14 Homology and homotopy groups

14/1. Homology groups and weak homotopy equivalence. Our goal here is to prove the following fact.

Theorem 14.1. *Let $f : X \longrightarrow Y$ be a weak homotopy equivalence. Then the induced homomorphism $f_* : H_q(X) \longrightarrow H_q(Y)$ is an isomorphism for all $q \geq 0$.*

We start with a preliminary lemma.

Lemma 14.2. *Let X be a topological space, $\alpha \in H_q(X)$. Then there exist a CW-complex K, a map $f : K \longrightarrow X$, an element $\beta \in H_q(K)$ such that $f_*(\beta) = \alpha$.*

Proof. Let $c = \sum_i \lambda_i f_i$, $f_i : \Delta_i^q \longrightarrow X$, be a chain representing $\alpha \in H_q(X)$. Consider the space

$$K = \bigsqcup_i \Delta_i^q.$$

Recall that the simplex $\Delta^q \subset \mathbf{R}^{q+1}$ is given by the vertices $\Delta^q = (v_0, \dots, v_q)$, where $v_0 = (1, 0, \dots, 0)$, $\dots$, $v_q = (0, \dots, 0, 1)$. We can describe all subsimplices of Δ^q as follows. Let $0 \leq t_1 < \cdots < t_{q-r} \leq q$. Then

$$\Gamma^r_{t_1,\dots,t_{q-r}}(\Delta^q) = (v_0, \dots, \widehat{v}_{t_1}, \dots, \widehat{v}_{t_{q-r}}, \dots, v_q)$$

is an r-dimensional simplex with the vertices $(v_0, \dots, \widehat{v}_{t_1}, \dots, \widehat{v}_{t_{q-r}}, \dots, v_q)$. We introduce the following equivalence relation in K:

$$\Gamma^r_{t_1,\dots,t_{q-r}}(\Delta_i^q) \equiv \Gamma^r_{s_1,\dots,s_{q-r}}(\Delta_j^q) \quad \text{iff} \quad f_i|_{\Gamma^r_{t_1,\dots,t_{q-r}}(\Delta_i^q)} = f_j|_{\Gamma^r_{s_1,\dots,s_{q-r}}(\Delta_j^q)}.$$

Let $K = K_1/\sim$. The maps $f_i :: \Delta_i^q \longrightarrow X$ determine a map $f : K \longrightarrow X$. Furthermore, let

$$g_j : \Delta_j^q \xrightarrow{\text{inclusion}} K_1 \xrightarrow{\text{projection}} K.$$

Then the chain $\bar{c} = \sum_j \lambda_j g_j \in C_q(K)$ maps to the chain c by construction. One has to notice that $\bar{c}$ is a cycle since c is a cycle. Then $\beta = [\bar{c}]$ maps to α under the induced homomorphism f_*. $\square$

There is the relative version of Lemma 14.2 which may be proved by slight modification of the above proof.

Lemma 14.3. *Let (X, A) be a pair of toplogical spaces, $\alpha \in H_q(X, A)$. The there exist a CW-pair (K, L), a map $f : (K, L) \longrightarrow (X, A)$, and element $\beta \in H_q(K, L)$, such that $f_*(\beta) = \alpha$.*

Exercise 14.1. Prove Lemma 14.3.

Proof of Theorem 14.1. Recall that $f : X \longrightarrow Y$ is a weak homotopy equivalence if for any CW-complex Z, the induced map $f_{\#} : [Z, X] \longrightarrow [Z, Y]$ is a bijection.

(1): f_* is an epimorphism. Let $\alpha \in H_q(Y)$. Then by Lemma 14.2 there exists a CW-complex K, a map $g : K \longrightarrow Y$ such that $g_*(\beta) = \alpha$. Consider a map $h \in f_{\#}^{-1}(g) \in [K, X]$. We have the following diagram

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ & K \arrow[u, "g"] \arrow[ul, "h"] \end{tikzcd}$$

which commutes up to homotopy. Now we obtain a commutative diagram:

$$\begin{tikzcd} H_q(X) \arrow[r, "f_*"] & H_q(Y) \\ & H_q(K) \arrow[u, "g"] \arrow[ul, "h_*"] \end{tikzcd}$$

Thus $f_*(h_*(\beta)) = \alpha$.

