$ acts on the group $\pi_n(X, x_0)$ for each $n = 1, 2, \ldots$ by isomorphisms, i.e. any element $\omega \in \pi$ determines an isomorphism $\omega_\# : \pi_n(X, x_0) \longrightarrow \pi_n(X, x_0)$. We consider the case $n \geq 2$. This action in turn gives to $\pi_n(X, x_0)$ the structure of $\mathbf{Z}[\pi]$-module as follows. Let $\omega = \sum_i^N k_i \omega_i \in \mathbf{Z}[\pi]$, where $\omega_i \in \pi$, and $k_i \in \mathbf{Z}$. Then the module map

$$\mathbf{Z}[\pi] \otimes \pi_n(X, x_0) \longrightarrow \pi_n(X, x_0)$$

is defined by $\omega(\alpha) = \sum_i^N k_i \omega_i(\alpha) \in \pi_n(X, x_0)$. The above definition may be rephrased as follows. A path-connected space X is n-simple if the $\mathbf{Z}[\pi]$-modules $\pi_k(X, x_0)$ are trivial (i.e. each element $\omega \in \pi$ acts on $\pi_k(X, x_0)$ identically).

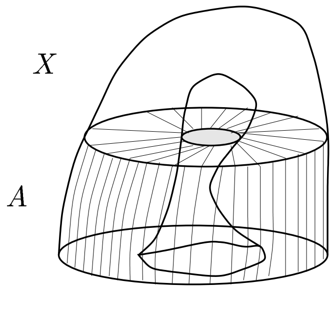

Fig. 8.3.

8/3. Relative homotopy groups. Let (X, A) be a pair of spaces and $x_0 \in A$ be a base point. A *relative homotopy group* $\pi_n(X, A; x_0)$ is a set of homotopy classes of maps $(D^n, S^{n-1}; s_0) \xrightarrow{f} (X, A; x_0)$, i.e. $f(S^{n-1}) \subset A$, $f(s_0) = x_0$, where a base point $s_0 \in S^{n-1}$, see Fig. 8.3. The other convenient geometric representation is to map cubes: $f : (I^n, \partial I^n) \longrightarrow (X, A)$, so that the base point $s_0 \in \partial I^n$ maps to x_0. We shall use both geometric interpretations.

Let $\alpha, \beta \in \pi_n(X, A; x_0)$ be represented by maps $f, g : (D^n, S^{n-1}) \longrightarrow (X, A)$. To define the sum $\alpha + \beta$ we construct a map $f + g$ as follows. First we define a map $c : D^n \longrightarrow D^n \vee D^n$ collapsing the equator disk to the base point, and the we compose c with the map $f \vee g$. Thus $f + g = (f \vee g) \circ c$, and $\alpha + \beta = [f + g]$, see Fig. 8.4.

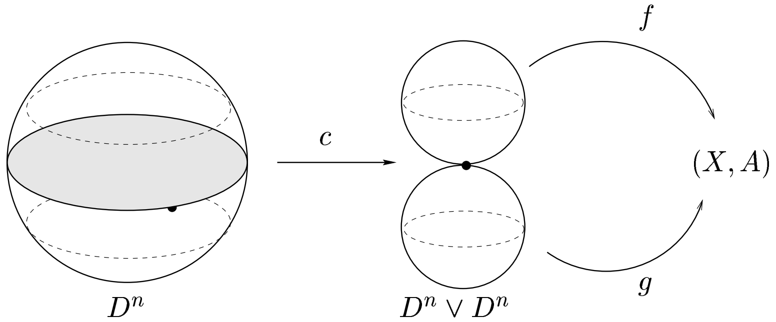

Fig. 8.4.

Again it is convenient to describe precisely the inverse element $-\alpha$. Let $f : (D^n, S^{n-1}) \longrightarrow (X, A)$ represent $\alpha \in \pi_n(X, A; x_0)$. We define a map $-f$ as follows. Consider a disk $D^n = D^n_- \cup_{D^{n-1}} D^n_+ \subset \mathbf{R}^n \subset \mathbf{R}^{n+1}$, see Fig. 8.5.

