3 Homotopy and homotopy equivalence

3/1. Definition of a homotopy. Let X and Y be topological spaces. Two maps $f_0 : X \longrightarrow Y$ and $f_1 : X \longrightarrow Y$ are *homotopic* (notation: $f_0 \sim f_1$) if there exists a map $F : X \times I \longrightarrow Y$ (continuous as usual!) such that the restriction $F|_{X\times\{0\}}$ coincides with f_0, and the restriction $F|_{X\times\{1\}}$ coincides with f_1.

The map $F : X \times I \longrightarrow Y$ is called a *homotopy*. We can think also that a homotopy between maps f_0 and f_1 is a continuous family of maps $\varphi_t : X \longrightarrow Y$, $0 \leq t \leq 1$, such that $\varphi_0 = f_0$, $\varphi_1 = f_1$, and the map $F : X \times I \longrightarrow Y$, $F(x,t) = \varphi_t(x)$ is a continuous map for every $t \in I$.

If the spaces X and Y are "good spaces" (like our examples S^n, $\mathbf{RP}^n$, $\mathbf{CP}^n$, $\mathbf{HP}^n$, $G(n,k)$ $V(n,k)$ and so on), then we can think about homotopy between f_0 and f_1 as a path in the space of continuous maps $\mathcal{C}(X,Y)$ joining f_0 and f_1. Furthemore, in such case, the set of homotopy classes $[X,Y]$ (see below) may be identified with the set of path-components of the space $\mathcal{C}(X,Y)$.

If a map $f : X \longrightarrow Y$ is homotopic to a constant map $X \longrightarrow pt \in Y$, we call the map f *null-homotopic.*

Example. Let $Y \subset \mathbf{R}^n$ (or $\mathbf{R}^\infty$) be a convex subset. Then for any space X any two maps $f_0 : X \longrightarrow Y$ and $f_1 : X \longrightarrow Y$ are homotopic. Indeed, the map

$$F : x \longrightarrow (1-t)f_0(x) + tf_1(x)$$

defines a corresponding homotopy.

3/2. Homotopy classes of maps. Clearly a homotopy determines an equivalence relation on the space of maps $\mathcal{C}(X,Y)$. The set of equivalence classes is denoted by $[X,Y]$ and it is called a *set of homotopy classes*.

Examples. 1. The set $[X,*]$ consists of one point for any space X.

2. The space $[*,Y]$ (where $*$ is a one-point set) is the set of path-connected components of Y.

Let $\varphi : X \longrightarrow X'$ be a map (continuous), then we define the map (not continuous since we do not have a topology on the set $[X,Y]$) $\varphi^* : [X',Y] \longrightarrow [X,Y]$ as follows. Let $a \in [X',Y]$ be a homotopy class. Choose any representative $f : X' \longrightarrow Y$ of the class a, then $\varphi^*(a)$ is a homotopy class contaning the map $f \circ \varphi : X \longrightarrow Y$.

Now let $\psi : Y \longrightarrow Y'$ be a map. Then the map $\psi_* : [X,Y] \longrightarrow [X,Y']$ is defined as follows. For any $b \in [X,Y]$ and a representative $g : X \longrightarrow Y$ the map $\psi \circ g : X \longrightarrow Y'$ determines a homotopy class $\psi(b) = [\psi \circ g]$.

Exercise 3.1. *Prove that the maps φ^* and ψ_* are well-defined.*

3/3. Homotopy equivalence. We will give three different definitions of homotopy equivalence.

Definition 3.1. (HE-I) *Two spaces X_1 and X_2 are homotopy equivalent ($X_1 \sim X_2$) if there exist maps $f : X_1 \longrightarrow X_2$ and $g : X_2 \longrightarrow X_1$ such that the compositions $g \circ f : X_1 \longrightarrow X_1$ and $f \circ g : X_2 \longrightarrow X_2$ are homotopy equivalent to the identity maps I_{X_1} and I_{X_2} respectively.*

In this case we call maps f and g *mutually inverse homotopy equivalences*, and both maps f and g are *homotopy equivalences.*

