5 CW-complexes and homotopy

Here we study the homotopy properties of CW-complexes.

5/1. Borsuk's Theorem on extension of homotopy. We call a pair (of topological spaces) (X, A) a *Borsuk pair*, if for any map $F : X \longrightarrow Y$ a homotopy $f_t : A \longrightarrow Y$, $0 \le t \le 1$, such that $f_0 = F|_A$ may be extended up to homotopy $F_t : X \longrightarrow Y$, $0 \le t \le 1$, such that $F_t|_A = f_t$ and $F_0 = F$.

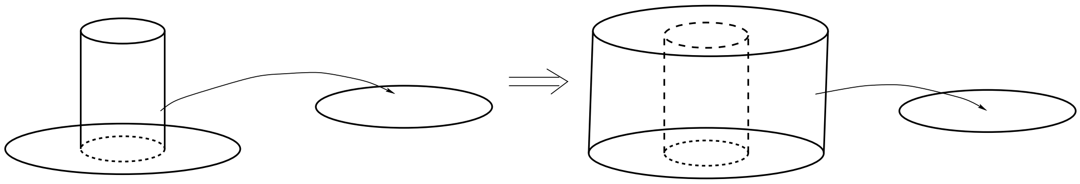

Fig. 5.1.

A major technical result of this subsection is the following theorem.

Theorem 5.1. (Borsuk) *A pair (X, A) of CW-complexes is a Borsuk pair.*

Proof. We are given a map $\Phi : A \times I \longrightarrow Y$ (a homotopy f_t) and a map $F : X \times \{0\} \longrightarrow Y$, such that $F|_{A\times\{0\}} = \Phi|_{A\times\{0\}}$. To extend a homotopy f_t up to homotopy F_t is the same as to construct a map $F' : X \times I \longrightarrow Y$ such that $F'|_{A\times I} = \Phi$. We construct F' by induction on dimension of cells of $X \setminus A$. The first step is to extend Φ to the space $X \cup (A \cup X^{(0)}) \times I$ as follows:

$$F'(x,t) = \begin{cases} F(x), & \text{if } x \text{ is a 0-cell from } X \text{ and if } x \notin A, \\ \Phi(x,t), & \text{if } x \in A. \end{cases}$$

Now assume that F' is already defined on $X \cup (A \cup X^{(n)}) \times I$. Let e^{n+1} be a $(n+1)$-cell, $e^{n+1} \subset X \setminus A$. By induction, the map F' is already given on the cylinder $(\bar{e}^{n+1} \setminus e^{n+1}) \times I$ since the boundary $\partial e^{n+1} = \bar{e}^{n+1} \setminus e^{n+1} \subset X^{(n)}$. Let $g : D^{n+1} \longrightarrow X^{(n+1)}$ be a characterictic map corresponding to the cell e^{n+1}. We have to define an extension of F' from the side $g(S^n) \times I$ and the bottom base $g(D^{n+1}) \times \{0\}$ to the cylinder $g(D^{n+1}) \times I$. By definition of CW-complex, it is the same as to construct an extension of the map

$$\psi = F' \circ g : (D^{n+1} \times \{0\}) \cup (S^n \times I) \longrightarrow Y$$

to a map of the cylinder $\psi' : D^{n+1} \times I \longrightarrow Y$. Let

$$\eta : D^{n+1} \times I \longrightarrow (D^{n+1} \times \{0\}) \cup (S^n \times I)$$

be a projection map of the cylinder $D^{n+1} \times I$ from a point s which is near and a bit above of the top side $D^{n+1} \times \{1\}$ of the cylinder $D^{n+1} \times I$, see the picture below:

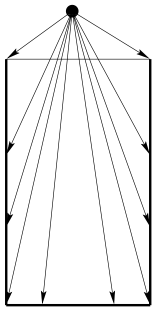

Fig. 5.2.