(2): f_* is a monomorphism. First we can change the map $f : X \longrightarrow Y$ to a homotopy equivalent map $f' : X' \longrightarrow Y'$, so that f' is an embedding. Thus we assume that $X \subset Y$ and $f = i : X \subset Y$ is an embedding. Let $\alpha \in H_q(X)$, and $i_*(\alpha) = 0$. Consider the long exact sequence in homology groups:

$$\cdots \to H_{q+1}(Y, X) \xrightarrow{\partial_*} H_q(X) \xrightarrow{i_*} H_q(Y) \to \cdots$$

The exactness implies that there is $\gamma \in H_{q+1}(Y, X)$ such that $\partial_*(\gamma) = \alpha$. By Lemma 14.3 there exist a pair (K, L), a map $g : (K, L) \longrightarrow (Y, X)$, and $\beta \in H_{q+1}(K, L)$ such that $g_*(\beta) = \gamma$. Then since $f : X \longrightarrow Y$ is a weak homotopy equivalence, there exists a map $h : K \longrightarrow X$ making the diagram

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ L \arrow[u, "g|_L"] \arrow[r, "i"] & K \arrow[u, "g"] \arrow[ul, "h"] \end{tikzcd}$$

is commutative up to homotopy. Furthermore, by Theorem 11.7 the map $h : K \longrightarrow Y$ may be chosen so that $(i \circ h)|_L = g|_L$. We have the commutative diagram

$$\begin{tikzcd}
H_{q+1}(Y,X) \arrow[r, "\partial_*"] & H_q(X) \arrow[r, "i_*"] & H_q(Y) \arrow[r] & \cdots \\
H_{q+1}(K,L) \arrow[r, "\partial_*"] \arrow[u, "g_*"] & H_q(L) \arrow[r, "i_*"] \arrow[u, "(g|_L)_* = ((i \circ h)|_L)_*"'] & H_q(K) \arrow[r] \arrow[u, "g_*"'] \arrow[ul, "h_*"'] & \cdots
\end{tikzcd}$$

Thus we have that $\alpha = \partial_*(\gamma) = \partial_*(g_*(\beta)) = (g|_L)_*(\partial_*(\gamma) = h_*(i_*(\partial_*(\beta))) = 0$ because of the exactness. Theorem 14.1 is proved. $\square$

Recall that we proved (Theorem 11.7) that a map $f : X \longrightarrow Y$ is weak homotopy equivalence if the induced homomorphism $f_* : \pi_q(X, x_0) \longrightarrow \pi_q(Y, f(x_0))$ is an isomorphism for all $q \geq 0$ and $x_0 \in X$. We reformulate Theorem 14.1:

Theorem 14.4. *Let a map $f : X \longrightarrow Y$ induce an isomorphism in homotopy groups $f_* : \pi_q(X, x_0) \longrightarrow \pi_q(Y, f(x_0))$ for all $q \geq 0$ and $x_0 \in X$. Then f induces isomorphism in the homology groups $f_* : H_q(X) \longrightarrow H_q(Y)$ for all $q \geq 0$.*

The following exercises show that some naive generalizations of Theorem 14.4 fail.

Exercise 14.2. Show that the spaces $\mathbf{CP}^\infty \times S^3$ and S^2 have isomorphic homotopy groups and different homology groups. Thus these spaces are not homotopy equivalent.

Exercise 14.3. Show that the spaces $\mathbf{RP}^n \times S^m$ and $S^n \times \mathbf{RP}^m$ $(n \neq m,\ m, n \geq 2)$ have isomorphic homotopy groups and different homology groups.

Exercise 14.4. Show that the spaces $S^1 \vee S^1 \vee S^2$ and $S^1 \times S^1$ have the same homology groups and different homotopy groups.

Exercise 14.5. Show that the Hopf map $h : S^3 \longrightarrow S^2$ induces trivial homomorphism in homology groups, and nontrivial homomorphism in homotopy groups.

Exercise 14.6. Show that the projection

$$S^1 \times S^1 \xrightarrow{\text{projection}} (S^1 \times S^1)/(S^1 \vee S^1) = S^2$$

induces trivial homomorphism in homotopy groups, and nontrivial homomorphism in homology groups.

14/2. Hurewicz homomorphism. Let X be a topological space with a base point $x_0 \in X$. Let s_n be a canonical generator of $H_n(S^n)$, $n = 1, 2, \ldots$, given by the homeomorphism $\partial\Delta^n \xrightarrow{\cong} S^n$. For any element $\alpha \in \pi_n(X, x_0)$ consider a representative $f : S^n \longrightarrow X$,

$[f] = \alpha$. We have the induced homomorphism $f_* : H_n(S^n) \longrightarrow H_n(X)$. Let

$$h(\alpha) = f_*(s_n) \in H_n(X).$$

Clearly the element $h(\alpha) \in H_n(X)$ does not depend on the choice of the representing map f. Furthermore, the correspondence $\alpha \mapsto h(\alpha)$ determines the homomorphism

$$h : \pi_n(X, x_0) \longrightarrow H_n(X), \quad n = 1, 2, \ldots.$$

The homomorphism h is the *Hurewicz homomorphism.* The Hurewicz homomorphism is natural with respect to maps $(X, x_0) \longrightarrow (Y, y_0)$ of based spaces.