Remark. If the maps $g \circ f$ and $f \circ g$ are the identity maps, then f and g are *mutually inverse homeomorphisms.*

Definition 3.2. (HE-II) *Two spaces X_1 and X_2 are homotopy equivalent ($X_1 \sim X_2$) if there is a rule assigning for any space Y a one-to-one map $\varphi_Y : [X_1, Y] \longrightarrow [X_2, Y]$ such that for any map $h : Y \longrightarrow Y'$ the diagram*

$$\begin{tikzcd}
{[X_1, Y]} \arrow[r, "\varphi_Y"] \arrow[d, "h_*"'] & {[X_2, Y]} \arrow[d, "h_*"'] \\
{[X_1, Y']} \arrow[r, "\varphi_{Y'}"] & {[X_2, Y']}
\end{tikzcd} \tag{5}$$

commutes, i.e. $\varphi_{Y'} \circ h_* = h_* \circ \varphi_Y$.

Definition 3.3. (HE-III) *Two spaces X_1 and X_2 are homotopy equivalent ($X_1 \sim X_2$) if there is a rule assigning for any space Y a one-to-one map $\varphi^Y : [Y, X_1] \longrightarrow [Y, X_2]$ such that for any map $h : Y \longrightarrow Y'$ the diagram*

$$\begin{tikzcd}
{[Y, X_1]} \arrow[r, "\varphi^Y"] & {[Y, X_2]} \\
{[X_1, Y']} \arrow[r, "\varphi_{Y'}"] \arrow[u, "h^*"] & {[X_2, Y']} \arrow[u, "h^*"']
\end{tikzcd} \tag{6}$$

commutes, i.e. $\varphi^Y \circ h^* = h^* \circ \varphi^{Y'}$.

Theorem 3.4. *Definitions 3.1, 3.2 and 3.3 are equivalent.*

Proof. Here we prove only that Definitions 3.1 and 3.2 are equivalent. Let $X_1 \sim X_2$ in the sence of Definition 3.2. Then there is one-to-one map $\varphi_{X_2} : [X_1, X_2] \longrightarrow [X_2, X_2]$.

Let I_{X_2} be the identity map, $I_{X_2} \in [X_2, X_2]$. Let $\alpha = \varphi^{-1}([I_{X_2}]) \in [X_1, X_2]$ and $f \in \alpha$, $f : X_1 \longrightarrow X_2$ be a representative.

There is also a one-to-one map $\varphi_{X_1} : [X_1, X_1] \longrightarrow [X_2, X_1]$. We let $\beta = \varphi_{X_1}([I_{X_1}])$ and we choose a map $g : X_2 \longrightarrow X_1$, $g \in \beta$. We shall show that $f \circ g \sim I_{X_2}$. The diagram

$$\begin{tikzcd}
{[X_1, X_1]} \arrow[r, "\varphi_{X_1}"] \arrow[d, "f_*"'] & {[X_2, X_1]} \arrow[d, "f_*"'] \\
{[X_1, X_2]} \arrow[r, "\varphi_{X_2}"] & {[X_2, X_2]}
\end{tikzcd} \tag{7}$$

commutes by Definition 3.2. It implies that $\varphi_{X_2} \circ f_* = f_* \circ \varphi_{X_1}$. Let us consider the image of the element $[I_{X_1}]$ in the diagram (7). We have:

$$f_*([I_{X_1}] = [f \circ I_{X_1}] = [f], \qquad \varphi_{X_2}([f]) = [I_{X_2}]$$

by definition and by the choice of f. It implies that

$$\varphi_{X_2} \circ f_*([I_{X_1}]) = [I_{X_2}].$$

On the other hand, we have:

$$f_* \circ \varphi_{X_1}([I_{X_1}]) = f_*([g]) = [f \circ g].$$

Commutativity of (7) implies that $[f \circ g] = [I_{X_1}]$, i.e. $f \circ g \sim I_{X_1}$.

A similar argument proves that $g \circ f \sim I_{X_1}$. It means that $X_1 \sim X_2$ in the sence of Definition 3.1.