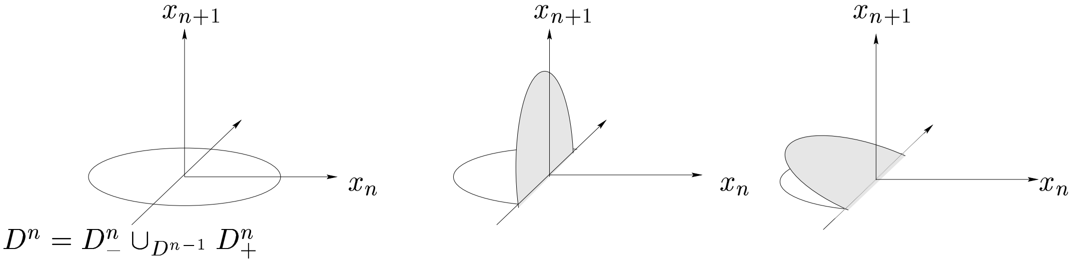

Fig. 8.5.

The disks D^n_- and D^n_+ are defined by the unequalities $\pm x_n \geq 0$. We consider a map $\varphi : D^n_- \cup_{D^{n-1}} D^n_+ \longrightarrow D^n_-$ flipping over the disk D^n_+ onto D^n_- as it is shown at Fig. 8.5. We may assume that the map $f : (D^n, S^{n-1}) \longrightarrow (X, A)$ is defined on the disk D^n_- so that $f|_{D^{n-1}}$ sends D^{n-1} to the base point x_0. Now we difine

$$-f : D^n_+ \xrightarrow{\varphi|_{D^n_+}} D^n_- \xrightarrow{f} X.$$

Exercise 8.4. *Prove that $f + (-f) \sim *$, where $-f$ as above.*

Exercise 8.5. *Prove that the group $\pi_n(X, A; x_0)$ is commutative for $n \geq 3$.*

Note that if we have a map of pairs $(X, A) \xrightarrow{f} (Y, B)$, such that $f(x_0) = y_0$, then there is a homomorphism

$$f_* : \pi_n(X, A; x_0) \longrightarrow \pi_n(Y, B; y_0).$$

Exercise 8.6. *Prove that if $f, g : (X, A) \xrightarrow{f} (Y, B)$ are homotopic maps, than $f_* = g_*$.*

Remark: Note the homotopy groups $\pi_n(X, x_0)$ may be interpreted as "relative homotopy groups": $\pi_n(X, x_0) = \pi_n(X, \{x_0\}; x_0)$. Moreover, one may construct a space Y such that $\pi_n(X, A; x_0) \cong \pi_{n-1}(Y, y_0)$. We will see this construction later.

The maps of pairs

$$(A, x_0) \xrightarrow{i} (X, x_0), \qquad (X, x_0) \xrightarrow{j} (X, A)$$

give the homomorphisms:

$$\pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0), \qquad \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A; x_0).$$

Exercise 8.7. *Let π be a group. Give definition of the center of π. Prove that the image of the homomorphism $j_* : \pi_2(X, x_0) \xrightarrow{j_*} \pi_2(X, A; x_0)$ belongs to the center of the group $\pi_2(X, A; x_0)$.*

Also we have a "connective homomorphism":

$$\partial : \pi_n(X, A; x_0) \longrightarrow \pi_{n-1}(A, x_0)$$

which maps the relative spheroid $f : (D^n, S^{n-1}) \longrightarrow (X, A)$, $f(s_0) = x_0$ to the spheroid $f|_{S^{n-1}} : (S^{n-1}, s_0) \longrightarrow (A, x_0)$.

Theorem 8.1. *The following sequence of groups is exact:*

$$\cdots \longrightarrow \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A; x_0) \xrightarrow{\partial} \pi_{n-1}(A, x_0) \longrightarrow \cdots \qquad (22)$$

First we remind that the sequence of groups and homomorphisms

$$\cdots \longrightarrow A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} A_3 \xrightarrow{\alpha_3} \cdots$$

is exact if $\mathrm{Ker}\ \alpha_{i+1} = \mathrm{Im}\ \alpha_i$.

Exercise 8.8. *Prove that the sequence (22) is exact*

- **(a)** *in the term $\pi_n(A, x_0)$,*
- **(b)** *in the term $\pi_n(X, x_0)$,*
- **(c)** *in the term $\pi_n(X, A; x_0)$.*

In the following exercises all groups are assumed to be abelian.