The map η is an identical map on $(D^{n+1}\times\{0\})\cup(S^n\times I)$. We define an extension ψ' as follows:

$$\psi' : D^{n+1}\times I \xrightarrow{\eta} (D^{n+1}\times\{0\})\cup(S^n\times I) \xrightarrow{\psi} Y.$$

This procedure may be carried out independently for all $(n+1)$-cells of X, so we obtain an extension

$$F' : X\cup(A\cup X^{(n+1)})\times I \longrightarrow Y.$$

Exercise 5.1. *Let $D^{n+1}\times I\subset\mathbf{R}^{n+1}$ given by:*

$$D^{n+1}\times I = \left\{(x_1,\dots,x_{n+1},x_{n+2})\ |x_1^2+\cdots+x^{n+1}\le 1,\quad x_{n+1}\in[0,1]\right\}.$$

Give a formula for the above map η.

Thus, going from the skeleton $X^{(n)}$ to the skeleton $X^{(n+1)}$, we construct an extension $F' : X\times I\longrightarrow Y$ of the map $\Phi : A\times I\longrightarrow Y$.

We should emphasize that if X is an infinite-dimensional complex, then our construction consists of infinite number of steps; in that case the axiom **(W)** implies that F' is a continuous map. □

Corollary 5.2. *Let X be a CW-complex and $A\subset X$ be its contractible subcomplex. Then X is homotopy equivalent to the complex X/A.*

Proof. Let $p : X\longrightarrow X/A$ be a projection map. Since A is a contractible there exists a homotopy $f_t : A\longrightarrow A$ such that $f_0 : A\longrightarrow A$ is an identity map, and $f_1(A)=x_0\in A$. By Theorem 5.1 there exists a homotopy $F_t : X\longrightarrow X$ such that $F|_{X\times\{0\}}=Id_X$ and $F_t|_A=f_t$. In particular, $F_1(A)=x_0$. It means that F_1 may be considered as a map given on X/A, (by definition of the quotient topology), i.e.

$$F_1=q\circ p : X\xrightarrow{p} X/A\xrightarrow{q} X,$$

where $q : X/A\longrightarrow X$ is some continuous map. By construction, $F_1\sim F_0$, i.e. $q\circ p\sim Id_X$. Now, $F(A)\subset A$ for any t, i.e. $p\circ F_t(A)=x_0$. It follows that $p\circ F_t=h_t\circ p$, where $h_t : X/A\longrightarrow X/A$ is some homotopy, such that $h_0=Id_{X/A}$ and $h_1=p\circ q$; it means that $p\circ q\sim Id_{X/A}$. □

Corollary 5.3. *Let X be a CW-complex and $A\subset X$ be its subcomplex. Then X/A is homotopy equivalent to the complex $X\cup C(A)$, where $C(A)$ is a cone over A.*

Exercise 5.2. *Prove Corollary 5.3.*

5/2. Cellular Approximation Theorem Let X and Y be CW-complexes. Recall that a map $f : X \longrightarrow Y$ is a *cellular map* if $f(X^{(n)}) \subset Y^{(n)}$ for any $n = 0, 1, \ldots$. We emphasize that it is not required that the image of n-cell belongs to a union of n-cells. For example, a constant map $* : X \longrightarrow x_0 = e^0$ is a cell map. The following theorem provides a very important tool in algebraic topology.

Theorem 5.4. *Any continuous map $f : X \longrightarrow Y$ of CW-complexes is homotopic to a cellular map.*

We shall prove the following stronger statement:

Theorem 5.5. *Let $f : X \longrightarrow Y$ be a continuous map of CW-complexes, such that a restriction $f|_A$ is a cellular map on a CW-subcomplex $A \subset X$. Then there exists a cell map $g : X \longrightarrow Y$ such that $g|_A = f|_A$ and, moreover, $f \sim g$ rel A.*

First of all, we should explain the notation $f \sim g$ rel A which we are using. Assume that we are given two maps $f, g : X \longrightarrow Y$ such that $f|_A = g|_A$. A notation $f \sim g$ rel A means that there exist a homotopy $h_t : X \longrightarrow Y$ such that $h_t(a)$ does not depend on t for any $a \in A$. Certainly $f \sim g$ rel A implies $f \sim g$, but $f \sim g$ does not imply $f \sim g$ rel A.

