4 CW-complexes

Algebraic topologists rarely study arbitrary topological spaces: there is not much one can prove about an abstract topological space. However, there is very well-developed area known as **general topology** which studies *simple properties* (such as conectivity, the Hausdorff property, compactness and so on) *of complicated spaces*. There is a giant Zoo out there of very complicated spaces endowed with all possible degrees of pathology, i.e. when one or another simple property fails or holds. Some of these spaces are extremely useful, such as the *Cantor set* or *fractals*, they help us to understand very delicate phenomenas observed in mathematics and physics. In algebraic topology we mostly study *complicated properties of simple spaces*.

It turns out that the most important spaces which are important for mathematics have some additional structures. The first algebraic topologist, Poincarè, studied mostly the spaces endowed with "analytic" structures, i.e. when a space X has natural differential structure or Riemannian metric and so on. The major advantage of these structures is that they all are natural, so we should not really care about their existence: they are given! There is the other type of natural structures on topological spaces: so called *combinatorial structures*, i.e. when a space X comes equipped with a decomposition into more or less "standard pieces", so that one could study the whole space X by examination the mutual geometric and algebraic relations between those "standard pieces". Below we formalize this concept: these spaces are known as CW-complexes. For instance, all examples we studied above are like that.

4/1. Basic definitions. We will call an open disk D^n (as well as any space homeomorphic to D^n) by n-cell. Notation: e^n. We will use the notation $\bar{e}^n$ for a "closed cell" which is homeomorphic to D^n. For $n = 0$ we let $e^0 = D^0$ (point). Let ∂e^n be a "boundary" of the cell e^n; ∂e^n is homeomorphic to the sphere S^{n-1}. Recall that if we have a map $\varphi : \partial e^n \longrightarrow K$, then we can construct the space $K \cup_\varphi e^n$, such that the diagram

$$\begin{tikzcd}
\bar{e}^n \arrow[r, "\Phi"] & K \cup_\varphi e^n \\
\partial e^n \arrow[u, "i"] \arrow[r, "\varphi"] & K \arrow[u, "i"]
\end{tikzcd}$$

commutes. We will call this procedure an *attaching of the cell e^n to the space K*. The map $\varphi : \partial e^n \longrightarrow K$ is the *attaching map*, and the map $\Phi : e^n \longrightarrow K \cup_\varphi e^n$ the *characteristic map of the cell e^n*. Notice that Φ is a homeomorphism of the open cell e^n on its image.

An example of this construction is the diagram (9), where the maps $c : S^n \longrightarrow \mathbf{RP}^n$ and $h : S^{2n+1} \longrightarrow \mathbf{CP}^n$ are the attaching maps of the corresponding cells e^{n+1} and e^{2n+2}. We return to these particular construction a bit later.

Definition 4.1. *A Hausdorff topological space X is a CW-complex (or cell-complex) if* *it is decomposed as a union of cells:*

$$X = \bigcup_{q=0}^{\infty} \left(\bigcup_{i \in I_q} e_i^q \right),$$

where the cells $e_i^q \cap e_j^p = \emptyset$ *unless* $q = p$, $i = j$, *and for each* e_i^q *there exists a characteristic map* $\Phi : D^q \longrightarrow X$ *such that its restriction* $\Phi_{\overset{o}{D}{}^q}$ *gives a homeomorphism* $\Phi|_{\overset{o}{D}{}^q} : \overset{o}{D}{}^q \longrightarrow e_i^q$. *It is required that the following axioms are satisfied:*

(C) (close finite) *The boundary* $\partial e_i^q = \bar{e}_i^q \setminus e_i^q$ *of the cell* e_i^q *is a subset of the union of finite number of cells* e_j^r, *where* $r < q$.

(W) (weak topology) *A set* $F \subset X$ *is closed if and only if the intersection* $F \cap \bar{e}^n$ *is closed for every cell* e_i^q.