Now assume that $X_1 \sim X_2$ in the sence of Definition 3.1, i.e. there are maps $f : X_1 \longrightarrow X_2$ and $g : X_1 \longrightarrow X_1$ such that $f \circ g \sim I_{X_2}$ and $g \circ f \sim I_{X_1}$. Let Y be any space and define

$$\varphi_Y = g^* : [X_1, Y] \longrightarrow [X_2, Y].$$

We shall show that this map is inverse to the map

$$f^*[X_2, Y] \longrightarrow [X_1, Y].$$

Indeed, let $h \in \mathcal{C}(X, Y)$, then

$f^* \circ g^*([h]) = f^*([h \circ g]) =$ (by definition of f^*) $= [h \circ (g \circ f)] = [h]$ (since $g \circ f \sim I_{X_1}$).

With a similar argument we prove that $g^* \circ f^* = I_{X_1}$.

Now let Y' be a space and $k : Y \longrightarrow Y'$ be a map. We show that the diagram

$$\begin{tikzcd}
{[X_1, Y]} \arrow[r, "\varphi_Y = g^*"] \arrow[d, "k_*"'] & {[X_2, Y]} \arrow[d, "k_*"'] \\
{[X_1, Y']} \arrow[r, "\varphi_{Y'} = g^*"] & {[X_2, Y']}
\end{tikzcd} \tag{8}$$

commutes. Let $h \in \mathcal{C}(X_1, Y)$ be a map. Then we have

$$k_*([h]) = [k \circ h], \quad g^*([k \circ h]) = [(k \circ h) \circ g],$$

and also

$$g^*([h]) = [h \circ g], \quad k_*([h \circ g]) = [k \circ (h \circ g)].$$

It means that (8) commutes. Thus Definitions 3.1 and 3.2 are equivalent. $\square$

Exercise 3.2. *Prove the equivalence of Definitions 3.1 and 3.3.*

We call a class of homotopy equivalent spaces a *homotopy type*. Obviously any homeomorphic spaces are homotopy equivalent. The simpest example of spaces which are homotopy equivalent, but not homeomorphic is the following: X_1 is a circle, and X_2 is an annulus, see Fig. 3.1.

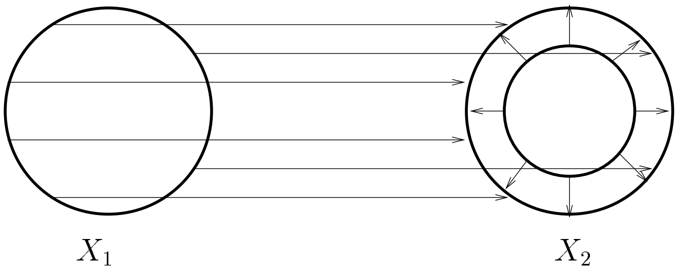

Fig. 3.1.

Exercise 3.3. *Give 3 examples of spaces homotopy equivalent and not homeomorphic spaces.*

We call a space X a *contractible space* if the identity map $I : X \longrightarrow X$ null-homotopic, i.e. it is homotopic to the "constant map" $* : X \longrightarrow X$, mapping all X to a single point.

Exercise 3.4. *Prove that a space X is contractible if and only if it is homotopy equivalent to a point.*

Exercise 3.5. *Prove that a space X is contractible if and only if every map $f : Y \longrightarrow X$ is null-homotopic.*

Exercise 3.6. *Prove that the space of paths $\mathcal{E}(X, x_0)$ is contractible for any X.*

Exercise 3.7. *Let X_1, X_2 be pointed spaces. Prove that if $X_1 \sim X_2$ then $\Sigma(X_1) \sim \Sigma(X_2)$ and $\Omega(X_1) \sim \Omega(X_2)$.*

3/4. Retracts. We call a subspace A of a topological space X its *retract* if there exists a map $r : X \longrightarrow X$ (a *retraction*) such that $r(X) = A$ and $r(a) = a$ for any $a \in A$.

Examples. 1. A single point $x \in X$ is a retact of the space X since a constant map $r : X \longrightarrow x$ is a retraction.

2. The subspace $A = \{0\} \cup \{1\}$ of the interval $I = [0, 1]$ *is not a retract of* I, otherwise we would map I to the disconnected space A.