Exercise 5.3. *Give an example of a map $f : [0, 1] \longrightarrow S^1$ which is homotopic to a constant map, and, at the same time f is not homotopic to a constant map relatively to $A = \{0\} \cup \{1\} \subset I$.*

Proof of Theorem 5.5. We assume that f is already a cellular map not just on A, but also on all cells of X of dimension less or equal to $(p - 1)$. Consider a cell $e^p \subset X \setminus A$. The image $f(e^p)$ has nonempty intersection only with a finite number of cells of Y: this is because $f(\bar{e}^p)$ is a compact. We choose a cell of maximal dimension ϵ^q of Y such that it has nonempty intersection with $f(e^p)$. If $q \leq p$, then we are done with the cell e^p and we pass to another one. Consider the case when $q > p$. Here we need the following lemma.

Lemma 5.6. (Free Point Lemma) *Let U be an open subset of $\mathbf{R}^p$, and $\varphi : U \to \overset{o}{D}{}^q$ be a continuous map such that the set $V = \varphi^{-1}(\mathbf{d}^q) \subset U$ is compact for some closed disk $\mathbf{d}^q \subset \overset{o}{D}{}^q$. If $q > p$ there exists a continuous map $\psi : U \longrightarrow \overset{o}{D}{}^q$ such that*

1. $\psi|_{U \setminus V} = \varphi|_{U \setminus V}$;
2. *the image $\psi(V)$ does not cover all disk $\mathbf{d}^q$, i.e. there exists a point $y_0 \in \mathbf{d}^q \setminus \psi(U)$.*

We postpone a proof of this Lemma for a while.

Remark. The maps φ and ψ from Lemma 5.6 are homotopic relatively to $U \setminus V$: it is enough to make a linear homotopy: $h_t(x) = (1 - t)\varphi(x) + t\psi(x)$ since the disk $\overset{o}{D}{}^q$ is a convex set.

Claim 5.1. *Lemma 5.6 implies the following statement: The map*

$$f|_{A\cup X^{(p-1)}\cup e^p} \quad \textit{is homotopic rel } (A\cup X^{(p-1)}) \textit{ to a map} \quad f' : A\cup X^{(p-1)}\cup e^p \longrightarrow Y,$$

such that the image $f'(e^p)$ does not cover all cell ϵ^q.

Proof. Indeed, let $h : D^q \longrightarrow X$, $k : D^q \longrightarrow Y$ be the characteristic maps of the cells e^p and ϵ^q respectively. Let

$$U = h^{-1}(f^{-1}(\epsilon)\cap e^p),$$

and let $\varphi : U \longrightarrow \overset{o\,q}{D}$ be the composition:

$$U \xrightarrow{h} e^p \cap f^{-1}\epsilon^q \xrightarrow{f} \epsilon^q \xrightarrow{k^{-1}} \overset{o\,q}{D}.$$

Let $\mathbf{d}^q$ be a small disk inside $\overset{o\,q}{D}$ (with the same center as $\overset{o\,q}{D}$). The set $V = \varphi^{-1}(\mathbf{d}^q)$ is compact (as a closed subset of the disk $\overset{o\,p}{D}$). Let $\psi : U \longrightarrow \overset{o\,q}{D}$ be a map from Lemma 5.6. We define a map f' on $h(U)$ as the composition:

$$h(U) \xrightarrow{h^{-1}} U \xrightarrow{\psi} \overset{o\,q}{D} \longrightarrow \epsilon^q \subset Y,$$

and $f'(x) = f(x)$ for $x \notin h(U)$. Clearly the map

$$f' : A\cup X^{(p-1)}\cup e^p \longrightarrow Y$$

is continuous (since it coincides with f on $h(U\setminus V)$) and

$$f' : A\cup X^{(p-1)}\cup e^p \longrightarrow Y \sim f|_{A\cup X^{(p-1)}\cup e^p} \ \text{ rel } \ (A\cup X^{(p-1)}),$$

moreover,

$$f' : A\cup X^{(p-1)}\cup e^p \longrightarrow Y \sim f|_{A\cup X^{(p-1)}\cup e^p} \ \text{ rel } \ (A\cup X^{(p-1)}\cup(e^p\setminus h(V)))$$

