6 Fundamental group

6/1. General definitions. Here we define the homotopy groups $\pi_n(X)$ for all $n \geq 1$ and examine their basic properties. Let (X, x_0) be a pointed space, and (S^n, s_0) be a pointed sphere. Remind that we have defined the set $[S^n, X]$ as a set of homotopy classes of maps $f : S^n \longrightarrow X$, such that $f(s_0) = x_0$, and homotopy should preserve this property. In different terms we can think of a representative of $[S^n, X]$ as a map $I^n \longrightarrow X$ such that the image of the boundary ∂I^n of the cube I^n maps to the point x_0.

The sum of two spheres $f, g : S^n \longrightarrow X$ is defined as the map $f + g : S^n \longrightarrow X$, constructing as follows. First we identify the equator of the sphere S^n (which contains the point s_0) to a single point, so we obtain a wedge of two spheres $S^n \wedge S^n$, and then we map the "top sphere" S^n with the map f, and the "bottom sphere" S^n with the map g, see the picture below:

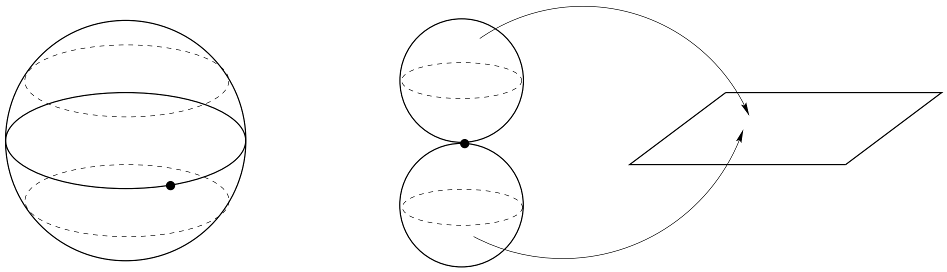

Fig. 6.1.

Exercise 6.1. *Prove that this operation is well-defined and induces a group structure on the set $\pi_n(X) = [S^n, X]$. In particular check assiciativity and existence of the unit.*

Lemma 6.1. *For $n \geq 2$ the homotopy group $\pi_n(X)$ is a commutative group.*

Proof. The corresonding homotopy is given below, where the black parts of the cube map to the point x_0:

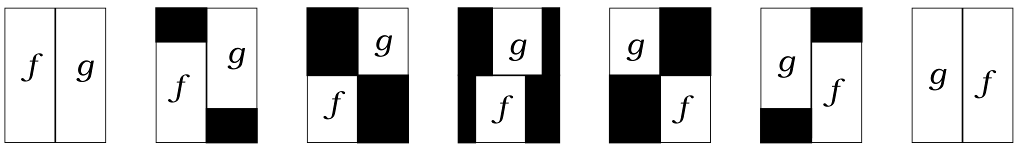

Fig. 6.2.

Remark. We note that the first homotopy group¶ $\pi_1(X)$ is not commutative in general. We will use $+$ for the operation in the homotopy groups $\pi_n(X)$ for $n \geq 2$ and product sign $\cdot$ for the fundamental group.

Now let $f : X \longrightarrow Y$ be a map; it induces a homomorphism $f_* : \pi_n(X) \longrightarrow \pi_n(Y)$.

¶ There is a special name for the group $\pi_1(X)$: *fundamental group of X*.

Exercise 6.2. *Prove that if $f, g : X \longrightarrow Y$ are homotopic maps of pointed spaces, than the homomorphisms $f_*, g_* : \pi_n(X) \longrightarrow \pi_n(Y)$ coincide.*

Exercise 6.3. *Prove that $\pi_n(X \times Y) \cong \pi_n(X) \times \pi_n(Y)$ for any spaces X, Y.*

6/2. One more definition of the fundamental group. The definition above was two general, we repeat it in more suitable terms again.