Example 1. The sphere S^n. There are two standard cell decompositions of S^n:

(a) Let e^0 be a point (say, the north pole $(0, 0, \ldots, 0, 1)$ and $e^n = S^n \setminus e^0$, so $S^n e^0 \cup e^n$. A characteristic map $D^n \longrightarrow S^n$ which corresponds to the cell e^n may be defined by

$$(x_1, x_2, \ldots, x_n) \longrightarrow (x_1 \sin \pi\rho, \ldots, x_n \sin \pi\rho, \cos \pi\rho), \quad \text{where } \rho = \sqrt{x_1^2 + \ldots + x_n^2}$$

(b) We define $S^n = \bigcup_{q=0}^n e_{\pm}^q$, where

$$e_{\pm}^q = \left\{ (x_1, \ldots x_{n+1}) \in S^n \mid x_{q+2} = \ldots = x_{n+1} = 0, \quad \text{and} \quad \pm x_{q+1} > 0 \right\}, \quad \text{see Fig. 4.1.}$$

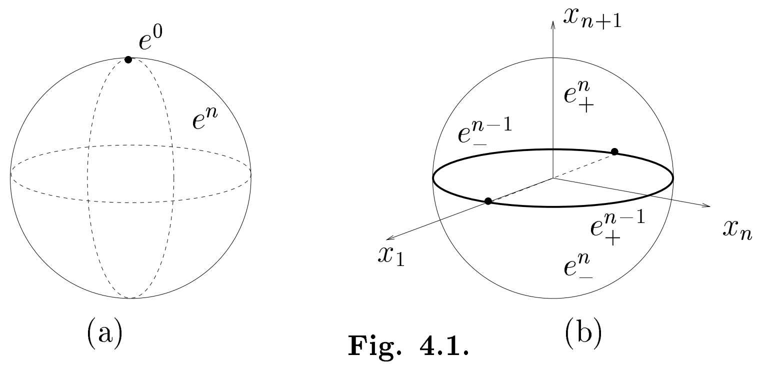


Fig. 4.1.

There exist a lot more cell decompositions of the sphere S^n: one can decompose S^n on $(3^{n+1} - 1)$ cells as a boundary of $(n+1)$-dimension simplex[§] Δ^{n+1}, or on $(2^{n+2} - 2)$ cells as a boundary of the cube I^n.

Exercise 4.1. *Describe these cell decompositions of* S^n.

[§] A simplex Δ^k is determined as follows:
$\Delta^k = \left\{ (x_1, \ldots, x_{k+1}) \in \mathbf{R}^{k+1} \mid x_1 \geq 0, \ldots, x_k \geq 0, \ \Sigma_{i=1}^{k+1} x_i = 1 \right\}$.

Example 2. Any of the above cell decompositions of the sphere S^{n-1} may be used to construct a cell decomposition of the disk D^n by adding one more cell: $Id : D^n \longrightarrow D^n$. The most simple one gives us three cells.

4/2. Some comments on the definition of a CW-complex. 1^o Let X be a CW-complex. We denote $X^{(n)}$ the union of all cells in X of dimension $\leq n$. This is the n-th skeleton of X. The n-th skeleton $X^{(n)}$ is an example (very important one) of a subcomplex of a CW-complex. A subcomplex $A \subset X$ is a closed subset of A which is a union of some cells of X. In particular, the n-th skeleton $A^{(n)}$ is a subcomplex of $X^{(n)}$ for each $n \geq 0$. A map $f : X \longrightarrow Y$ of CW-complexes is a cellular map if $f|_{X^{(n)}}$ maps the n-th skeleton to the n-skeleton $Y^{(n)}$ for each $n \geq 0$. In particular, the inclusion $A \subset X$ of a subcomplex is a cellular map. A CW-complex is called *finite* if it has a finite number of cells. A CW-complex is called *locally finite* if X has a finite number of cells in each dimension. Finally (X, x_0) is a pointed CW-complex, if x_0 is a 0-cell.

Exercise 4.2. *Prove that a CW-complex compact if and only if it is finite.*

2^o It turns out that a closure of a cell within a CW-complex may be not a CW-complex.

Exercise 4.3. *Construct a cellular decomposion of the wedge $X = S^1 \vee S^2$ (with a single 2-cell e^2) such that a closure of the cell e^2 is not a CW-subcomplex of X.*

3^o (Warning) The axiom (W) does not imply the axiom (C). Indeed, consider a decomposition of the disk D^2 into 2-cell e^2 which is the interior of the disk D^2 and each point of the circle S^1 is considered as a zero cell.