(the latter follows from a homotopy $\varphi \sim \psi$ rel $(U\setminus V)$). Also it is clear that $f'(e^p)$ does not cover all cell ϵ^q.

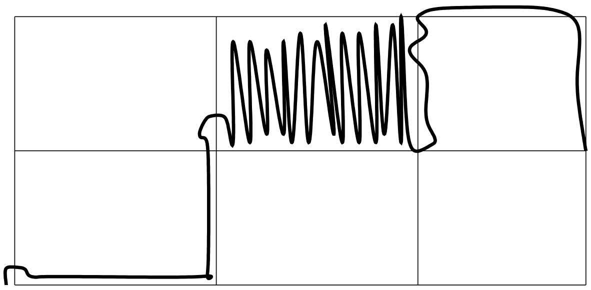

Fig. 5.3.

Completion of the proof of Theorem 5.5. Now the argument is simple. Firstly, a homotopy between the maps

$$f|_{A\cup X^{(p-1)}\cup e^p} \quad \text{and} \quad f' \quad \text{rel} \quad (A\cup X^{(p-1)})$$

can be extend on all X by Borsuk Theorem. In particular, we can assume that f' with all above properties is defined on all X.

Secondly, we consider a point $y_0 \in \epsilon^q \subset Y$ which does not belong to the image $f'(e^p)$, and "blow out" the map $f|_{e^p}$ from this point:

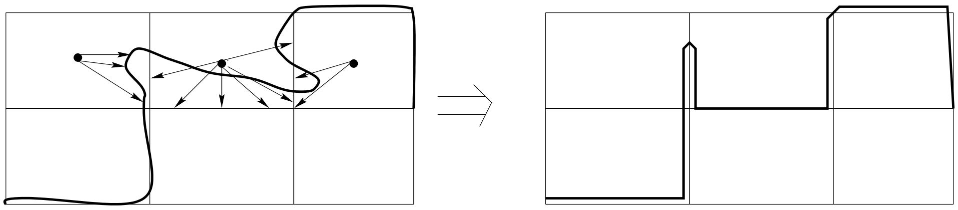

Fig. 5.4.

This is a homotopy which may be described as follows:

If $x \in e^p$, and $x \notin {f'}^{-1}(\epsilon^q)$, then $H_t(x) = x$ for all t.

If $x \in e^p$, and $x \in {f'}^{-1}(\epsilon^q)$, then $f'(x)$ moves along the ray connecting y_0 and the boundary of ϵ^q to a point on the boundary of ϵ^q.

We extend this homotopy to a homotopy of the map $f'|_{A\cup X^{(p-1)}\cup e^p}$ (relatively to e^p), and then up to homotopy the map $f' : X \longrightarrow Y$. The resulting map f'' is homotopic to f' (and f), and $f''(e^p)$ does not touch the cell ϵ^q and any other cell of dimension $> q$. Now we can apply the procedure just described several times and we obtain a map f_1 homotopic to f, such that f_1 is a cellular map on the subcomplex $A \cup X^{(p-1)} \cup e^p$. Note that each time we applied homotopy it was fixed on (relative to) $A \cup X^{(p-1)}$. It justifies the induction step, and proves the theorem. $\square$

Exercise 5.4. *Find all points in the argument from "Completion of the proof of Theorem 5.5" where we have used Borsuk Theorem.*

Remark. Again, if the CW-complex X is infinite, then the axiom **(W)** takes care for the resulting cellular map to be continuous.