We consider loops of the space X, i.e. such maps $\varphi : I \longrightarrow X$ that $\varphi(0) = \varphi(1) = x_0$. The loops φ, φ' are homotopic if there is a homotopy $\varphi_t : I \longrightarrow X$, $(0 \le t \le 1)$ such that $\varphi_0 = \varphi$, $\varphi_1 = \varphi'$. A "product" of the loops φ, ψ is the loop ω, difined by the formula:

$$\omega(t) = \begin{cases} \varphi(2t), & \text{for } 0 \le t \le 1/2, \\ \psi(2t-1), & \text{for } 1/2 \le t \le 1. \end{cases}$$

This product operation induces a group structure of $\pi_1(X)$. It is easy to check that a group operation is well-defined. Note that the loop $\bar{\varphi}(t) = \varphi(1-t)$ defines a homotopy class $[\varphi]^{-1} = [\bar{\varphi}]$.

6/3. Dependence of the fundamental group on the base point.

Theorem 6.2. *Let X be a path-connected space, then $\pi_1(X, x_0) \cong \pi_1(X, x_1)$.*

Proof. Since X is path-connected, there exist a path $\alpha : I \longrightarrow X$, such that $\alpha(0) = x_0$, $\alpha(1) = x_1$. We define a homomorphism $\alpha_\# : \pi_1(X, x_0) \longrightarrow \pi_1(X, x_1)$ as follows. Let $[\varphi] \in \pi_1(X, x_0)$. We define $\alpha_\#([\varphi]) = (\alpha\varphi)\alpha^{-1}$. ‖ It is very easy to check that $\alpha_\#$ is well-defined and is a homomorphism. Moreover, the homomorphism $\alpha_\#^{-1} : \pi_1(X, x_1) \longrightarrow \pi_1(X, x_0)$ defined by the formula $\alpha_\#^{-1}([\psi]) = [(\alpha^{-1}\psi)\alpha]$, gives a homomorphism which is inverse to $\alpha_\#$. The rest of the proof is left to you. $\square$

Perhaps the isomorphism $\alpha_\#$ depends on α. Let β be the other path, $\beta(0) = x_0$, $\beta(1) = x_1$. Let $\gamma = \beta\alpha^{-1}$ which defines an element $[\gamma] \in \pi_1(X, x_1)$.

Exercise 6.4. *Prove that $\beta_\# = [\gamma]\alpha_\#[\gamma]^{-1}$.*

Exercise 6.5. *Let $f : X \longrightarrow Y$ be a homotopy equivalence, and $x_0 \in X$. Prove that $f_* : \pi_1(X) \longrightarrow \pi_1(Y, f(x_0))$ is an isomorphism.*

6/4. Fundamental group of circle. Here we will compute the fundamental group of the circle. In fact, we will be using a "universal covering space" of the circle which we did not defined yet.

‖ here we "multiply" not just loops, but paths as well: we can always do that if the second path starts at the same point where the first ends

Theorem 6.3. $\pi_1 S^1 \cong \mathbf{Z}$.

Proof. Consider the map $\exp : \mathbf{R} \longrightarrow S^1$ defined by the formula: $x \longrightarrow e^{ix}$. We can think about the circle S^1 as the quotient group $\mathbf{R}/\mathbf{Z}$ (where $\mathbf{Z}$ is embedded as the set of the numbers $2\pi k$, $k = 0, \pm 1, \pm 2, \ldots$). A loop $\varphi : I \longrightarrow S^1$ ($\varphi(0) = \varphi(1) = e^0$) may be lifted to a map $\widetilde{\varphi} : I \longrightarrow \mathbf{R}$. It means that φ is decomposed as

$$\varphi : I \xrightarrow{\widetilde{\varphi}} \mathbf{R} \xrightarrow{\exp} \mathbf{R}/\mathbf{Z} = S^1,$$

where $\widetilde{\varphi}(0) = 0$ and $\widetilde{\varphi}(1) = 2\pi k$ for some integer k. Note that a lifting $\widetilde{\varphi} : I \longrightarrow \mathbf{R}$ with the above properties is unique.