Exercise 4.4. *Prove that the disk D^2 with the cellular decomposition described above satisfies (W), and does not satisfy (C).*

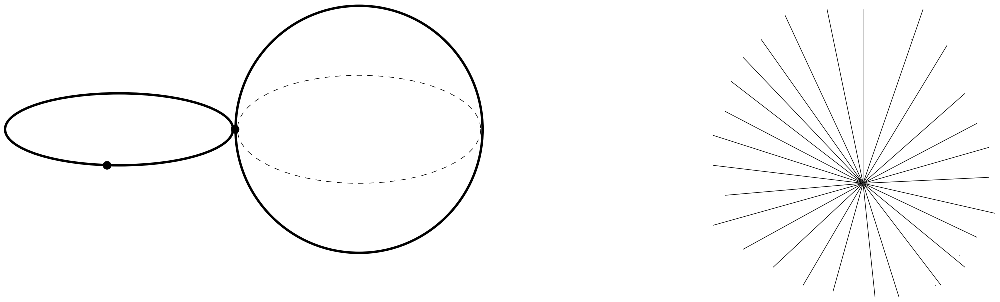


Fig. 4.2.

4^o (Warning) The axiom (C) does not imply the axiom (W). Indeed, consider the following space X. We start with an infinite (even countable) family I_α of unit intervals. Let $X = \bigvee_\alpha I_\alpha$, where we identify zero points of all intervals I_α. We define a topology on X by means of the following metric. Let $t' \in I_{\alpha'}$ and $t'' \in I_{\alpha''}$. Then a distance is defined by

$$\rho(t', t'') = \begin{cases} |t' - t''| & \text{if } \alpha' = \alpha'' \\ t' + t'' & \text{if } \alpha' \neq \alpha'' \end{cases}$$

Exercise 4.5. *Check that a natural cellular decomposition of X into the interiors of I_α and remaining points (zero cells) does not satisfy the axiom (W).*

4/3. Operations on CW-complexes. All operations we considered are well-defined on the category of CW-complexes, however we have to be a bit careful. If one of the CW-complexes X and Y is locally finite, then the product $X \times Y$ has a canonical CW-structure. The same holds for a smash-product $X \wedge Y$ of pointed CW-complexes. The cone $C(X)$, cylinder $X \times I$, and suspension $\Sigma(X)$ has canonical CW-structure determined by X. We can glue CW-complexes $X \cup_f Y$ if $f : A \longrightarrow Y$ a cellular map, and $A \subset X$ is a subcomplex. Also the quotient space X/A is a CW-complex if (A, X) is a CW-pair. The functional spaces $\mathcal{C}(X, Y)$ are two big to have natural CW-structure, however, a space $\mathcal{C}(X, Y)$ is homotopy equivalent to a CW-complex if X and Y are CW-complexes. The last statement is a nontrivial result due to J. Milnor (1958).

4/4. More examples of CW-complexes. Real projective space $\mathbf{RP}^n$. Here we choose in $\mathbf{RP}^n$ a sequence of projective subspaces:

$$* = \mathbf{RP}^0 \subset \mathbf{RP}^1 \subset \ldots \subset \mathbf{RP}^{n-1} \subset \mathbf{RP}^n.$$

and set $e^0 = \mathbf{RP}^0$, $e^1 = \mathbf{RP}^1 \setminus \mathbf{RP}^0, \ldots e^n = \mathbf{RP}^n \setminus \mathbf{RP}^{n-1}$. The diagram (9) shows that the map $c : S^{k-1} \longrightarrow \mathbf{RP}^k$ is an attaching map, and its extension to the cone over S^{k-1} (the disk D^k) is a characteristic map of the cell e^k. Alternatively this decomposition may be described in the homogeneous coordinates as follows. Let

$$e^q = \{(x_0 : x_1 : \cdots : x_n) \mid x_q \neq 0, x_{q+1} = 0, \ldots x_n = 0\}.$$

Exercise 4.6. *Prove that e^q is homeomorphic to $\mathbf{RP}^q \setminus \mathbf{RP}^{q-1}$.*

Exercise 4.7. *Construct cell decompositions of $\mathbf{CP}^n$ and $\mathbf{HP}^n$.*

Exercise 4.8. *Represent as CW-complex every 2-dimensional manifold. Try to find a CW-strucute with a minimal number of cells.*

