2 Constructions

2/1. Product. Recall that a product $X \times Y$ of X, Y is a set of pairs (x, y), $x \in X, y \in Y$. If X, Y are topological spaces then a *basis* for *product topology* on $X \times Y$ is given by the products $U \times V$, where $U \subset X$, $V \subset Y$ are open. Here are the first examples:

Example. The torus $T^n = S^1 \times \cdots \times S^1$. Note that the torus T^n may be identified with $U(1) \times \cdots \times U(1) \subset U(n)$ (diagonal orthogonal complex matrices).

Exercise 2.1. *Consider the surface X in S^5, given by the equation $x_1x_6 - x_2x_5 + x_3x_4 = 0$ (where $S^5 \subset \mathbf{R}^6$ is given by $x_1^2 + \cdots + x_6^2 = 1$). Prove that $X \cong S^2 \times S^2$.*

Exercise 2.2. *Prove that the space $SO(4)$ is homeomorphic to $S^3 \times \mathbf{RP}^3$.*

Hint: *Consider carefully the map $SO(4) \longrightarrow S^3 = SO(4)/SO(3)$ and take advantage of the fact that S^3 has a natural group structure: it is a group of unit quaternions. It should be emphasized that it is not true that $SO(n) \cong S^{n-1} \times SO(n-1)$ if $n > 4$.*

We note also that there are standard projections $X \times Y \xrightarrow{pr_X} X$ and $X \times Y \xrightarrow{pr_Y} Y$, and to give a map $f : Z \longrightarrow X \times Y$ is the same as to give two maps $f_X : Z \longrightarrow X$ and $f_Y : Z \longrightarrow Y$.

2/2. Cylinder, suspension. Let $I = [0, 1] \subset \mathbf{R}$. The space $X \times I$ is called a *cylinder over X*, and the subspaces $X \times \{0\}$, $X \times \{1\}$ are the bottom and top "bases". Now we will construct new spaces out of the cylinder $X \times I$.

Remark: quotient topology. Let "$\sim$" be an equivalence relation on the topological space X. We denote by $X/\sim$ the set of equivalence classes. There is a natural map (not continuos so far) $p : X \longrightarrow X/\sim$. We define the following topology on $X/\sim$: the set $U \subset X/\sim$ is open if and only if $p^{-1}(U)$ is open. This topology is called a *quotient topology.*

The first example: let $A \subset X$ be a closed set. Then we define the relation "$\sim$" on X as follows ([ ] denote an equivalence class):

$$[x] = \begin{cases} \{x\} & \text{if } x \notin A, \\ A & \text{if } x \in A. \end{cases}$$

The space $X/\sim$ is denoted by X/A.

The space $C(X) = X \times I/X \times \{1\}$ is a *cone over X*. A *suspention* ΣX over X is the space $C(X)/X \times \{0\}$.

Exercise 2.3. *Prove that the spaces $C(S^n)$ and ΣS^n are homeomorphic to D^{n+1} and S^{n+1} respectively.*

Here is a picture of these spaces:

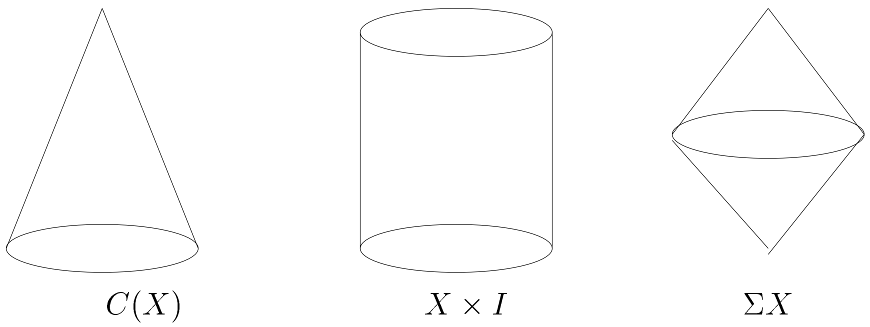

Fig. 2.1.

