7 Covering spaces

Here I will be very brief, referring (when it is possible) to [Massey] and [Hatcher, p.56-82].

7/1. Definition and examples. A path-connected space T is a *covering space* over a path-connected space X, if there is a map $p : T \longrightarrow X$ such that for any point $x \in X$ there exists a path-connected neighbourhood $U \subset X$, such that $p^{-1}(U)$ is homeomorphic to $U \times \Gamma$ (where Γ is a descrete set), futhermore the following diagram commutes

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "\cong"] \arrow[dr, "p"'] & & U \times \Gamma \arrow[dl, "pr"] \\ & U & \end{tikzcd} \tag{16}$$

The neighbourhood U from the above definition is called an *elementary neighborhood.*

Examples. 1. $p : \mathbf{R} \longrightarrow S^1$, where $S^1 = \{ z \in \mathbf{C} \mid |z| = 1 \}$, and $p(\varphi) = e^{i\varphi}$.

2. $p : S^1 \longrightarrow S^1$, where $p(z) = z^k$, $k \in \mathbf{Z}$, and $S^1 = \{ z \in \mathbf{C} \mid |z| = 1 \}$.

3. $p : S^n \longrightarrow \mathbf{RP}^n$, where p maps a point $x \in S^n$ to the line in $\mathbf{R}^{n+1}$ going through the origin and x.

7/2. Theorem on covering homotopy. The following result is a key fact allowing to classify coverings.

Theorem 7.1. *Let $p : T \longrightarrow X$ be a covering space and Z be a CW-complex, and $f : Z \longrightarrow X$, $\widetilde{f} : Z \longrightarrow T$ such that the diagram*

$$\begin{tikzcd} & T \arrow[d, "p"] \\ Z \arrow[ur, "\widetilde{f}"] \arrow[r, "f"] & X \end{tikzcd} \tag{17}$$

commutes; futhermore it is given a homotopy $F : Z \times I \longrightarrow X$ such that $F|_{Z\times\{0\}} = f$. Then there exists a unique homotopy $\widetilde{F} : Z \longrightarrow T$ such that $\widetilde{F}|Z \times \{0\} = \widetilde{f}$ and the following diagram commutes:

$$\begin{tikzcd} & T \arrow[d, "p"] \\ Z \times I \arrow[ur, "\widetilde{F}"] \arrow[r, "F"] & X \end{tikzcd}$$

We prove first the following lemma:

Lemma 7.2. *For any path $s : I \longrightarrow X$ and any point $\tilde{x}_0 \in T$, such that $p(\tilde{x}_0) = x_0 = s(0)$ there exists a unique path $\tilde{s} : I \longrightarrow T$, such that $\tilde{s}(0) = \tilde{x}_0$ and $p \circ \tilde{s} = s$.*

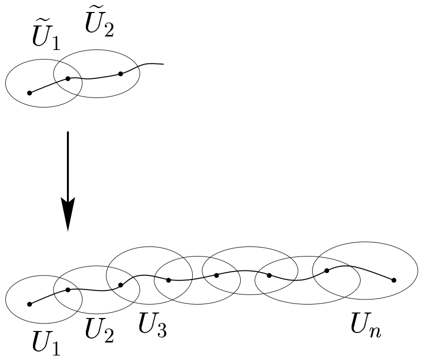

Fig. 7.1.

Proof. For each $t \in I$ we find an elementary neighbourhood U_t of the point $s(t)$. A compactness of $I = [0,1]$ implies that there exists a finite number of points $0 = t_1 < t_2 < \ldots < t_n = 1$, such that $U_j \supset s([t_j, t_{j+1}])$. The inverse image $p^{-1}(U_1)$ is homeomorphic to $U_1 \times \Gamma$ Let $\widetilde{U}_1$ be such that $\tilde{x}_0 \in \widetilde{U}_1$. Then the path $s|_{[0,t_2]} : [0,t_2] \longrightarrow X$ has a unique lifting $\tilde{s} : [0,t_2] \longrightarrow T$ covering the path $s|_{[0,t_2]}$. Then we do the same in the neighbourhood U_2 and so on. Note that we have a finite number of U_j, and in each neighbourhood U_j a "lifting" is unique, see Fig. 7.1.