Exercise 4.9. *Prove that a finite CW-complex (with finite number of cells) may be embedded into Euclidean space of finite dimension.*

4/5. CW-structure of the Grassmanian manifolds. We describe here the *Schubert decomposition*, and the cells of this decomposition are known as the *Schubert cells.* We consider the space $G(n, k)$. We choose the standard basis $e_1, \ldots, e_n$ of $\mathbf{R}^n$. Let $\mathbf{R}^q = \langle e_1, \ldots, e_q \rangle$. It is convenient to denote $\mathbf{R}^0 = \{0\}$. We have the inclusions:

$$\mathbf{R}^0 \subset \mathbf{R}^1 \subset \mathbf{R}^2 \subset \cdots \subset \mathbf{R}^n.$$

Let $\pi \in G(n, k)$. Clearly π determines a collection of nonnegative numbers

$$0 \leq \dim(\mathbf{R}^1 \cap \pi) \leq \dim(\mathbf{R}^2 \cap \pi) \leq \cdots \leq \dim(\mathbf{R}^n \cap \pi) = k.$$

We note that $\dim(\mathbf{R}^j \cap \pi) \le \dim(\mathbf{R}^{j-1} \cap \pi) + 1$. Indeed, we have linear maps

$$0 \longrightarrow \mathbf{R}^{j-1} \cap \pi \xrightarrow{i} \mathbf{R}^j \cap \pi \xrightarrow{j\text{-th coordinate}} \mathbf{R} \tag{10}$$

where the first one, $i : \mathbf{R}^{j-1} \cap \pi \longrightarrow \mathbf{R}^j \cap \pi$, is an embedding, and the map

$$j\text{-th coordinate} : \mathbf{R}^j \cap \pi \longrightarrow \mathbf{R}$$

is either onto or zero. In the first case $\dim(\mathbf{R}^j \cap \pi) = \dim(\mathbf{R}^{j-1} \cap \pi) + 1$, and in the second case $\dim(\mathbf{R}^j \cap \pi) = \dim(\mathbf{R}^{j-1} \cap \pi)$. Thus there are exactly k "jumps" in the sequence $(0, \dim(\mathbf{R}^1 \cap \pi), \ldots, \dim(\mathbf{R}^n \cap \pi))$.

A *Schubert symbol* $\sigma = (\sigma_1, \ldots, \sigma_n)$ is a collection of integers, such that

$$1 \le \sigma_1 < \sigma_2 < \cdots < \sigma_k \le n.$$

Let $e(\sigma) \subset G(n,k)$ be the following set of the following k-planes in $\mathbf{R}^n$

$$e(\sigma) = \left\{ \pi \in G(n,k) \mid \dim(\mathbf{R}^{\sigma_j} \cap \pi) = j \ \& \ \dim(\mathbf{R}^{\sigma_j - 1} \cap \pi) = j-1, \ j = 1, \ldots, k \right\}.$$

Notice that every $\pi \in G(n,k)$ belongs to exactly one subset $e(\sigma)$. Indeed, in the sequence of subspaces

$$\mathbf{R}^1 \cap \pi \subset \mathbf{R}^2 \cap \pi \subset \cdots \subset \mathbf{R}^n \cap \pi = \pi$$

their dimensions "jump" by one exactly k times. Clearly $\pi \in e(\sigma)$, where $\sigma = (\sigma_1, \ldots, \sigma_n)$ and

$$\sigma_t = \min \left\{ j \mid \dim(\mathbf{R}^j \cap \pi) = t \right\}.$$

Our goal is to prove that the set $e(\sigma)$ is homeomorphic to an open cell of dimension $d(\sigma) = (\sigma_1 - 1) + (\sigma_2 - 2) + \cdots + (\sigma_k - k)$. Let $H^j \subset \mathbf{R}^n$ denote an open "half j-plane of $\mathbf{R}^j$:

$$H^j = \{(x_1, \ldots, x_j, 0, \ldots, 0) \mid x_j > 0\}.$$

It will be convenient to denote $\overline{H}^j = \{(x_1, \ldots, x_j, 0, \ldots, 0) \mid x_j \ge 0\}$.