2/3. Glueing. Let X and Y be topological spaces, $A \subset Y$ and $\varphi : A \longrightarrow X$ be a map. We consider a disjoint union $X \cup Y$, and then we identify a point $a \in A$ with the point $\varphi(a) \in X$. The quotient space $X \cup Y/\sim$ under this identification will be denoted as $X \cup_\varphi Y$, and this procedure will be called *glueing* X *and* Y *by means of* φ.

There are two special cases of this construction.

Let $f : X \longrightarrow Y$ be a map. We identify X with the bottom base $X \times \{0\}$ of the cylinder $X \times I$. The space $X \times I \cup_f Y = Cyl(f)$ is called a *cylinder of the map* f. The space $C(X) \cup_f Y$ is called a *cone of the map* f. Note that the space $Cyl(f)$ contains X and Y as subspaces, and the space $C(f)$ contains X.

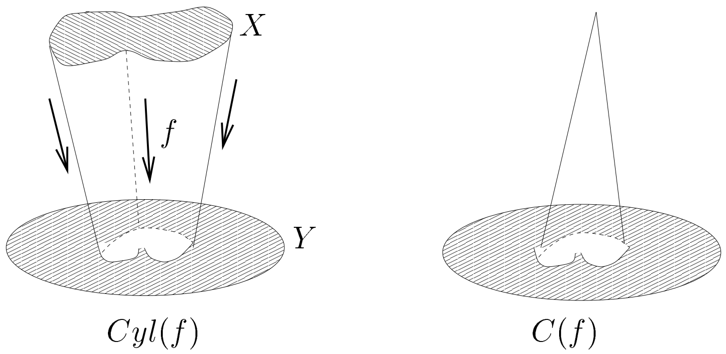

Fig. 2.2.

Let $f : S^n \longrightarrow \mathbf{RP}^n$ be the (we have studied before) map which takes a vector $\vec{v} \in S^n$ to the line $\ell = \langle \vec{v} \rangle$ spanned by $\vec{v}$.

Claim 2.1. *The cone* $C(f)$ *is homeomorphic to the projective space* $\mathbf{RP}^{n+1}$.

Proof (outline). Consider the cone over S^n, clearly $C(S^n) \cong D^{n+1}$ (Exercise 2.3). Now the cone $C(f)$ is a disk D^{n+1} with the opposite points of S^n identified, see Fig. 2.3. $\square$

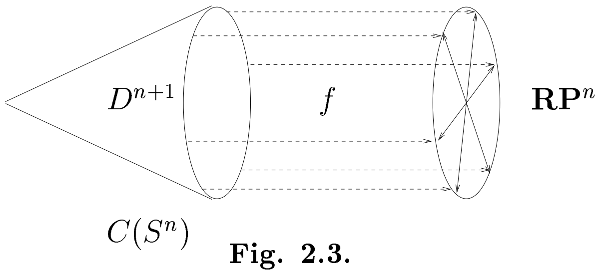

Fig. 2.3.

In particular, a cone of the map $f : S^1 \longrightarrow S^1 = \mathbf{RP}^1$ (given by the formula $e^{i\varphi} \mapsto e^{2i\varphi}$) coincides with the projective plane $\mathbf{RP}^2$.