Proof of Theorem 7.1. Let $z \in Z$ be any point. The formula $t \longrightarrow F(z,t)$ defines a path in X. Lemma 7.2 gives a unique lifting of this path to T, such that it starts at $\tilde{f}(z)$. It gives a map $Z \times I \longrightarrow T$. This is our homotopy $\widetilde{F}$. □

7/3. Covering spaces and fundamental group.

Theorem 7.3. *Let $p : T \longrightarrow X$ be a covering space, then $p_* : \pi_1(T, \tilde{x}_0) \longrightarrow \pi_1(X, x_0)$ is a monomorphism (injective).*

Proof: Let $\tilde{s} : I \longrightarrow T$ be a loop, where $\tilde{s}(0) = \tilde{s}(1) = \tilde{x}_0$. Denote $x_0 = p(\tilde{x}_0)$. Assume that the loop $s = p \circ \tilde{s} : I \longrightarrow X$ is homotopic to zero. Let $s_t : I \longrightarrow X$ be such a homotopy: $s_0 = s$, $s_t(0) = s_t(1) = x_0$, and $s_1(I) = x_0$.

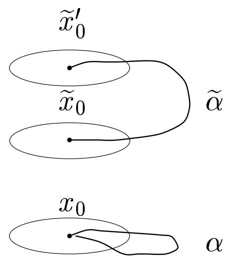

Fig. 7.2.

Theorem 7.1 implies that there is a homotopy $\tilde{s}_t : I \longrightarrow T$ covering the homotopy s_t. Since the inverse image $p^{-1}(x_0)$ is a descrete set, then $\tilde{s}_t(0) = \tilde{s}_t(1) = \tilde{x}_0$. □

The subgroup $p_*(\pi_1(T, \tilde{x}_0)) \subset \pi_1(X, x_0)$ is called the *covering group of $T \xrightarrow{p} X$*. Let $\tilde{x}_0' \neq \tilde{x}_0$, $p(\tilde{x}_0') = p(\tilde{x}_0) = x_0$. Consider a path $\tilde{\alpha} : I \longrightarrow T$ such that $\tilde{\alpha}(0) = \tilde{x}_0$, $\tilde{\alpha}(1) = \tilde{x}_0'$. Then the projection $\alpha = p(\tilde{\alpha})$ is a loop in X, see Fig. 7.2. Clearly $\alpha_{\#} : p_*(\pi_1(T, \tilde{x}_0)) \longrightarrow p_*(\pi_1(T, \tilde{x}_0'))$ given by $\alpha_{\#}(g) = \alpha g \alpha^{-1}$ is an isomorphism.

Consider the coset $\pi_1(X, x_0)/p_*(\pi_1(T, \tilde{x}_0))$ (the subgroup $p_*(\pi_1(T, \tilde{x}_0)) \subset \pi_1(X, x_0)$ is not normal subgroup in general).

Claim 7.1. *There is one-to-one correspondence $p^{-1}(x_0) \longleftrightarrow \pi_1(X, x_0)/p_*(\pi_1(T, \tilde{x}_0))$.*

Proof. Let $[\gamma] \in \pi_1(X, x_0)$, where $\gamma : I \longrightarrow X$, $\gamma(0) = \gamma(1) = x_0$. There exists a unique lifting $\widetilde{\gamma} : I \longrightarrow T$ of γ, so that $\widetilde{\gamma}(0) = \widetilde{x}_0$. We define $A([\gamma]) = \widetilde{\gamma}(1) \in p^{-1}(x_0)$, see Fig. 7.3. The homotopy lifting property implies that if $\gamma \sim \gamma'$ then $\widetilde{\gamma} \sim \widetilde{\gamma}'$ and $\widetilde{\gamma}(1) = \widetilde{\gamma}'(1)$. Now let $A([\gamma]) = A([\gamma'])$. Then the loop $\beta = (\gamma')^{-1}\gamma$ is covered by the loop $\widetilde{\beta} = (\widetilde{\gamma}')^{-1}\widetilde{\gamma}$, see Fig. 7.4. Thus $[\beta] = [(\gamma')^{-1}\gamma] \in p_*(\pi_1(T, \widetilde{x}_0))$.

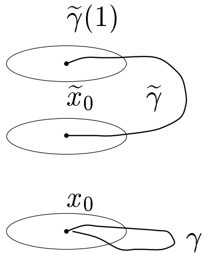

Fig. 7.3.

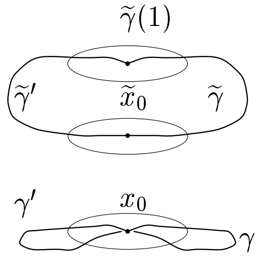

Fig. 7.4.