Note that if the loops $\varphi, \varphi' : I \longrightarrow S^1$ are homotopic, then the paths $\widetilde{\varphi}$, $\widetilde{\varphi}'$ have the same end point $2\pi k$ (since we cannot "jump" from $2\pi k$ to $2\pi l$ if $l \neq k$ by means of continuous homotopy!). Now the isomorphism $\pi_1 \cong \mathbf{Z}$ becomes almost obvious: $[\varphi] \longrightarrow k \in \mathbf{Z}$. It remains to see that the loop $\varphi : I \longrightarrow S^1$ ($\widetilde{\varphi} : I \longrightarrow \mathbf{R}$, where $\widetilde{\varphi}(0) = 0$ and $\widetilde{\varphi}(1) = 2\pi k$) is homotopic to the "standatrd loop" $\widetilde{h}_k$ going from 0 to $2\pi k$, see picture below:

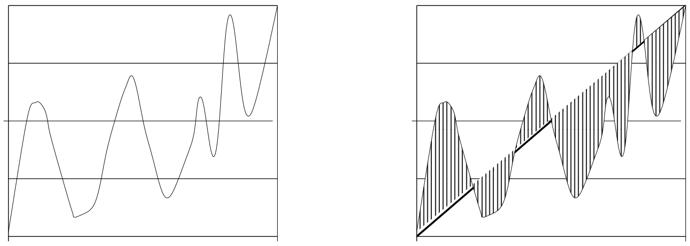

Fig. 6.3.

It remains to observe that $\widetilde{h}_k \widetilde{h}_l \sim \widetilde{h}_{k+l}$. $\square$

Theorem 6.4. *Let* $X_A = \bigvee_{\alpha \in A} S^1_\alpha$. *Then* $\pi_1(X_A)$ *is a free group with generators* η_α, $\alpha \in A$.

Proof. Let $i_\alpha : S^1 \longrightarrow X_A$ be an embedding of the corresponding circle. Let $\eta_\alpha \in \pi_1(X_A)$ be the element given by i_α. We prove the following statement.

Claim 6.1. 1^o *Any element* $\beta \in \pi_1(X_A)$ *may be represented as a finite product of elements* η_α, η_α^{-1}, $\alpha \in A$:

$$\beta = \eta_{\alpha_1}^{\epsilon_1} \cdots \eta_{\alpha_s}^{\epsilon_s}, \qquad \epsilon_j = \pm 1. \tag{12}$$

2^o *The presentation 12 is unique up to cancelation of the elements* $\eta_\alpha \eta_\alpha^{-1}$ *or* $\eta_\alpha^{-1}\eta_\alpha$.

Claim 6.1 is equivalent to Theorem 6.4. Now we prove **1^o**, and we postpone **2^o** to the next section.

Let I_α, J_α be two closed intervals in the circle, $J_\alpha \subset \mathrm{Int} I_\alpha$, and I_α does not contain the base point.

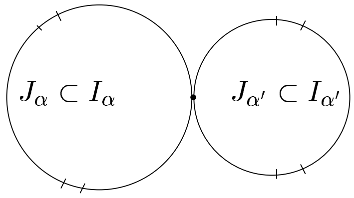

Fig. 6.4.

Now let $\varphi : I \longrightarrow X_A$ be a loop. We find n such that for any interval J of the length $1/n$ if the intersection $\varphi(J) \cap J_\alpha \neq \emptyset$, then $\varphi(J) \subset I_\alpha$. Let K be the following union:

$$K = \bigcup_{\varphi([k/n,(k+1)/n]) \cap (\cup_\alpha J_\alpha) \neq \emptyset} [k/n, (k+1)/n].$$

Now we construct a map $\varphi_1 : I \longrightarrow X_A$ which coincides with φ outside of K and in all points with the coordinates k/n, and it is linear on each interval $[k/n, (k+1)/n] \subset K$.**

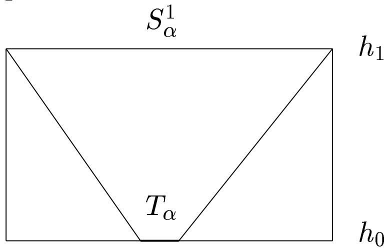

Fig. 6.5. Homotopy h_t

Clearly the loop φ_1 is homotopic to φ. Now we find an interval $T_\alpha \subset J_\alpha$, so that T_α does not contain points $\varphi_1(k/n)$. We can do this since there is only finite number of points like that inside of each J_α. We notice that $\varphi_1^{-1}(T_\alpha) \subset I$ is a finite number of disjoint intervals $S^{(1)}_\alpha, \ldots, S^{(r)}_\alpha$ so that the map $\varphi_1|_{S^{(j)}_\alpha} : S^{(j)}_\alpha \longrightarrow T_\alpha$ is linear for each j. The last step: we define a homotopy $h_t : X_A \longrightarrow X_A$ which stretches linearly each interval T_α on the circle S^1_α and taking $S^1_\alpha \setminus T_\alpha$ to the base point.