Claim 4.1. *A k-plane π belongs to $e(\sigma)$ if and only if there exists its basis $v_1, \ldots, v_k$, such that $v_1 \in H^{\sigma_1}$, ..., $v_k \in H^{\sigma_k}$.*

Proof. Indeed, if there is such a basis $v_1, \ldots, v_k$ then

$$\dim(\mathbf{R}^{\sigma_j} \cap \pi) > \dim(\mathbf{R}^{\sigma_j - 1} \cap \pi)$$

for $j = 1, \ldots, k$. Thus $\pi \in e(\sigma)$. The following lemma proves Claim 4.1 in the other direction. $\square$

Lemma 4.2. *Let $\pi \in e(\sigma)$, where $\sigma = (\sigma_1, \ldots, \sigma_n)$. Then there exists a unique orthonormal basis $v_1, \ldots, v_k$ of π, so that $v_1 \in H^{\sigma_1}$, ..., $v_k \in H^{\sigma_k}$.*

Proof. We choose v_1 to be a unit vector which generates the line $\mathbf{R}^{\sigma_1} \cap \pi$. There are only two choices here, and the condition that the σ_1-th coordinate is positive determines v_1 uniquely. Then the unit vector $v_2 \in \mathbf{R}^{\sigma_2} \cap \pi$ should be chosen so that $v_2 \perp v_1$. There are two choices like that, and again the positivity of the σ_2-th coordinate determines v_2 uniquely. By induction one obtains the required basis. This completes proof of Lemma 4.2 and Claim 4.1. $\square$

We define the following subset of the Stiefel manifold $V(n,k)$:

$$E(\sigma) = \{(v_1, \ldots, v_k) \in V(n,k) \mid v_1 \in H^{\sigma_1}, \ldots, v_k \in H^{\sigma_k} \}.$$

Lemma 4.2 gives a well-defined map $q : e(\sigma) \longrightarrow E(\sigma)$. It is convenient to denote $\overline{E}(\sigma) = \left\{(v_1, \ldots, v_k) \in V(n,k) \mid v_1 \in \overline{H}^{\sigma_1}, \ldots, v_k \in \overline{H}^{\sigma_k} \right\}$.

Claim 4.2. *The set $\overline{E}(\sigma) \subset V(n,k)$ is homeomorphic to the closed cell of dimension $d(\sigma) = (\sigma_1 - 1) + (\sigma_2 - 2) + \cdots + (\sigma_k - k)$. Furthermore the map $q : e(\sigma) \longrightarrow E(\sigma)$ is a homeomorphism.*

Proof. Induction on k. If $k = 1$ the set $\overline{E}(\sigma_1)$ consists of the vectors

$$v_1 = (x_{11}, \ldots, x_{1\sigma_1}, 0, \ldots, 0), \quad \text{such that} \quad \sum x_{1j}^2 = 1, \quad \text{and} \quad x_{1\sigma_1} \geq 0.$$

Clearly $\overline{E}(\sigma_1)$ is a closed hemisphere of dimension $(\sigma_1 - 1)$, i.e. $\overline{E}(\sigma_1)$ is homeomorphic to the disk $D^{\sigma_1 - 1}$.

To make an induction step, consider the following construction. Let $u, v \in \mathbf{R}^n$ be two unit vectors such that $u \neq -v$. Let $T_{u,v}$ an orthogonal transformation $\mathbf{R}^n \longrightarrow \mathbf{R}^n$ such that

(1) $T_{u,v}(u) = v$;

(2) $T_{u,v}(w) = w$ if $w \in \langle u, v\rangle^{\perp}$.

In other words, $T_{u,v}$ is a rotation in the plane $\langle u, v\rangle$ taking the vector v to u, and is identity on the orthogonal complement to the plane $\langle u, v\rangle$ generated by u and v.

Claim 4.3. *The transformation $T_{u,v}$ (where $u, v \in \mathbf{R}^n$, $u \neq -v$) has the following properties:*

(a) $T_{u,u} = Id$;

(b) $T_{v,u} = -T_{u,u}$;

(c) $T_{v,u} : \mathbf{R}^n \longrightarrow \mathbf{R}^n$ *is be given by*

$$T_{u,u}(x) = x - \frac{\langle u + v, x\rangle}{1 + \langle u, v\rangle}(u + v) + 2\langle u, x\rangle v;$$

(d) *a vector $T_{u,v}(x)$ depends continuously on u, v, x;*

(e) $T_{u,v}(x) = x$ *(mod* $\mathbf{R}^j$*) if* $u, v \in \mathbf{R}^j$.