This proves that $A : \pi_1(X, x_0)/p_*(\pi_1(T, \widetilde{x}_0)) \longrightarrow p^{-1}(x_0)$ is an injection. Clearly A is onto since T is path-connected, and if $\widetilde{x} \in p^{-1}(x_0)$ there exists a path connecting x_0 and x which projects to a loop in X. $\square$

Claim 7.2. *Let $p : T \longrightarrow X$ be a covering and $x_0, x_1 \in X$. There is one-to-one correspondence $p^{-1}(x_0) \longleftrightarrow p^{-1}(x_1)$.*

Exercise 7.1. *Prove Claim 7.2.* **Hint:** *consider a path connecting x_0 and x_1.*

7/4. Observation. Let γ be a loop in X, $\gamma(0) = \gamma(1) = x_0$, and $\widetilde{\gamma} : I \longrightarrow T$ γ, be its lifting. Then if $\widetilde{\gamma}(1) \neq \widetilde{\gamma}(0)$ then the loop γ is not homotopic to zero. Indeed, if such homotopy would exist, then necessarily it implies that $\widetilde{\gamma}(1) = \widetilde{\gamma}(0)$.

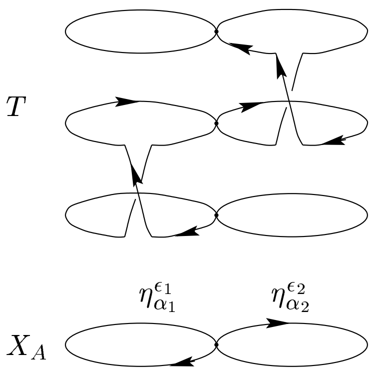

Fig. 7.5.

We use this observation to complete a proof of Theorem 6.4, or, to be precise, a proof of Claim 2^o 6.1. Indeed, let $\beta = \eta_{\alpha_1}^{\epsilon_1} \cdots \eta_{\alpha_s}^{\epsilon_s}$, $\epsilon_j = \pm 1$, where all elements $\eta_\alpha \eta_\alpha^{-1}$, $\eta_\alpha^{-1}\eta_\alpha$ are canceled. Recall that η_α is given by the inclusion $S^1_\alpha \longrightarrow \bigvee_{\alpha \in A} S^1_\alpha = X_A$. It is enough to construct a covering space $p : T \longrightarrow X_A$ so that the loop β is covered by a loop $\widetilde{\beta}$ with the property that $\widetilde{\beta}(0) \neq \widetilde{\beta}(1)$. Consider $s+1$ copies of the wedge X_A placed over X_A, see Fig. 7.5. We assume that these copies of X_A project vertically on X_A. Consider the word $\beta = \eta_{\alpha_1}^{\epsilon_1} \cdots \eta_{\alpha_s}^{\epsilon_s}$. Then we delete small intervals of the circles $S^1_{\alpha_1}$ at the first and second levels and "braid" these two circles together as it is shown at Fig. 7.5. We extend the verical projection to the "braid" in the obvious way. Then we join by a braid the circles $S^1_{\alpha_2}$ at the second and the third levels, and so on. In this way we construct a covering space T so that the loop $\beta = \eta_{\alpha_1}^{\epsilon_1} \cdots \eta_{\alpha_s}^{\epsilon_s}$ is covered by $\widetilde{\beta}$ which starts at the first level and ends at the last level. Thus $\beta = \eta_{\alpha_1}^{\epsilon_1} \cdots \eta_{\alpha_s}^{\epsilon_s} \neq 0$. $\square$

7/5. Lifting to a covering space. Consider the following situation. Let $p : T \longrightarrow X$ be a covering space, $x_0 \in X$, $\widetilde{x}_0 \in p^{-1}(x_0) \in T$. Let $f : Z \longrightarrow X$ be a map, so that $f(z_0) = x_0$. There is a natural question:

Question: Does there exist a map $\widetilde{f} : Z \longrightarrow T$ covering the map $f : Z \longrightarrow X$, such that $\widetilde{f}(z_0) = \widetilde{x}_0$? In other words, the lifting map $\widetilde{f}$ should make the following diagram commutative:

$$\begin{tikzcd} & T \arrow[d, "p"] \\ Z \arrow[ur, "\widetilde{f}"] \arrow[r, "f"'] & X \end{tikzcd} \tag{18}$$

where $f(z_0) = x_0$, $\widetilde{f}(z_0) = \widetilde{x}_0$. Clearly the diagram (18) gives the following commutative diagram of groups:

$$\begin{tikzcd} & \pi_1(T, \widetilde{x}_0) \arrow[d, "p_*"] \\ \pi_1(Z, z_0) \arrow[ur, "\widetilde{f}_*"] \arrow[r, "f_*"] & \pi_1(X, x_0) \end{tikzcd} \tag{19}$$

It is clear that commutativity of the diagram (19) implies that

$$f_*(\pi_1(Z, z_0)) \subset p_*(\pi_1(T, \widetilde{x}_0)). \tag{20}$$

Thus (20) is a necessary condition for the existence of the map $\widetilde{f}$. It turns out that (20) is also a sufficient condition.

Theorem 7.4. *Let $p : T \longrightarrow X$ be a covering space, and Z be a path-connected space, $x_0 \in X$, $\widetilde{x}_0 \in T$, $p(\widetilde{x}_0) = x_0$. Given a map $f : (Z, z_0) \longrightarrow (X, x_0)$ there exists a lifting $\widetilde{f} : (Z, z_0) \longrightarrow (T, \widetilde{x}_0)$ if and only if $f_*(\pi_1(Z, z_0)) \subset p_*(\pi_1(T, \widetilde{x}_0))$.*

Proof (outline). We have to define a map $\widetilde{f} : (Z, z_0) \longrightarrow (T, \widetilde{x}_0)$. Let $z \in Z$. Consider a path $\omega : I \longrightarrow Z$, so that $\omega(0) = z_0$, $\omega(1) = z$. Then the path $f(\omega) = \gamma$ has a unique lift $\widetilde{\gamma}$ so that $\widetilde{\gamma}(0) = \widetilde{x}_0$. We define $\widetilde{f}(z) = \widetilde{\gamma}(1) \in T$. We have to check that the construction does not depend on the choice of ω. Let ω' be another path such that $\omega'(0) = z_0$, $\omega'(1) = z$, see Fig. 7.6.

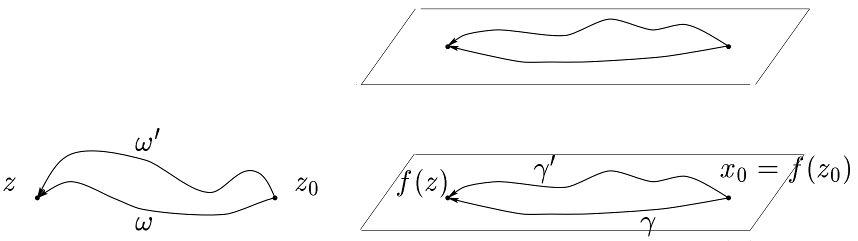

Fig. 7.6.

Let $\gamma' = f(\omega')$. Then we have a loop $\beta = (\gamma')^{-1}\gamma$, and $[\beta] \in f_*(\pi_1(Z, z_0))$. Since $f_*(\pi_1(Z, z_0)) \subset p_*(\pi_1(T, \widetilde{x}_0))$, the loop β may be lifted to the loop $\widetilde{\beta}$ in T. In particular, it follows that $\widetilde{\gamma}(1) = \widetilde{\gamma}'(1)$ because of uniqueness of the liftings $\widetilde{\gamma}$ and $\widetilde{\gamma}'$ and $\widetilde{\gamma}$.

Exercise 7.2. *Prove that the map $\widetilde{f}$ we constructed is continuous.*

Exercise 7.3. *Let $p : T \longrightarrow X$ be a covering, and $f, g : Y \longrightarrow T$ be two maps so that $p \circ f = p \circ g$, where Y is path-connected. Assume that $f(y) = g(y)$ for some point $y \in Y$. Prove that $f = g$.* **Hint:** *consider the set $V = \{ \ y \in Y \mid f(y) = g(y) \ \}$ and prove that V is open and closed in Y.*

Exercise 7.2 completes the proof. Exercise 7.3 implies that the lifting $\widetilde{f}$ is unique. $\square$