Exercise 2.4. *Prove that a cone $C(h)$ of the Hopf map $h : S^{2n+1} \longrightarrow \mathbf{CP}^n$ is homeomorphic to the projective space $\mathbf{CP}^{n+1}$.*

Here is the construction which should help you with Exercise 2.4. Let us take one more look at the Hopf map $h : S^{2k+1} \longrightarrow \mathbf{CP}^k$: we take a point $(z_1, \cdots, z_{k+1}) \in S^{2k+1}$, (where $|z_1|^2 + \cdots + |z_{k+1}|^2 = 1$), then h takes it to the line $(z_1 : \cdots : z_{k+1}) \in \mathbf{CP}^k$. Moreover $h(z_1, \cdots, z_{k+1}) = (z'_1, \cdots, z'_{k+1})$ if and only if $z'_j = e^{i\varphi} z_j$. Thus we can identify $\mathbf{CP}^k$ with the following quotient space:

$$\mathbf{CP}^k = S^{2k+1}/\sim, \quad \text{where } (z_1, \cdots, z_{k+1}) \sim (e^{i\varphi} z_1, \cdots, e^{i\varphi} z_{k+1}). \tag{2}$$

Now consider a subset of lines in $\mathbf{CP}^k$ where the last homogeneous coordinate is nonzero:

$$U_{k+1} = \{(z_1 : \cdots : z_{k+1}) \mid z_{k+1} \neq 0\}.$$

We already know that U_{k+1} is homeomorphic to $\mathbf{C}^k$ by means of the map

$$(z_1 : \cdots : z_{k+1}) \mapsto (\frac{z_1}{z_{k+1}}, \ldots, \frac{z_k}{z_{k+1}}$$

Now we use the identification (2) to identify U_{k+1} with an open disk $D^{2k} \subset \mathbf{C}^k$ as follows. Let us think about $U_{k+1} \subset S^{2k+1}/\sim$ as above. Let $\ell \in U_{k+1}$. Choose a point $(z_1, \cdots, z_{k+1}) \in S^{2k+1}$ representing ℓ. Then we have that

$$|z_1|^2 + \cdots + |z_{k+1}|^2 = 1, \quad \text{and} \quad z_{k+1} \neq 0.$$

A complex number z_{k+1} has a unique representation $z_{k+1} = re^{i\alpha}$, where $r = |z_{k+1}|$. Notice that $0 < r \leq 1$. Then the point

$$(e^{-i\alpha} z_1, e^{-i\alpha} z_k \cdots, e^{-i\alpha} z_{k+1}) = (e^{-i\alpha} z_1, e^{-i\alpha} z_k \cdots, r) \in S^{2k+1}$$

represents the same line $\ell \in U_{k+1}$. Moreover, this representation is unique. We have:

$$|z_1|^2 + \cdots + |z_k|^2 = 1 - r^2$$

which describes the sphere $S^{2k-1}\sqrt{1-r^2} \subset \mathbf{C}^k$ of radius $\sqrt{1-r^2}$. The union of the spheres $S^{2k-1}\sqrt{1-r^2}$ over $0 < r \leq 1$ is nothing but an open unit disk in $\mathbf{C}^k$. Then we notice that we can let z_{k+1} to be equal to zero: corresponds to the points

$$(z_1, \cdots, z_k, 0) \in S^{2k+1} \quad \text{with} \quad |z_1|^2 + \cdots + |z_k|^2 = 1,$$

i.e. the sphere $S^{2k-1} \subset \mathbf{C}^k$ modulo the equivalence relation $(z_1, \cdots, z_k, 0) \sim (e^{i\varphi}z_1, \cdots, e^{i\varphi}z_k, 0)$. This is nothing but the projective space $\mathbf{CP}^{k-1}$. We summarize our construction:

Lemma 2.1. *There is a homeomorphism*

$$\mathbf{CP}^k \equiv D^{2k}/\sim, \quad \textit{where} \begin{cases} (z_1, \cdots, z_k) \sim (z_1', \cdots, z_k') \ \textit{if and only if} \\ |z_1|^2 + \cdots + |z_k|^2 = 1, \ |z_1'|^2 + \cdots + |z_k'|^2 = 1, \quad \textit{and} \\ z_j' = e^{i\varphi} z_j \quad \textit{for all } j = 1, \dots, k. \end{cases}$$