5/3. Fighting the Phantom: Proof of Lemma 5.6. There are two well-known ways to prove our Lemma. The first one is to approximate our map by a smooth one, and then apply so called Sard Theorem. The second way is to use a *simplicial approximation* of continuous maps. The first way is more elegant, but the second is elementary, so we prove our Lemma following the second idea. First we need some new "standard spaces" which live happily inside the Euclidian space $\mathbf{R}^n$.

Let $q \leq n+1$, and $\vec{v}_1, \ldots, \vec{v}_{q+1}$ be vectors those endpoints do not belong to any $(q-1)$-dimensional subspace. We call the set

$$\Delta^q(\vec{v}_1, \ldots, \vec{v}_{q+1}) = \{ \ t_0\vec{v}_1 + \ldots + t_q\vec{v}_{q+1} \mid t_1 + \ldots t_{q+1} = 1, \quad t_1 \geq 0, \ldots, t_{q+1} \geq 0 \ \}$$

a q-dimensional simplex.

Exercise 5.5. *Prove that the simplex $\Delta(\vec{v}_1, \ldots, \vec{v}_{q+1})$ is homeomorphic (moreover, by means of a linear map) to the standard simplex*

$$\Delta^q = \left\{(x_1, \ldots, x_{q+1}) \in \mathbf{R}^{q+1} \mid x_1 \geq 0, \ldots, x_q \geq 0, \ \textstyle\sum_{i=1}^{q+1} x_i = 1 \right\}.$$

Example. A 0-simplex is a point; a simplex Δ^1 is the interval connecting two points; a simplex δ^2 is a nondegenerated triangle in the space $\mathbf{R}^n$; a simplex Δ^3 is a pyramid in $\mathbf{R}^n$ with the vertices $\vec{v}_0, \vec{v}_1, \vec{v}_2, \vec{v}_3$, see the picture below:

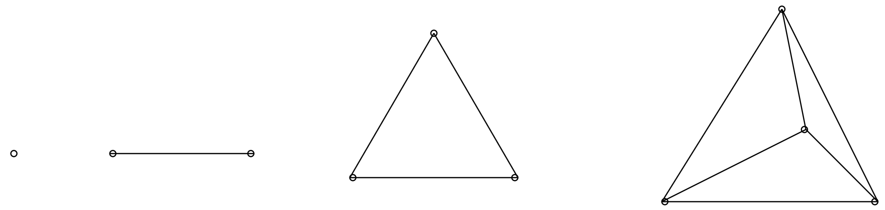

Fig. 5.5.

A j-th side of the simplex $\Delta^q(\vec{v}_1, \ldots, \vec{v}_{q+1})$ is the following $(q-1)$-simplex:

$$\Delta^{q-1}(\vec{v}_1, \ldots, \vec{v}_{j-1}, \vec{v}_{j+1}, \ldots, \vec{v}_q)_j = \{t_1\vec{v}_1 + \ldots + t_{q+1}\vec{v}_{q+1} \in \Delta^q(\vec{v}_1, \ldots, \vec{v}_{q+1}) \mid t_j = 0 \}.$$

We are not going to develop a theory of *simplicial complexes* (this theory is parallel to the theory of CW-complexes), however we need the following definition

Definition 5.7. *A finite triangulation of a subset $X \subset \mathbf{R}^n$ is a finite covering of X by simplices $\{\Delta^n(i)\}$ such that the intersection $\Delta^n(i) \cap \Delta^n(j)$ either empty, or $\Delta^{n-1}(i) \cap \Delta^n(j) = \Delta^n(i)_k$ for some $k = 0, \ldots, n$.*

Exercise 5.5. Let $\Delta_1^n, \ldots, \Delta_s^n$ be a finite set of n-dimensional simplexes in $\mathbf{R}^n$. Prove that the union $K = \Delta_1^n \cup \Delta_2^n \cup \cdots \cup \Delta_s^n$ is a finite simplicial complex.

Exercise 5.6. Let Δ_1^p, Δ_2^q be two simplices. Prove that $K = \Delta_1^p \times \Delta_2^q$ is a finite simplicial complex.