Exercise 8.9. *Prove the following statements*

(a) *The sequence $0 \longrightarrow A \longrightarrow B$ is exact if and only if $A \longrightarrow B$ is a monomorphism; and the sequence $A \longrightarrow B \longrightarrow 0$ is exact if and only if $A \longrightarrow B$ is an epimorphism.*

(b) *The sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is exact if and only if $C \cong B/A$.*

Corollary 8.2. **1.** *Let $A \subset X$ be a contractible subspace. Then $\pi_n(X, x_0) \cong \pi_n(X, A; x_0)$ for $n \geq 1$.*

2. *Let X be contractible, and $A \subset X$. Then $\pi_n(X, A; x_0) \cong \pi_{n-1}(A, x_0)$ for $n \geq 1$.*

3. *Let $A \subset X$ be a deformational retract of X. Then $\pi_n(X, A; x_0) = 0$.*

Exercise 8.10. *Prove Corollary 8.2.*

Exercise 8.11. *Let $A \subset X$ be a retract. Prove that*

- $i_* : \pi_n(A, x_0) \longrightarrow \pi_n(X, x_0)$ *is monomorphism,*
- $j_* : \pi_n(X, x_0) \longrightarrow \pi_n(X, A; x_0)$ *is epimorphism,*
- $\partial : \pi_n(X, A; x_0) \longrightarrow \pi_{n-1}(A, x_0)$ *is zero homomorphism.*

Exercise 8.12. *Let A be contractible in X. Prove that*

- $i_* : \pi_n(A, x_0) \longrightarrow \pi_n(X, x_0)$ *is zero homomorphism,*
- $j_* : \pi_n(X, x_0) \longrightarrow \pi_n(X, A; x_0)$ *is monomorphism,*
- $\partial : \pi_n(X, A; x_0) \longrightarrow \pi_{n-1}(A, x_0)$ *is epimorphism.*

Exercise 8.13. *Let $f_t : X \longrightarrow X$ be a homotopy such that $f_0 = Id_X$, and $f_1(X) \subset A$. Prove that*

- $i_* : \pi_n(A, x_0) \longrightarrow \pi_n(X, x_0)$ *is epimorphism.,*
- $j_* : \pi_n(X, x_0) \longrightarrow \pi_n(X, A; x_0)$ *is zero homomorphism,*
- $\partial : \pi_n(X, A; x_0) \longrightarrow \pi_{n-1}(A, x_0)$ *is monomorphism.*

Lemma 8.3. (Five-Lemma) *Let the following diagram be commutative:*

$$\begin{tikzcd}
A_1 \arrow[r] \arrow[d, "\varphi_1"] & A_2 \arrow[r] \arrow[d, "\varphi_2"] & A_3 \arrow[r] \arrow[d, "\varphi_3"] & A_4 \arrow[r] \arrow[d, "\varphi_4"] & A_5 \arrow[d, "\varphi_5"] \\
B_1 \arrow[r] & B_2 \arrow[r] & B_3 \arrow[r] & B_4 \arrow[r] & B_5
\end{tikzcd} \tag{23}$$

Furthermore, let the rows be exact and the homomorphisms $\varphi_1, \varphi_2, \varphi_4, \varphi_5$ be isomorphisms. Then φ_3 is isomorphism.

Exercise 8.14. *Prove Lemma 8.3.*

Exercise 8.15. *Let exclude the homomorphism φ_3 from the diagram (23) and keep all other conditions of Lemma 8.3 the same. Does it follow then that $A_3 \cong B_3$? If not, give a counter example.*

Exercise 8.16. *Let* $0 \longrightarrow A_1 \longrightarrow A_2 \longrightarrow \cdots \longrightarrow A_n \longrightarrow 0$ *be an exact sequence of finitely generated abelian groups, then* $\sum_{i=1}^{n}(-1)^i \text{rank } A_i = 0$.

Exercise 8.17. *Let* $1 \longrightarrow G_1 \longrightarrow G_2 \longrightarrow \cdots \longrightarrow G_n \longrightarrow 1$ *be an exact sequence of finite groups (not necessarily abelian), then* $\sum_{i=1}^{n}(-1)^i \text{order } G_i = 0$.

Corollary 8.4. *Let* $f : (X, A) \longrightarrow (Y, B)$ *be a map of pairs,* $f(x_0) = y_0$*, where* $x_0 \in A$*,* $y_0 \in B$*. Then any two following statements imply the third one:*

- $f_* : \pi_n(X, x_0) \longrightarrow \pi_n(Y, y_0)$ *is an isomorphism for all* n.
- $f_* : \pi_n(A, x_0) \longrightarrow \pi_n(A, y_0)$ *is an isomorphism for all* n.
- $f_* : \pi_n(X, A; x_0) \longrightarrow \pi_n(Y, B; y_0)$ *is an isomorphism for all* n.

Exercise 8.18. *Prove Corollary 8.4.*