Exercise 6.6. *Prove that the inverse image $\varphi_1^{-1}(\cup_\alpha T_\alpha) \subset I$ consists of finite number of disjoint intervals.*

Then the map $\psi = \varphi_1 \circ h_1$ gives a loop which maps I as follows. For each $\alpha \in A$ there is finite number of disjoint intervals $S^{(j)}_\alpha \subset I$ so that $S^{(j)}_\alpha$ maps linearly on the circle S^1_α and the rest of the interval I maps to the base point. $\square$

6/5. Fundamental group of a finite CW-complex. Here we prove a general result showing how to compute the fundamental group $\pi_1(X)$ for arbitrary CW-complex X.

Remark. Let X be a path-connected. If a map $S^1 \longrightarrow X$ sends a base point s_0 to a base point x_0 then it determines an element of $\pi_1(X, x_0)$; if f sends s_0 somethere else, then it defines an element of the group $\pi(X, f(s_0))$, which is isomorphic to $\pi_1(X, x_0)$ with an isomorphism $\alpha_\#$. The images of the element $[f] \in \pi(X, f(s_0))$ in the group $\pi_1(X, x_0)$ under all possible isomorphisms $\alpha_\#$ define a class of conjugated elements. So we can say that a map $S^1 \longrightarrow X$ to a path-connected space X determines an element of $\pi_1(X, x_0)$ up to conjugation.

Let X be a CW-complex with a single zero-cell e^0, one-cells e^1_i, $i \in I$, and two-cells e^2_j, $j \in J$. The characteristic map $g_j : D^2 \longrightarrow X$ of the cell e^2_j determines the attaching

**A linear map $I \longrightarrow S^1$ is given by $t \mapsto (\cos(\lambda t + \mu), \sin(\lambda t + \nu))$ for some constants λ, μ, ν.

map $f_j : S^1 \longrightarrow X^1$ which gives some element $\beta_j \in \pi_1(X^1, x_0)$ up to conjugation. By Theorem 6.4 $\pi_1(X^1, x_0)$ is a free group on generators α_i, $i \in I$.

Theorem 6.5. *Let X be a CW-complex with a single zero cell e^0, one-cells e_i^1 $(i \in I)$, and two-cells e_j^2 $(j \in J)$. Let α_i, be generators corresponding of each e_i^1, and $\beta_j \in \pi_1(X^1, x_0) = F(\alpha_i \mid i \in I)$ be elements determined by attaching maps $f_j : S^1 \longrightarrow X^1$ of the cells e_j^2. Then*

1. $\pi_1(X, x_0) \cong \pi_1(X^2, x_0)$;

2. $\pi_1(X, x_0)$ *is a group on generators α_i, $i \in I$, and relations $\beta_j = 1$, $j \in J$.*

Proof. We consider the circle S^1 as 1-dimensional CW-complex. Cellular Approximation Theorem implies then that any loop $S^1 \longrightarrow X$ homotopic to a loop in the first skeleton, i.e. the homomorphism

$$i_* : \pi_1(X^{(1)}, x_0) \longrightarrow \pi_1(X, x_0)$$

induced by the inclusion $i : X^{(1)} \longrightarrow X$, is an epimorphism. It is enough to prove that Ker i_* is generated by β_j, $j \in J$. It is clear that $\beta_j \in$ Ker i_*. Indeed, the attaching map $f_j : S^1 \longrightarrow X^{(1)}$ is extended to the characteristic map $g_j : D^2 \longrightarrow X$, and determines a trivial element in the group $\pi_1(X, f_j(s_0))$; and this element corresponds to β_j under some isomorphism $\pi_1(X, f_j(s_0)) \cong \pi_1(X, x_0)$.