A *barycentric subdivision* of a q-simplex Δ^q is a subdivision of this simplex on $(q+1)!$ smaller simplices as follows. First let us look at the example:

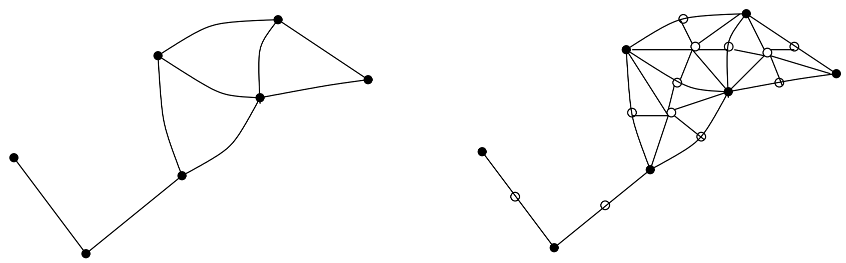

Fig. 5.6.

In general, we can proceed by induction. The picture above shows a barycentric subdivision of the simplices Δ^1, and Δ^2. Assume by induction that we have defined a barycentric subdivision of the simplices Δ^j for $j \leq q-1$. Now let x^* be a weight center of the simplex Δ^q. We already have a barycentric subdivision of each j-the side Δ^q_j by $(q-1)$-simplices $\Delta_j^{(1)}, \ldots, \Delta_j^{(n)}$, $n = q!$. The cones over these simplices, $j = 0, \ldots, q$, with a vertex x^* constitute a barycentric subdivision of Δ^q. Now we will prove the following "Approximation Lemma":

Lemma 5.8. *Let $V \subset U$ be two open sets of $\mathbf{R}^n$ such that their closure $\bar{V}$, $\bar{U}$ are compact sets and $\bar{V} \subset U$. Then there exists a finite triangulation of V by n-simplices $\{\Delta^n(i)\}$ such that $\Delta^n(i) \subset U$.*

Proof. For each point $x \in \bar{V}$ there exists a simplex $\Delta^n(x)$ with a center at x and $\Delta^n(x) \subset U$. By compactness of $\bar{V}$ there exist a finite number of simplices $\Delta^n(x_i)$ covering $\bar{V}$. It remains to use Exercise 5.6 to conclude that a union of finite number of $\Delta^n(x_i)$ has a finite triangulation. $\square$

Back to the Proof of Lemma 5.6. We consider carefully our map $\varphi : U \longrightarrow \overset{o}{D}{}^q$. First we construct the disks $\mathbf{d}_1$, $\mathbf{d}_2$, $\mathbf{d}_3$, $\mathbf{d}_4$ inside the disk $\mathbf{d}$ with the same center and of radii $r/5$, $2r/5$, $3r/5$, $4r/5$ respectively, where r is a radios of $\mathbf{d}$. Then we cover $V = \varphi^{-1}(\mathbf{d})$ by finite number of p-simplexes $\Delta^p(j)$, such that $\Delta^n(j) \subset U$. Making, if necessary, a barycentric subdivion (a finite number of times) of these simplices, we can assume that each simplex Δ^p has a diameter $\mathrm{d}(\varphi(\Delta^p)) < r/5$. Let K_1 be a union of all simlices Δ^q such that the intersection $\varphi(\Delta) \cap \mathbf{d}_4$ is not empty. Then $\mathbf{d}_4 \cap \varphi(U) \subset \varphi(K_1) \subset \mathbf{d}$.

Now we consider the map $\varphi' : K_1 \longrightarrow \mathbf{d}_4$ which coincides with φ on all vertices of our triangulation, and is linear on each simplex $\Delta \subset K_1$. The maps $\varphi|_{K_1}$ and φ' are homotopic, i.e. there is a homotopy $\varphi_t : K_1 \longrightarrow \mathbf{d}_4$, such that $\varphi_0 = \varphi|_{K_1}$ and $\varphi_1 = \varphi'$.