7/6. Classification of coverings over given space. Consider a category of covering over a space X. The objects of this category are covering spaces $T \xrightarrow{p} X$, and a morphism of covering $T_1 \xrightarrow{p_1} X$ to $T_2 \xrightarrow{p_2} X$ is a map $\varphi : T_1 \longrightarrow T_2$ so that the following diagram commutes:

$$\begin{tikzcd} T_1 \arrow[rr, "\varphi"] \arrow[dr, "p_1"'] & & T_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd} \tag{21}$$

Claim 7.3. *Let $\varphi, \varphi' : T_1 \longrightarrow T_2$ be two morphisms, and $\varphi(t) = \varphi'(t)$ for some $t \in T_1$. Then $\varphi = \varphi'$.*

Exercise 7.4. *Use Exercise 7.3 to prove Claim 7.3.*

Claim 7.4. *Let $T_1 \xrightarrow{p_1} X$, $T_2 \xrightarrow{p_2} X$ be two coverings, $x_0 \in X$, $\widetilde{x}_0^{(1)} \in p_1(x_0)$, $\widetilde{x}_0^{(2)} \in p_2(x_0)$. There exists a morphism $\varphi : T_1 \longrightarrow T_2$ such that $\varphi(\widetilde{x}_0^{(1)}) = \widetilde{x}_0^{(2)}$ if and only if $(p_1)_*(\pi_1(T_1, \widetilde{x}_0^{(1)})) \subset (p_1)_*(\pi_1(T_2, \widetilde{x}_0^{(2)}))$.*

Exercise 7.5. *Prove Claim 7.4.*

A morphism $\varphi : T \longrightarrow T$ is *automorphism* if there exists a morphism $\psi : T \longrightarrow T$ so that $\psi \circ \varphi = Id$ and $\varphi \circ \psi = Id$. Now consider the group $\mathrm{Aut}(T \xrightarrow{p} X)$ of automorphisms of a given covering $p : T \longrightarrow X$. The group operation is a composition and the identity element is the identity map $Id : T \longrightarrow T$. An element $\varphi \in \mathrm{Aut}(T \xrightarrow{p} X)$ acts on the space T.

Claim 7.5. *The group $Aut(T \xrightarrow{p} X)$ acts on the space T without fixed points.*

Exercise 7.6. *Prove Claim 7.5.* **Hint:** *A point $t \in T$ is a fixed point if $\varphi(t) = t$.*

Claim 7.6. *Let $T \xrightarrow{p} X$ be a covering, $x_0 \in X$, $\tilde{x}_0, \tilde{x}'_0 \in p^{-1}(x_0)$. Then there exists an automorphism $\varphi \in Aut(T \xrightarrow{p} X)$ such that $\varphi(\tilde{x}_0) = \tilde{x}'_0$ if and only if $p_*(\pi_1(T, \tilde{x}_0)) = p_*(\pi_1(T, \tilde{x}'_0))$.*

Exercise 7.7. *Prove Claim 7.6.*

Theorem 7.5. *Two coverings $T_1 \xrightarrow{p_1} X$ and $T_2 \xrightarrow{p} X$ are isomorphic if and only if for any two points $\tilde{x}_0^{(1)} \in p_1(x_0)$, $\tilde{x}_0^{(2)} \in p_2(x_0)$ the subgroups $(p_1)_*(\pi_1(T_1, \tilde{x}_0^{(1)}))$ $(p_1)_*(\pi_1(T_2, \tilde{x}_0^{(2)}))$ belong to the same conjugation class.*

Exercise 7.8. *Prove Theorem 7.5.*

Let $H \subset G$ be a subgroup. Recall that a normalizer $N(H)$ of H is a maximal subgroup of G such that H is a normal subgroup of that group. The subgroup $N(H)$ of the group G may be described as follows:

$$N(H) = \left\{ g \in G \mid gHg^{-1} = H \right\}.$$

Recall also that the group $\pi_1(X, x_0)$ acts on the set $\Gamma = p^{-1}(x_0)$, and Γ may be considered as a *right* $\pi_1(X, x_0)$*-set*; the subgroup $p_*(\pi_1(T, \tilde{x}_0))$ is the "isotropy group" of the point $\tilde{x}_0 \in p^{-1}(x_0)$. Again, we have seen that coset $\pi_1(X, x_0)/p_*(\pi_1(T, \tilde{x}_0))$ is isomorphic to $p^{-1}(x_0)$.