The properties (a), (b), (e) follow from the definition.

Exercise 4.10. *Prove (c), (d) from Claim 4.3.*

Let $\epsilon_i \in H^{\sigma_i}$ be a vector which has σ_i-coordinate equal to 1, and all others are zeros. Thus $(\epsilon_1, \ldots, \epsilon_k) \in E(\sigma)$. For each k-frame $(v_1, \ldots, v_k) \in \overline{E}(\sigma)$ consider the transformation:

$$T = T_{\epsilon_k, v_k} \circ T_{\epsilon_{k-1}, v_{k-1}} \circ \cdots \cdots \circ T_{\epsilon_1, v_1} : \mathbf{R}^n \longrightarrow \mathbf{R}^n \tag{11}$$

First we notice that $v_i \neq \epsilon_i$ since $v_i \in \overline{H}^{\sigma_i}$. Thus the transformations T_{ϵ_i, v_i} are well-defined.

Exercise 4.11. *Prove that the transformation T takes the k-frame $(\epsilon_1, \ldots, \epsilon_k)$ to the frame $(v_1, \ldots, v_k)$.*

Consider the following subspace $D \subset \overline{H}^{\sigma_{k+1}}$:

$$D = \left\{ u \in \overline{H}^{\sigma_{k+1}} \mid |u| = 1, \quad \langle \epsilon_j, u \rangle = 0, \ j = 1, \ldots, k \right\}.$$

Exercise 4.12. *Prove that D is homeomorphic to the hemisphere of the dimension $\sigma_{k+1} - k - 1$.*

Thus D is a closed cell of dimension $\sigma_{k+1} - k - 1$. Now we make an induction step to complete a proof of Claim 4.2. We define the map

$$f : \overline{E}(\sigma_1, \ldots, \sigma_k) \times D \longrightarrow \overline{E}(\sigma_1, \ldots, \sigma_k, \sigma_{k+1})$$

by the formula $f((v_1, \ldots, v_k), u) = (v_1, \ldots, v_k, Tu)$ where T is given by (11). We notice that

$$\langle v_i, Tu \rangle = \langle T\epsilon_i, Tu \rangle = \langle \epsilon_i, u \rangle = 0, \quad i = 1, \ldots, k,$$

and $\langle Tu, Tu \rangle = \langle u, u \rangle = 1$ by definition of T and since $T \in O(n)$.

Exercise 4.13. *Recall that $\sigma_k < \sigma_{k+1}$. Prove that $Tu \in \overline{H}^{\sigma_{k+1}}$ if $u \in D$.*

The inverse map $f^{-1} : \overline{E}(\sigma_1, \ldots, \sigma_k, \sigma_{k+1}) \longrightarrow \overline{E}(\sigma_1, \ldots, \sigma_k) \times D$ is defined by

$$v_j = f^{-1} v_j, j = 1, \ldots, k,$$

$$u = f^{-1} v_{k+1} = (T^{-1} v_{k+1}) = T_{\epsilon_k, v_k} \circ T_{\epsilon_{k-1}, v_{k-1}} \circ \cdots \cdots \circ T_{\epsilon_1, v_1}(v_{k+1}) \in D.$$

Both maps f and f^{-1} are continuous, thus f is a homeomorphism. This concludes induction step in the proof of Claim 4.2. Lemma 4.2 implies that $e(\sigma_1, \ldots, \sigma_k)$ is homeomorphic to an open cell of dimension $d(\sigma) = (\sigma_1 - 1) + (\sigma_2 - 2) + \cdots + (\sigma_k - k)$.