2/4. Join. A *join* $X * Y$ of spaces X Y is a union of all linear paths $I_{x,y}$ which startin at $x \in X$ and ending at $y \in Y$; the union is taken over all points $x \in X$ and $y \in Y$. For example, a joint of two intervals I_1 and I_2 lying on two non-parallel and non-intersecting lines is a tetrahedron:

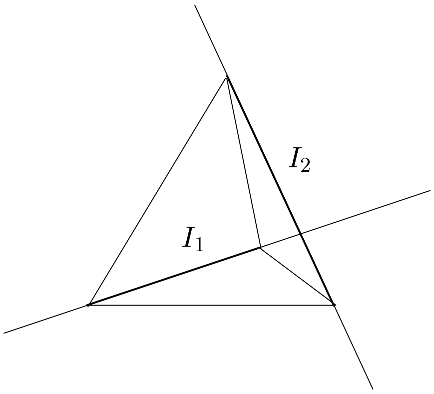

Fig. 2.4.

A formal definition of $X * Y$ is the following. We start with the product $X \times Y \times I$: here there is a linear path (x, y, t), $t \in I$ for given points $x \in X$, $y \in Y$. Then we identify the following points:

$$\begin{array}{ll} (x, y', 1) \sim (x, y'', 1) & \text{for any } x \in X,\ y', y'' \in Y, \\ (x', y, 0) \sim (x'', y, 0) & \text{for any } x', x'' \in X,\ y \in Y. \end{array}$$

Exercise 2.5. *Prove the homeomorphisms*

(a) $X * \{\textit{one point}\} \cong C(X)$;

(b) $X * \{\textit{two points}\} \cong \Sigma(X)$;

(c) $S^n * S^k \cong S^{n+k+1}$. *Hint: prove first that* $S^1 * S^1 \cong S^3$.

2/5. Space of maps, loop spaces, path spaces. Let X, Y are topological spaces. We consider the space $\mathcal{C}(X,Y)$ of all continuous maps from X to Y. To define a topology of the functional space $\mathcal{C}(X,Y)$ it is enough to describe a basis. The basis of the *compact-open topology* is given as follows. Let $K \subset X$ be a compact set, and $O \subset Y$ be an open set. We denote by $\mathcal{U}(K,O)$ the set of all continuous maps $f : X \longrightarrow Y$ such that $f(K) \subset Y$, this is (by definition) a basis for the compact-open topology on $\mathcal{C}(X,Y)$.

Examples. Let X be a point. Then the space $\mathcal{C}(X,Y)$ is homeomorphic to Y. If X be a space consisting of n points, then $\mathcal{C}(X,Y) \cong Y \times \cdots \times Y$ (n times).

Let X, Y, and Z be Hausdorff and locally compact* topological spaces. There is a natural map

$$T : \mathcal{C}(X,\mathcal{C}(Y,Z)) \longrightarrow \mathcal{C}(X \times Y, Z),$$

given by the formula: $\{f : X \longrightarrow \mathcal{C}(Y,Z)\} \longrightarrow \{(x,y) \longrightarrow (f(x))(y)\}$.

Exercise 2.6. *Prove that the map $T : \mathcal{C}(X,\mathcal{C}(Y,Z)) \longrightarrow \mathcal{C}(X \times Y, Z)$ is a homeomorphism.*

Recall we call a map $f : I \longrightarrow X$ a path, and the points $f(0) = x_0$ $f(1) = x_1$ are the beginning and the end points of the path f. The space of all paths $\mathcal{C}(I,X)$ contains two important subspaces:

1. $\mathcal{E}(X,x_0,x_1)$ is the subspace of paths $f : I \longrightarrow X$ such that $f(0) = x_0$ and $f(1) = x_1$;

2. $\mathcal{E}(X,x_0)$ is the space of all paths with x_0 the begining point.

3. $\Omega(X,x_0) = \mathcal{E}(X,x_0,x_0)$ is the loop space with the begining point x_0.