Corollary 7.6. *The group of automorphisms $Aut(T \xrightarrow{p} X)$ is isomorphic to the group $N(H)/H$, where $H = p_*(\pi_1(T, \tilde{x}_0)) \subset \pi_1(X, x_0)$ for any points $x_0 \in X$, $\tilde{x}_0 \in p^{-1}(x_0)$.*

Exercise 7.9. *Prove Corollary 7.6.*

Now remind that a covering space $p : T \longrightarrow X$ is a *regular covering space* if the group $p_*(\pi_1(T, \tilde{x}_0))$ is a normal subgroup of the group $\pi_1(X, x_0)$.

Exercise 7.10. *Prove that a covering space $p : T \longrightarrow X$ is regular if and only if there is no loop in X which is covered by a loop and a path (starting and ending in different points) in the same time.*

Exercise 7.11. *Prove that if a covering space $p : T \longrightarrow X$ is regular then there exists a free action of the group $G = \pi_1(X, x_0)/\pi_1(T, \tilde{x}_0)$ on the space T such that $X \cong T/G$.*

Exercise 7.12. *Prove that a two-folded covering space $p : T \longrightarrow X$ is always a regular one.*

We complete this section with the classification theorem:

Theorem 7.7. *Let X be a "good" path-connected space (in particular, CW-complexes are "good" spaces), $x_0 \in X$. Then for any subgroup $G \subset \pi_1(X, x_0)$ there exist a covering space $p : T \longrightarrow X$ and a point $\tilde{x}_0 \in T$, such that $p_*(\pi_1(T, \tilde{x}_0)) = G$.*

The idea of the proof: We consider the following equivalence relation on the space of paths $\mathcal{E}(X, x_0)$: two paths $s \sim s_1$ if $s(1) = s_1(1)$ and a homotopy class of the loop ss_1^{-1} belongs to G. We define $T = \mathcal{E}(X, x_0)/\sim$. The projection $p : T \longrightarrow X$ maps a path s to a point $s(1)$. The details are left to you. $\square$

Exercise 7.13. *Prove that in the above construction $p_*(\pi_1(T, \tilde{x}_0)) = G$.*

In particular, Theorem 7.7 claims the existence of the *universal covering space* $\widehat{T} \xrightarrow{\widehat{p}} X$ (i.e. such that $\pi_1(\widehat{T}, \tilde{x}_0) = 0$).

Exercise 7.14. *Let $\widehat{T} \xrightarrow{\widehat{p}} X$ be a universal covering over X, and $T \longrightarrow X$ be a covering. Prove that there exists a morphism $\varphi : \widehat{T} \longrightarrow T$ so that it is a covering over T.*

7/7. Homotopy groups and covering spaces. First, we have the following result:

Theorem 7.8. *Let $p : T \longrightarrow X$ be a covering space and $n \geq 2$. Then the homomorphism $p_* : \pi_n(T, \tilde{x}_0) \longrightarrow \pi_n(X, x_0)$ is an isomorphism.*

Exercise 7.15. *Prove Theorem 7.8.*

Theorem 7.8 allows us to compute homotopy groups of several important spaces. Actually there are only few spaces where all homotopy groups are known. Believe me or not, here we have at least half of those examples.

Theorem 7.9. $\pi_n(S^1) = \begin{cases} \mathbf{Z} & \text{if } n = 1, \\ 0 & \text{if } n \geq 2. \end{cases}$

One may prove Theorem 7.9 by applying Theorem 7.8 to the covering space $\mathbf{R} \xrightarrow{\exp} S^1$; of course, one should be able to prove $\pi_n(\mathbf{R}) = 0$ for all $n \geq 0$.

Corollary 7.10. *Let $X = \bigvee S^1$. Then $\pi_n(X) = 0$ for $n \geq 2$.*

Exercise 7.16. *Prove Theorem 7.9 and Corollary 7.10.* **Hint:** *Construct a universal covering space over $\bigvee S^1$; see the pictures given in [Hatcher, p.59].*

The next example is T^2: here we have a universal covering $\mathbf{R}^2 \longrightarrow T^2$, so it follows from Theorem 7.7 that $\pi_n(T^2) = 0$ for $n \geq 2$.