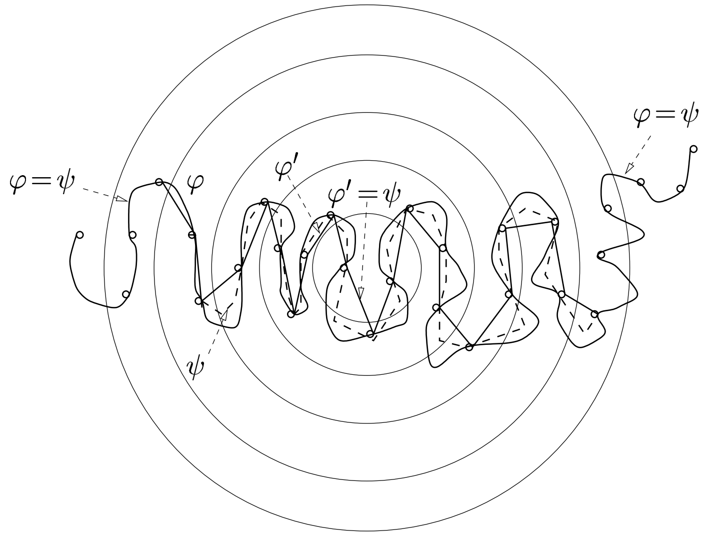

Fig. 5.7.

Exercise 5.7. Construct a homotopy φ_t as above.

Now we construct a map $\psi : U \longrightarrow \overset{o}{D}{}^q$ out of maps φ, φ_t and φ' as follows:

$$\psi(u) = \begin{cases} \varphi(u) & \text{if } \varphi(u) \notin \mathbf{d}_3, \\ \varphi'(u) & \text{if } \varphi(u) \in \mathbf{d}_2, \\ \varphi_{3-\frac{5r(u)}{r}}(u) & \text{if } \varphi(u) \in \mathbf{d}_3 \setminus \mathbf{d}_2. \end{cases}$$

Here $r(u)$ is a distance from $\varphi(u)$ to a center of the disk $\mathbf{d}$, see Fig. 5.7.

Now we notice that ψ is a continuous map, and it coincides with φ on $U \backslash V$. Furthermore, the intersection of its image with $\mathbf{d}_1$, the set $\psi(U) \cup \mathbf{d}_1$, is a union of finite number of pieces of p-dimensional planes, i.e. there is a point $y \in \mathbf{d}$ which $y \notin \psi(U)$.

Exercise 5.8. Let $\pi_1, \dots, \pi_s$ be a finite number of p-dimensional planes in $\mathbf{R}^q$, where $p < q$. Prove that the union $\pi_1 \cup \cdots \pi_s$ does cover any open subset $U \subset \mathbf{R}^n$.

Thus Cellular Approximation Theorem proved. □

5/4. First applications of Cellular Approximation Theorem. We start with the following important result.

Theorem 5.9. *Let X be a CW-complex with only one zero-cell and without cells of dimension $q < n$, and Y be a CW-complex of dimension $< q$. Then any map $Y \longrightarrow X$ is homotopic to a constant map. The same statement holds for "pointed" spaces and "pointed" maps.*

Exercise 5.9. Prove Theorem 5.9 using the Cellular Approximation Theorem.

Remark. For each pointed space (X, x_0) define $\pi_k(X, x_0) = [S^k, X]$ (where we consider homotopy classes of maps $f : (S^k, s_0) \longrightarrow (X, x_0)$). Very soon we will learn a lot about $\pi_k(X, x_0)$, in particular, that there is a natural group structure on $\pi_k(X, x_0)$ which are called *homotopy groups* of X. The following statement is a particular case of Theorem 5.9:

Corollary 5.10. *The homotopy groups $\pi_k(S^n)$ are trivial for $1 \le k < n$.*

We call a space X *n-connected* if it is path-connected and $\pi_k(X) = 0$ for $k = 1, \dots, n-1$.