Exercise 14.7. Prove that $h : \pi_n(X, x_0) \longrightarrow H_n(X)$ is a homomorphism.

Exercise 14.8. Let $x_0, x_1 \in X$, and $\gamma : I \longrightarrow X$ be a path connecting the points x_0, x_1: $\gamma(0) = x_0$, and $\gamma(1) = x_1$. The path γ determines the isomorphism $\gamma_\# : \pi_n(X, x_0) \longrightarrow \pi_n(X, x_1)$. Prove that the following diagram commutes:

$$\begin{tikzcd}
\pi_n(X, x_0) \arrow[rr, "\gamma_\#"] \arrow[dr, "h"'] & & \pi_n(X, x_1) \arrow[dl, "h"] \\
& H_n(X) &
\end{tikzcd}$$

Theorem 14.5. (Hurewicz) *Let (X, x_0) be a based space, such that*

$$\pi_0(X, x_0) = 0, \ \pi_1(X, x_0) = 0, \cdots, \pi_{n-1}(X, x_0) = 0, \tag{63}$$

where $n \geq 2$. Then

$$H_1(X) = 0, \quad H_2(X) = 0, \cdots, H_{n-1}(X) = 0,$$

and the Hurewicz homomorphism $h : \pi_{n-1}(X, x_0) \longrightarrow H_n(X)$ is an isomorphism.

Proof. By Theorem 11.10 there exist a CW-complex K and a weak homotopy equivalence $f : K \longrightarrow X$. Theorem 14.1 guarantees that f induces an isomorphism in homology groups. Thus it is enough to prove the statement in the case when X is a CW-complex. Then the condition (63) means that X is $(n-1)$-connected CW-complex. Theorem 5.9 implies that up to homotopy equivalence X may be chosen so that it has a single zero-cell, and it does not have any cells of dimensions $1, 2, \ldots, n-1$. In particular, this implies that $H_1(X) = 0$, $H_2(X) = 0$, $\cdots$, $H_{n-1}(X) = 0$. Now the n-th skeleton of X is a wedge of spheres: $X^{(n)} = \bigvee_i S_i^n$. Let $g_i : S_i^n \longrightarrow \bigvee_i S_i^n$ be the embedding of the i-th sphere, and let $r_j : S^n \longrightarrow \bigvee_i S_i^n$ be the attaching maps of the $(n+1)$-cells e_j^{n+1}. The maps g_i determine the generators of the group $\pi_n(X^{(n)})$, and let $\rho_j \in \pi_n(X^{(n)})$ be the elements determined by the maps r_j.

Theorem 11.6 describes the first nontrivial homotopy group $\pi_n(X, x_0)$ as a factor-group of the homotopy group $\pi_n(X^{(n)}) \cong \mathbf{Z} \oplus \cdots \oplus \mathbf{Z}$ by the subgroup generated by ρ_j. Notice that the cellular chain group

$$\mathcal{E}_n(X) = H_n(X^{(n)}) = H_n\left(\bigvee_i S_i^n\right),$$

and $H_n(X) = \mathcal{E}_n(X)/\mathrm{Im}\ \overset{b}{\partial}_{n+1}$. Finally we notice that the Hurewicz homomorphism $h : \pi_n(S^n) \longrightarrow H_n(S^n)$ is an isomorphism. Thus we have the commutative diagram

$$\begin{tikzcd}
\pi_n\left(\bigvee S_i^n\right) \arrow[r, "h"] & H_n\left(\bigvee S_i^n\right) \\
\pi_n(S_j^n) \arrow[r, "h"] \arrow[u, "(\rho_j)_*"] & H_n(S_j^n) \arrow[u, "(\rho_j)_*"']
\end{tikzcd}$$

where the horizontal homomorphisms are isomorphisms. Hence h induces an isomorphism $\pi_n(X, x_0) \longrightarrow H_n(X)$. $\square$

Corollary 14.6. *Let X be a simply connected space, and $H_1(X) = 0$, $H_2(X) = 0$ $\cdots$ $H_{n-1}(X) = 0$. Then $\pi_1(X) = 0$, $\pi_2(X) = 0$ $\cdots$ $\pi_{n-1}(X) = 0$ and the Hurewicz homomorphism $h : \pi_n(X, x_0) \longrightarrow H_n(X)$ is an isomorphism.*

Corollary 14.7. *Let X be a simply-connected CW-complex with $\widetilde{H}_n(X) = 0$ for all n. Then X is contractible.*

Exercise 14.9. Prove Corollary 14.6.

Exercise 14.10. Prove Corollary 14.7.

Remark. Let X be a CW-complex. The above results imply that if $\pi_q(X, x_0) = 0$ for all $q \geq 0$ or $H_q(X) = 0$ for all $q \geq 0$, then X is homotopy equivalent to a point. However for a given map $f : X \longrightarrow Y$ the fact that f induces trivial homomorphism in homotopy or homology groups does not imply that f is homotopic to a constant map. The following exercises show that even if f induces trivial homomorphism in both homotopy and homology groups, it does not imply that f is homotopic to a constant map.