Exercise 2.7. *Prove that the spaces $\Omega(S^n,x)$ and $\Omega(S^n,x')$ are homeomorphic for any points x, $x' \in S^n$.*

Exercise 2.8. *Give examples of a space X other than S^n for which $\Omega(X,x)$ and $\Omega(X,x')$ are homeomorphic for any points x, $x' \in X$. Why does it fail for an arbitrary space X? Give an example when this is not true.*

The loop spaces $\Omega(X,x)$ are rather difficult to describe even in the case of $X = S^n$, however, the spaces X and $\Omega(X,x)$ are intimately related. To see that, consider the following map

$$p : \mathcal{E}(X,x_0) \longrightarrow X \tag{3}$$

* A topological space X is called *locally compact* if for each point $x \in X$ and an open neighborhood U of X there exists an open neighbourhood $V \subset U$ such that the closure $\overline{V}$ of V is compact.

which sends a path $f : I \longrightarrow X$, $f(0) = x_0$, to the point $x = f(1)$. Notice that $p^{-1}(x_0) \cong \Omega(X, x_0)$. The map (3) may be considered as a map of pointed spaces (see the definitions below):

$$p : (\mathcal{E}(X, x_0), *) \longrightarrow (X, *),$$

where the path $* : I \longrightarrow X$ sends the interval to the point $*(t) = x_0$ for all $t \in I$. Clearly $p(*) = x_0$.

2/6. Pointed spaces. Pointed space $X, x_0)$ is a topological space X with a chosen point $x_0 \in X$ (a base point x_0). A map $f : (X, x_0) \longrightarrow (Y, y_0)$ is a continuous map $f : X \longrightarrow Y$ such that $f(x_0) = y_0$. Many operations preserve base points, for example the product $X \times Y$ of pointed spaces (X, x_0), (Y, y_0) have the base point $(x_0, y_0) \in X \times Y$. Some other operations have to be modified.

The cone $C(X, x_0) = C(X)/\{x_0\} \times I$: here we identify with the point all interval over the base point x_0, and the image of $\{x_0\} \times I$ in $C(X, x_0)$ is the base point of this space.

The suspension:

$$\Sigma(X, x_0) = \Sigma(X)/\{x_0\} \times I = C(X)/(X \times \{0\} \cup x_0 \times I) = C(X, x_0)/(X \times \{0\}).$$

The space of maps $\mathcal{C}(X, x_0, Y, y_0)$ for pointed spaces[†] (X, x_0) and (Y, y_0) is the space of continuous maps $f : X \longrightarrow Y$ such that $f(x_0) = y_0$ (with the same compact-open topology). The base point in the space $\mathcal{C}(X, Y)$ is the map $c : X \longrightarrow Y$ which sends all space X to the point $y_0 \in Y$.

If X is a pointed space, then $\Omega(X, x_0)$ is the space of loops begining and ending at the base point $x_0 \in X$, and the space $\mathcal{E}(X, x_0)$ is the space of paths starting at the base point x_0.

Exercise 2.9. *Let X and Y are pointed space. Prove that the space $C(\Sigma(X), Y)$ and the space $C(X, \Omega(Y))$ are homeomorphic.*

Exercise 2.10. *Let $S^1 = \{e^{i\varphi}\}$ be a circle and $s_0 = 1$ ($\varphi = 0$) be a base point. How many path-connected components does the space $\Omega(S^1)$ (a space of loops with s_0 the begining point) have? Try the same question for $\Omega(\mathbf{RP}^2)$.*

[†] We will denote this space by $\mathcal{C}(X, Y)$ when it is clear what the base points are.

There are two more operations which are specific for pointed spaces.

1. A sum (or a bouquet) $X \vee Y$ of pointed spaces (X, x_0) and (Y, y_0) is a disjoint union $X \cup Y$ with the points x_0 and y_0 identified, see Fig. 2.5.

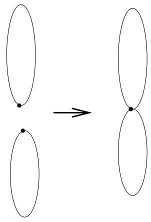

Fig. 2.5.