It is more difficult to prove that if $\gamma \in$ Ker i_* then x may be presented as a product of elements $\beta_j^{\pm}$: here we will apply again the Cellular Approximation Theorem. We identify each cell e_j^2 with the open disk D_j^2 in $\mathbf{R}^2$, so we can construct disks $d^{(j)} \subset D_j^2$ of radius $r^{(j)}$, and disks $d_1^{(j)}$, $d_2^{(j)}$, $d_3^{(j)}$ and $d_4^{(j)}$ (with the same center) of radius $r^{(j)}/5$, $2r^{(j)}/5$, $3r^{(j)}/5$ and $4r^{(j)}/5$ respectively.

Now let $\varphi : S^1 \longrightarrow X^{(1)}$ be a representative of an element $\gamma \in$ Ker i_*. Clearly there is an extension $\Phi : D^2 \longrightarrow X$ of the map φ. By the Cellular Approximation Theorem we can assume that $\Phi(D^2) \subset X^2$. We triangulate D^2 in such way that if Δ is a triangle from this triangulation and if $\Phi(\Delta) \cap d_4^{(j)} \neq \emptyset$ then

(a) $\Phi(\Delta) \subset d^{(j)}$ and

(b) diam$(\Phi(\Delta) < r^{(j)}/5$.

Let K be a union of all triangles Δ of our triangulation such that

$$\Phi(\Delta) \cap \left(\bigcup_{j \in J} d_4^{(j)} \right) \neq \emptyset.$$

Now we make the map $\Phi' : K \longrightarrow X^2$ which concides with Φ on the vertexes of each simplex and is linear on each simplex Δ. The maps Φ' and $\Phi|_K$ are homotopic (inside the cell e_j^2) by means of a homotopy $\Phi_t = (1-t)\Phi + t\Phi'$, with $\Phi_0 = \Phi|_K$, $\Phi_1 = \Phi'$. Now we use a familiar formula

$$\Phi'' = \begin{cases} \Phi(u), & \text{if } \ \Phi(u) \notin \bigcup_j d_3^{(j)} \\ \Phi'(u), & \text{if } \ \Phi(u) \in \bigcup_j d_2^{(j)} \\ \Phi_{\frac{3-5r(u)}{r_j}}(u), & \text{if } \ \Phi(u) \in d_3^{(j)} \setminus d_2^{(j)} \end{cases}$$

to define a map Φ'', which is a piece-wise linear on the inverse image of $\bigcup_j d_1^{(j)}$.

Now we choose a small disk $\delta^{(j)} \subset d_1^{(j)}$ which does not intersect with images of any vertex and a side of all simlexes Δ. There are two possibilities:

1. $\delta^{(j)} \subset \Phi''(\Delta)$ for some simplex Δ;

2. $(\Phi'')^{-1}(\delta^{(j)}) = \emptyset$.

Let $\omega_j : X^2 \longrightarrow X^2$ be a map identical on $X^{(1)}$ and mapping each disk $\delta^{(j)}$ on the cell e_j^2 (by pushing $e^2 \setminus \delta^{(j)}$ to the boundary of e_j^2). The map

$$\Psi : D^2 \xrightarrow{\Phi''} X^{(2)} \xrightarrow{\omega} X^{(2)},$$

extends the same map $\varphi : S^1 \longrightarrow X^{(1)}$.

Note that in the case **1.** the inverse image of $\delta^{(j)}$ under the map Φ'' is a finite number of ovals $E_1, \dots, E_s$ (bounded by an ellips), and in the case **2.** the inverse image of $\delta^{(j)}$ is empty. We see that the map Ψ maps the complement $D^2 \setminus (\bigcup_s E_s)$ to $X^{(1)}$, and maps each oval $E_1 \dots, E_k$ linearly on one of the cells e_j^2.

We join now a point $s_0 \in S^1 \subset D^2$ with each oval $E_1, \dots, E_k$ by paths $s_1, \dots s_k$, which do not intersect with each other, see the picture below:

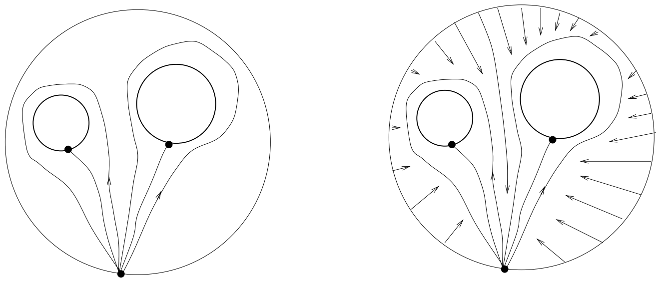

Fig. 6.6.