Exercise 7.17. *Let Kl^2 be the Klein bottle. Construct two-folded covering space $Kl^2 \longrightarrow T^2$. Compute $\pi_n(Kl^2)$ for all n.*

Theorem 7.11. *Let M^2 be a two-dimensional manifold without boundary, $M^2 \neq S^2, \mathbf{RP}^2$. Then $\pi_n(M^2) = 0$ for $n \geq 2$.*

Exercise 7.18. *Prove Theorem 7.11.* **Hint:** *One way is to construct a universal covering space over M^2; this universal covering space turns our to be $\mathbf{R}^2$. The second way may be as follows: Let M^2 be a sphere with two handles, and $X \longrightarrow M^2$ be the covering space pictured below:*

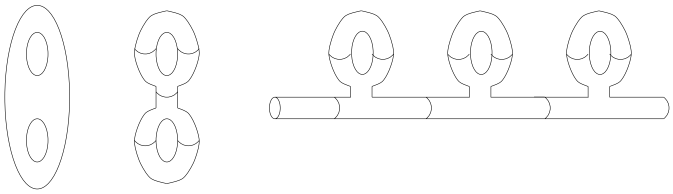

Fig. 7.7

Theorem 7.8 shows that $\pi_n(X) = \pi_n(M^2)$. Now let $f : S^n \longrightarrow X$, you may observe that $f(S^n)$ lies in the compact part of X; after cutting down the rest of X it becomes two-dimensional manifold with boundary and homotopy equivalent to its one-skeleton (Prove it!). Now it remains to make an argument in a general case.

7/8. Lens spaces. We conclude with important examples. Let $S^1 = \{z \in \mathbf{C} \mid |z| = 1\}$. The group S^1 acts freely on the sphere $S^{2n-1} \subset \mathbf{C}^n$ by $(z_1, \ldots, z_n) \mapsto (e^{i\varphi}z_1, \ldots, e^{i\varphi}z_n)$. The group $\mathbf{Z}/m$ may be thought as a subgroup of S^1:

$$\mathbf{Z}/m = \left\{ e^{2i\pi m/k} \mid k = 0, \ldots, m-1 \right\} \subset S^1.$$

Thus $\mathbf{Z}/m$ acts freely on the sphere S^{2n-1}. The space $L^{2n-1}(\mathbf{Z}/m) = S^{2n-1}/(\mathbf{Z}/m)$ is called a *lens space*. Thus S^{2n-1} is a universal covering space over the lens space $L^{2n-1}(\mathbf{Z}/m)$. Clearly $\pi_1(L^{2n-1}(\mathbf{Z}/m)) \cong \mathbf{Z}/m$, and $\pi_j(L^{2n-1}(\mathbf{Z}/m)) \cong \pi_j(S^{2n-1})$ for $j \geq 2$. The case $m = 2$ is well-known to us: $L^{2n-1}(\mathbf{Z}/2) = \mathbf{RP}^{2n-1}$.

Exercise 7.19. *Describe a cell decomposition of the lens space $L^{2n-1}(\mathbf{Z}/m)$.*

Consider the sphere $S^3 \subset \mathbf{C}^2$. Let p be a prime number, and $q \neq 0 \mod p$. We define the lens spaces $L^3(p, q)$ as follows. We consider the action of $\mathbf{Z}/p$ on $S^3 \subset \mathbf{C}^2$ given by the formula: $T : (z_1, z_2) \mapsto (e^{2\pi i/p}z_1, e^{2\pi iq/p}z_2)$. Let $L^3(p, q) = S^3/T$.

Exercise 7.20. *Prove that $\pi_1(L^3(p, q)) \cong \mathbf{Z}/p$.*

Certainly the lens spaces $L^3(p,q)$ are 3-dimensional manifolds, and for given p they all have the same fundamental group and the same higher homotopy groups $\pi_j(L^3(p,q))$ for $j \geq 2$ since S^3 is a universal covering space for all of them. Clearly one may suspect that some of these spaces are homeomorphic or at least homotopy equivalent. The following theorem gives classification of the lens spaces $L^3(p,q)$ up to homotopy equivalence. The result is rather surprising.

Theorem 7.12. *The lens spaces $L^3(p,q)$ and $L^3(p,q')$ are homotopy equivalent if and only if $q' \equiv \pm kq$ mod p for some integer k.*

We are not ready to prove Theorem 7.12. For instance the lenses $L^3(5,1)$ and $L^3(5,2)$ are not homotopy equivalent, and $L^3(7,1)$ and $L^3(7,2)$ are homotopy equivalent. However it is known that the lenses $L^3(7,1)$ and $L^3(7,2)$ are not homeomorphic, and the classification of the lenses $L^3(p,q)$ is completely resolved.