2. A smash-product $X \wedge Y$ is the factor-space:

$$X \wedge Y = X \times Y/((x_0 \times Y) \cup (X \times y_0)),$$

see Fig.2.6.

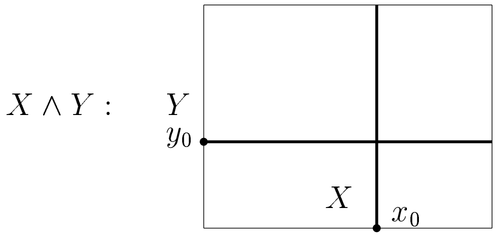

Fig. 2.6.

Exercise 2.11. *Prove that the space $S^n \wedge S^n$ is homeomorphic to S^{n+m} as pointed spaces.*

Exercise 2.12. *Prove that $X \wedge S^1$ is homeomorphic to $\Sigma(X)$ as pointed spaces.*

Remark. We have mentioned several *natural* homeomorphisms, for instance, the homeomorphisms

(a) $\mathcal{C}(\Sigma(X), Y) \xrightarrow{F_{X,Y}} \mathcal{C}(X, \Omega(Y))$,

(b) $X \wedge S^1 \xrightarrow{G} \Sigma(X)$

are *natural*. It means the following. If $f : X' \longrightarrow X$, and $g : Y \longrightarrow Y'$ are maps of pointed spaces, then there the maps

$$f^* : \mathcal{C}(X', Y) \longrightarrow \mathcal{C}(X, Y),$$

$$g_* : \mathcal{C}(X, Y) \longrightarrow \mathcal{C}(X, Y'),$$

given by the formula:

$$f^* : (h : X' \longrightarrow Y) \mapsto (X \xrightarrow{f} X' \xrightarrow{h} Y),$$

$$g_* : (h : X \longrightarrow Y) \mapsto (X \xrightarrow{h} Y \xrightarrow{g} Y').$$

Furthermore, the maps $f : X' \longrightarrow X$, and $g : Y \longrightarrow Y'$ induce the maps

$$\Sigma f : \Sigma X' \longrightarrow \Sigma X, \quad \Omega g : \Omega Y \longrightarrow \Omega Y'$$

given by the formula

$$\Sigma f(x,t) = (f(x),t), \quad \Omega(g) : (\gamma : I \longrightarrow Y) \mapsto (g \circ \gamma : I \longrightarrow Y').$$

We call the homeomorphism **(a)** *natural* since for any maps $f : X' \longrightarrow X$, $g : Y \longrightarrow Y'$ the following diagram of pointed spaces and maps commutes:

$$\begin{tikzcd}
 & \mathcal{C}(\Sigma(X),Y') \arrow[rr, "F_{X,Y'}"] & & \mathcal{C}(X,\Omega(Y')) \\
\mathcal{C}(\Sigma(X),Y) \arrow[ur, "g_*"] \arrow[rr, "F_{X,Y}"] & & \mathcal{C}(X,\Omega(Y)) \arrow[ur, "\Omega g_*"] & \\
 & \mathcal{C}(\Sigma(X'),Y') \arrow[uu, "\Sigma f^*"] \arrow[rr, "F_{X',Y'}"] & & \mathcal{C}(X',\Omega(Y')) \arrow[uu, "f^*"] \\
\mathcal{C}(\Sigma(X'),Y) \arrow[uu, "\Sigma f^*"] \arrow[ur, "g_*"] \arrow[rr, "F_{X',Y}"] & & \mathcal{C}(X',\Omega(Y)) \arrow[uu, "f^*"] \arrow[ur, "\Omega g_*"] &
\end{tikzcd} \tag{4}$$

Exercise 2.13. *Check commutativity of the diagram (4).*

Exercise 2.14. *Show that the homeomorphism $X \wedge S^1 \xrightarrow{G} \Sigma(X)$ is natural.*