We denote by $\sigma_1, \dots, \sigma_k$ the loops, going clock-wise around each oval. Then the loop σ going clock-wise along the circle $S^1 \subset D^2$ is homotopic in $D^2 \setminus \bigcup_t \mathrm{Int}(E_t)$ to the loop:

$$(s_1\sigma_1 s_1^{-1})(s_2\sigma_2 s_2^{-1}) \cdots (s_k\sigma_k s_k^{-1}),$$

see Fig. 6.6.

It means that the loop $\varphi : S^1 \longrightarrow X^{(2)}$ is homotopic (in $X^{(1)}$) to the loop

$$[\Psi \circ (s_1\sigma_1 s_1^{-1})][\Psi \circ (s_2\sigma_2 s_2^{-1})] \cdots [\Psi \circ (s_k\sigma_k s_k^{-1})].$$

It remains to observe that the loop $[\Psi \circ (s_j\sigma_j s_j^{-1})]$ determines an element in $\pi_1(X, x_0)$, conjugate to $\beta_j^{\pm}$.

We see now that the element γ belongs to a normal subgroup of $F(\alpha_i \mid i \in I)$, generated by β_j. □

Theorem 6.5 helps to compute fundamental groups of all classic spaces. In the case of S^n $(n \geq 2)$ and $\mathbf{CP}^n$, $n \geq 1$ we see that the fundamental group is trivial. However, there are several interesting cases:

Theorem 6.6. *Let M^2 be a two-dimensional manifold, the sphere with g handles (oriented manifold of genus g). Then $\pi_1(M^2)$ is generated by $2g$ generators $a_1, \dots a_g, b_1, \dots, b_g$ with a single relation:*

$$a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} = 1.$$

Exercise 6.7. *Prove Theorem 6.6.*

Remark. We note that in particular $\pi_1(T^2) \cong \mathbf{Z} \oplus \mathbf{Z}$, which is obvious from the product formula $\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$.

Recall that a non-oriented two-dimensional manifold of genus g is heomeomorphic either to $M_g^2(1)$, a connective sum of a projective plane $\mathbf{RP}^2$ and g tori $T^2 \# \cdots \# T^2$, or to $M_g^2(2)$, a connective sum of the Klein bottle Kl^2 and g tori $T^2 \# \cdots \# T^2$.

Theorem 6.7. 1. *The group $\pi_1(M_g^2(1))$ is isomorphic to a group on generators $c_1, \dots, c_{2g+1}$ wit a single relation*

$$c_1^2 \cdots c_{2g+1}^2 = 1.$$

2. *The group $\pi_1(M_g^2(2))$ is isomorphic to a group on generators $c_1, \dots, c_{2g+2}$ wit a single relation*

$$c_1^2 \cdots c_{2g+1}^2 c_{2g+2}^2 = 1.$$

Exercise 6.8. *Prove Theorem 6.7.*

Exercise 6.9. *Compute $\pi_1(\mathbf{RP}^n)$, $\pi_1(Kl^2)$.*

Exercise 6.10. *Prove that the fundamental groups computed in Theorems 6.6, 6.7 are pair-wise nonisomorphic. Make the conclusion that any two manifolds above are not homeomorphic and even are not homotopy equivalent to each other.*

6/6. Theorem of Seifert and Van Kampen. Here we will need some algebraic material, we give only basic definition and refer to [Massey, Chapter 3] and [Hatcher, 1.2] for detailes.

Let G_1, G_2 be two groups with system of generators A_1, A_2respectively and relations R_1, $\mathbf{R}_2$. A group with a system of generators $A_1 \cup A_2$ (disjoint union) and system of relations $R_1 \cup R_2$ is called a *free product* of G_1 and G_2 and is denoted as $G_1 * G_2$.

Exercise 6.11. *Prove that the group* $\mathbf{Z}_2 * \mathbf{Z}_2$ *contains a subgroup isomorphic to* $\mathbf{Z}$ *and* $(\mathbf{Z}_2 * \mathbf{Z}_2)/\mathbf{Z} \cong \mathbf{Z}_2$.