Exercise 5.10. Prove that a space X is 0-connected if and only if it is path-connected.

Theorem 5.11. *Let $n \ge 1$. Any n-connected CW-complex homotopy equivalent to a CW-complex with a single zero cell and without cells of dimensions $1, 2, \dots, n$.*

Proof. Let us choose a cell e^0 and for each zero cell e_i^0 choose a path s_i connecting e_i^0 and e^0 (these paths may have nonempty intersections). By Cellular Approximation

Theorem we can choose these paths inside 1-skeleton. Now for each path s_i we glue a 2-disk, identifying a half-circle with s_i, see the picture:

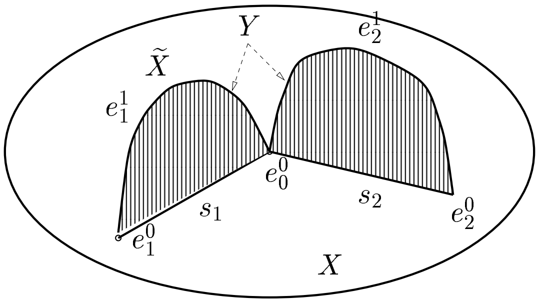

Fig. 5.8.

We denote the resulting CW-complex by $\widetilde{X}$. The CW-complex $\widetilde{X}$ has the same cells as X and new cells e_i^1, e_i^2 (the top half-circles and interior of 2-disks). A boundary of each cell e_i^2 belongs to the first skeleton since the paths s_i are in the first skeleton.

Clearly the complex $\widetilde{X}$ is a deformational retract of X (one can deform each cell e_i^2 to the path s_i). Let Y be a closure of the union $\bigcup_i e_i^1$. Obviously Y is contractible. Now note that $\widetilde{X}/Y \sim \widetilde{X} \sim X$, and the complex $\widetilde{X}/Y$ has only one zero cell. Now we use induction. Let us assume that we already constructed the CW-complex X' such that $X' \sim X$ and X' has a single zero cell, and it does not have cells of dimensions $1, 2, \dots, k-1$, where $k \leq n$. Note that a closure of each k-cell of X' is a sphere S^k. Since X' is still k-connected, then the embedding $S^k \longrightarrow X'$ (corresponding to a cell e_i^k) may be extended to a map $D^{k+1} \longrightarrow X'$. Again, Cellular Approximation Theorem implies that we can choose such extention that the image of D^{k+1} belongs to the $(k+1)$-skeleton of X'. Now we glue the disk D^{k+2} to X' using the map $D^{k+1} \longrightarrow X'^{(k+1)}$ (we identify the disk D^{k+1} with a bottom half-sphere S_-^{k+1} of the boundary of D^{k+2}). We do this procedure for each cell e^{k+1} and construct the complex $\widetilde{X}'$. Certainly $\widetilde{X}' \sim X' \sim X$. Now let Y' is a closure of the union $\bigcap_i e_i^{k+1}$, (where e_i^{k+1} are top half-spheres of new $(k+2)$ cells e^{k+2}. Clearly Y is contractible, and we have: $\widetilde{X}'/Y' \sim \widetilde{X}' \sim \widetilde{X} \sim X$. $\square$

Corollary 5.12. *Let Y be n-connected CW-complex, and X be an n-dimensional CW-complex. Then the set $[X, Y]$ consists of a single element.*

A pair of spaces (X, A) is n-connected if for any $k \leq n$ and any map of pairs $f : (D^k, S^{k-1}) \longrightarrow (X, A)$ homotopic to a map $g : (D^k, S^{k-1}) \longrightarrow (X, A)$ (as a map of pairs) so that $g(D^k) \subset A$.

Exercise 5.11. What does it mean geometrically that a pair (X, A) is 0-connected? 1-connected? Give some alternative description.

Exercise 5.12. Let (X, A) be an n-connected pair of CW-complexes. Prove that (X, A) is homotopy equivalent to a CW-pair (Y, B) so that $B \subset Y^{(n)}$.