3. In general, the sphere S^n *is not a retract of the disk* D^{n+1} for any n, however we do not have enough tools in our hands to prove it now.

4. The "base" $X \times \{0\}$ is a retract of the cylinder $X \times I$.

Exercise 3.8. *Prove that the "base" $X \times \{0\}$ of the cone $C(X)$ is a retract of $C(X)$ if and only if the space X is contractible.*

Sometimes a retraction $r : X \longrightarrow X$ (where $r(X) = A$) is homotopic to the identity map $Id : X \longrightarrow X$, in that case we call A a *deformation retract of* X; moreover if this homotopy may be chosen to be the identity map on A,[‡] then we call A a *strict deformation retract of* X.

Obviously X is homotopy equivalent to its deformation retract, moreover the following fact holds

Lemma 3.5. *A subspace A is a deformation retract of X if and only if the inclusion $A \longrightarrow X$ is a homotopy equivalence.*

Exercise 3.9. *Prove Lemma 3.5.*

Lemma 3.5 shows that actually a concept of deformation retract is hardly new for us; a concept of strict deformation retract is more restrictive, however these two concepts are different only in some pathological cases.

Exercise 3.10. *Let $A \subset X$, and $r^{(0)} : X \longrightarrow A$, $r^{(1)} : X \longrightarrow A$ be two deformation retractions. Prove that the retractions $r^{(0)}$, $r^{(1)}$ may be joined by a continuous family of deformation retractions $r^{(s)} : X \longrightarrow A$, $0 \leq s \leq 1$. Note: It is important here that $r^{(0)}$, $r^{(1)}$ are both homotopic to the identity map I_X.*

3/5. The case of "pointed" spaces. The definitions of homotopy, homotopy equivalence have to be changed (in an obvious way) for spaces with a base point. The set of homotopy clases of "pointed" maps $f : X \longrightarrow Y$ will be also denoted as $[X, Y]$. We need one more generalization.

[‡] i.e. a homotopy $h : X \times I \longrightarrow X$ between $r : X \longrightarrow X$ and the identity map $Id : X \longrightarrow X$ has the following property: $h(a, t) = a$ for any $a \in A$.

Definition 3.6. *A pair (X,A) is just a space X with a labeled subspace $A \subset X$; a map of pairs $f : (X,A) \longrightarrow (Y,B)$ is a continuous map $f : X \longrightarrow Y$ such that $f(A) \subset B$. Two maps $(X,A) \longrightarrow (Y,B)$, $f_1 : (X,A) \longrightarrow (Y,B)$ are homotopic if there exist a map $F : (X \times I, A \times I) \longrightarrow (Y,B)$ such that*

$$F|_{(X\times\{0\},A\times\{0\}} = f_0, \quad F|_{(X\times\{1\},A\times\{1\}} = f_1.$$

We have seen already the example of pairs and their maps. Let me recall that the cones of the maps $c : S^n \longrightarrow \mathbf{RP}^n$ and $h : S^{2n+1} \longrightarrow \mathbf{CP}^n$ give us the commutative diagrams:

$$\begin{tikzcd}
D^{n+1} \arrow[r, "f"] & \mathbf{RP}^{n+1} & D^{2n+2} \arrow[r, "g"] & \mathbf{CP}^{n+1} \\
S^n \arrow[u, "i"] \arrow[r, "c"] & \mathbf{RP}^n \arrow[u, "i"] & S^{2n+1} \arrow[u, "i"] \arrow[r, "h"] & \mathbf{CP}^n \arrow[u, "i"]
\end{tikzcd} \tag{9}$$

which are the maps of pairs:

$$f : (D^{n+1}, S^n) \longrightarrow (\mathbf{RP}^{n+1}, \mathbf{RP}^n), \quad g : (D^{2n+2}, S^{2n+1}) \longrightarrow (\mathbf{CP}^{n+1}, \mathbf{CP}^n).$$

Exercise 3.11. *Consider a map of pairs $(D^n, S^{n-1}) \longrightarrow (S^n, x_0)$ contracting the sphere S^{n-1} to the point x_0. Prove that f is a homotopy equivalence of pairs.*