Exercise 6.12. *Prove that* $\pi_1(X \vee Y) = \pi_1(X) * \pi_1(Y)$.

Remark. As it is defined in [Massey, Ch. 3], the group $G = G_1 * G_2$ may be characterized as follows. Let $\varphi_1 : G_1 \longrightarrow G$ and $\varphi_2 : G_2 \longrightarrow G$ be natural homomorphisms and let L be a group and $\psi_1 : G_1 \longrightarrow H$, $\psi_2 : G_2 \longrightarrow H$, then there exist a unique homomorphism $\psi : G \longrightarrow L$, such that the diagram

$$\begin{tikzcd}
& G \arrow[dd, "\psi"] & \\
G_1 \arrow[ur, "\varphi_1"] \arrow[dr, "\psi_i"'] & & G_2 \arrow[ul, "\varphi_2"'] \arrow[dl, "\psi_2"] \\
& L &
\end{tikzcd} \tag{13}$$

is commutative. The above definition may be generalized as follows. Assume that we also are given two homomorphisms $\rho_1 : H \longrightarrow G_1$, $\rho_2 : H \longrightarrow G_2$. Let us choose generators $\{h_\alpha\}$ of H and define a group $G_1 *_H G_2$ by adding the relations $\rho_1(h_\alpha) = \rho_2(h_\alpha)$ to relations of $G_1 * G_2$.

In different terms we may define the group $G_1 *_H G_2$ as follows. Assume that we are given a commutative diagram:

$$\begin{tikzcd}
& G_1 \arrow[dr, "\psi_1"] & \\
H \arrow[ur, "\rho_1"] \arrow[rr, "\psi_{1,2}"] \arrow[dr, "\rho_2"'] & & L \\
& G_2 \arrow[ur, "\psi_2"'] &
\end{tikzcd} \tag{14}$$

The group $G_1 *_H G_2$ is characterized by the following property: There are such homomorphisms $\sigma : H \longrightarrow G_1 *_H G_2$, $\sigma_1 : G_1 \longrightarrow G_1 *_H G_2$ and $\sigma_2 : G_2 \longrightarrow G_1 *_H G_2$ that for each homomorphisms $\psi_1 : G_1 \longrightarrow L$, $\psi_2 : G_2 \longrightarrow L$ and $\psi_{1,2} : H \longrightarrow L$ such that the diagram (14) is commutative, there exists a unique homomorphism $\psi : G_1 *_H G_2 \longrightarrow L$ such that the following diagram is commutative:

$$\begin{tikzcd}
& & G_1 *_H G_2 \arrow[dd, "\psi"] & & \\
H \arrow[r, "\rho_1"] \arrow[urr, "\sigma"] \arrow[drr, "\psi_{1,2}"'] & G_1 \arrow[ur, "\sigma_1"] \arrow[dr, "\psi_1"] & & G_2 \arrow[ul, "\sigma_2"'] \arrow[dl, "\psi_2"'] & H \arrow[l, "\rho_2"'] \arrow[ull, "\sigma"'] \arrow[dll, "\psi_{1,2}"] \\
& & L & &
\end{tikzcd} \tag{15}$$

Exercise 6.13.* *Prove that the group* $SL_2(\mathbf{Z})$ *of integral* 2×2*-matrices is isomorphic to* $\mathbf{Z}_4 *_{\mathbf{Z}_2} \mathbf{Z}_6$.

Theorem 6.8. (Seifert, Van Kampen) *Let* Y_1, Y_2 *and* $Z = Y_1 \cap Y_2$ *be connected subcomplexes of a CW-complex* X, $x_0 \in Y_1 \cap Y_2 \subset X$, *and* $\rho_1 : \pi_1(Y_1) \longrightarrow \pi_1(X)$, $\rho_2 : \pi_1(Y_2) \longrightarrow \pi_1(X)$. *Then*

$$\pi_1(X) \cong \pi_1(Y_1) *_{\pi_1(Z)} \pi_1(Y_2).$$

Exercise 6.14. *Prove Theorem 6.8 in the case of finite CW-complexes using induction on the number of cells of* $Y_1 \cap Y_2$.

Remark. There is more general version of Van Kampen Theorem, see [Massey, Ch. 4] and [Hatcher, 1.2].