□

Remark. Let $(v_1,\dots,v_k)\in \overline{E}(\sigma)\setminus E(\sigma)$, then the k-plane $\pi=\langle v_1,\dots,v_k\rangle$ does not belong to $e(\sigma)$. Indeed, it means that at least one vector $v_j\in \mathbf{R}^{\sigma_j-1}=\partial\left(\overline{H}^{\sigma_j}\right)$. Thus $\dim(\mathbf{R}^{\sigma_j-1}\cap\pi)\geq j$, hence $\pi\notin e(\sigma)$. □

Theorem 4.3. *A collection of* $\begin{pmatrix} k \\ n \end{pmatrix}$ *cells* $e(\sigma)$ *gives* $G(n,k)$ *a cell-decomposition.*

Proof. We should show that any point x of the boundary of the cell $e(\sigma)$ belongs to some cell $e(\tau)$ of dimension less than $d(\sigma)$. We use the map $q: e(\sigma)\longrightarrow E(\sigma)$ to see that $q(\overline{e}(\sigma))=\overline{E}(\sigma)$. Thus we can describe $\pi\in\overline{e}(\sigma)\setminus e(\sigma)$ as a k-plane $\langle v_1,\dots,v_k\rangle$, where $v_j\in\overline{H}^{\sigma_j}$. Clearly $v_j\in\mathbf{R}^{\sigma_j}$, thus $\dim(\mathbf{R}^{\sigma_j}\cap\pi)\geq j$ for each $j=1,\dots,k$. Hence $\tau_1\leq\sigma_1$, …, $\tau_k\leq\sigma_k$. However, at least one vector v_j belongs to the subspace $\mathbf{R}^{\sigma_j-1}=\partial\left(\overline{H}^{\sigma_j}\right)$, and corresponding $\tau_j<\sigma_j$. Thus $d(\tau)<d(\sigma)$. The number of all cells is equal to $\begin{pmatrix} k \\ n \end{pmatrix}$ by counting. □

Now we count a number of cells of dimension r in the cell decomposition of $G(n,k)$. Recall that a *partition* of an integer r is an unordered collection $(i_1,\dots,i_s)$ such that $i_1+\dots+i_s=r$. Let $\rho(r)$ be a number of partitions of r. This are values of $\rho(r)$ for $r\leq 10$:

r	0	1	2	3	4	5	6	7	8	9	10
$\rho(r)$	1	1	2	3	5	7	11	15	22	30	42

Each Schubert symbol $\sigma=(\sigma_1,\dots,\sigma_k)$ of dimension $d(\sigma)=(\sigma_1-1)+(\sigma_2-2)+\dots+(\sigma_k-k)=r$ gives a partition $(i_1,\dots,i_s)$ of r which is given by deleting zeros from the sequence $(\sigma_1-1),(\sigma_2-2),\dots,(\sigma_k-k)$.

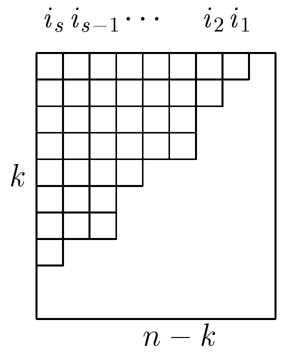
Exercise 4.14. *Show that*

$$1\leq i_1\leq i_2\leq\dots\leq i_s\leq k,\quad \textit{and}\quad s\leq n-k.$$

Prove that a number of r-dimensional cells of $G(n,k)$ is equal to a number of partitions $(i_1,\dots,i_s)$ of r such that $s\leq n-k$ and $i_t\leq k$.

Remark. There is a natural chain of embeddings $G(n,k)\longrightarrow G(n+1,k)\longrightarrow\cdots\longrightarrow G(n+l,k)$. It is easy to notice that these embeddings preserve the Schubert cell decomposition, and if l and k are large enough, the number of cells of dimension r is equal to $\rho(r)$. In particular, the Schubert cells give a cell decomposition of $G(\infty,k)$ and $G(\infty,\infty)$. □

Remark. Let $\iota=(i_1,\dots,i_s)$ be a partition of r as above (i.e. $s\leq n-k$ and $1\leq i_1\leq i_2\leq\dots\leq i_s\leq k$). The partition ι may be represented as a *Young diagram.*



This Young diagram gives a parametrization of the corresponding cell $e(\sigma)$. Clearly the Schubert symbols σ are in one-to-one correspondence with the Yaung diagrams corresponding to the partitions $(i_1, \ldots, i_s)$ as above. The Young diagrams were invented in the representation theory of the symmetric group S_n. This is not an accident, it turns out that there is a deep relationship between the Grasmannian manifolds and the representation theory.