Exercise 14.11. Consider the map

$$f : S^1 \times S^1 \times S^1 \xrightarrow{\text{projection}} (S^1 \times S^1 \times S^1)/(S^1 \times S^1 \times S^1)^{(2)} = S^3 \xrightarrow{\text{Hopf}} S^2.$$

Prove that f induces trivial homomorphism in homology and homotopy groups, however f is not homotopic to a constant map.

Exercise 14.12. Consider the map

$$g: S^{2n-2}\times S^3 \xrightarrow{\text{projection}} (S^{2n-2}\times S^3)/(S^{2n-2}\vee S^3) = S^{2n+1} \xrightarrow{\text{Hopf}} \mathbf{CP}^n.$$

Prove that g induces trivial homomorphism in homology and homotopy groups, however g is not homotopic to a constant map.

14/3. Hurewicz homomorphism in the case $n = 1$.

Theorem 14.8. (Poincarè) *Let X be a connected space. Then the Hurewicz homomorphism $h: \pi_1(X, x_0) \longrightarrow H_1(X)$ is epimorphism, and the kernel of h is the commutator $[\pi_1(X, x_0), \pi_1(X, x_0)] \subset \pi_1(X, x_0)$. Thus $H_1(X) \cong \pi_1(X, x_0)/[\pi_1(X, x_0), \pi_1(X, x_0)]$.*

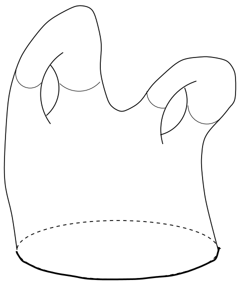

Fig. 14.1.

Exercise 14.13. Prove Theorem 14.8.

Exercise 14.14. We say that a map $f: S^1 \longrightarrow X$ is cobordant to zero if there is an oriented surface M with boundary $\partial M = S^1$ and a map $F: M \longrightarrow X$ such that $F|_{\partial M} = f$, see Fig. 14.1. Let $f: S^1 \longrightarrow X$ be a map representing an element $\alpha \in \pi_1(X, x_0)$. Prove that $\alpha \in \operatorname{Ker} h$ if and only if the map $f: S^1 \longrightarrow X$ is cobordant to zero.

Exercise 14.15. Let M_g^2 be an oriented surface of genus g. As we know, $H_2(M_g^2) \cong \mathbf{Z}$, and a generator $s \in H_2(M_g^2)$ may be represented by the identity map $M_g^2 \longrightarrow M_g^2$. Let X be a simply connected space. Prove that for any class $\alpha \in H_2(X)$ there exist a surface M_g^2, and a map $f: M_g^2 \longrightarrow X$ such that $f_*(s) = \alpha$.

14/4. Relative version of the Hurewicz Theorem. One defines the relative Hurewicz homomorphism $h: \pi_n(X, A, x_0) \longrightarrow H_n(X, A)$ similarly to the regular Hurewicz homomorphism. Let $s \in H_n(D^n, S^{n-1})$ be a canonical generator given by the homeomorphism $(\Delta^n, \partial\Delta^n) \xrightarrow{\cong} (D^n, S^{n-1})$. Let $f: (D^n, S^{n-1}) \longrightarrow (X, A)$ be a map representing an element $\alpha \in \pi_n(X.A)$. Then $h(\alpha) = f_*(s_n) \in H_n(X, A)$. There is a relative version of the above Hurewics Theorem:

Theorem 14.9. *Let (X, A) be a pair of simply connected spaces, $x_0 \in A$, such that*

$$\pi_0(X, A, x_0) = 0, \ \pi_1(X, A, x_0) = 0, \cdots, \pi_{n-1}(X, A, x_0) = 0, \tag{64}$$

where $n \geq 2$. Then

$$H_1(X, A) = 0, \quad H_2(X, A) = 0, \cdots, H_{n-1}(X, A) = 0,$$

and the Hurewicz homomorphism $h: \pi_{n-1}(X, A, x_0) \longrightarrow H_n(X, A)$ is an isomorphism.

We do not give a proof of Theorem 63, however it is very similar to the proof of the above Hurewicz Theorem.

Theorem 14.10. (Whitehead Theorem-II) *Let X, Y be simply connected spaces and $f : X \longrightarrow Y$ be a map.*

(a) *If the induced homomorphism in homotopy groups $f_* : \pi_q(X, x_0) \longrightarrow \pi_q(Y, f(x_0))$ is isomorphism for $q = 2, 3, \ldots, n-1$, and epimorphism if $q = n$, then the homomorphism in homology groups $f_* : H_q(X) \longrightarrow H_q(Y)$ is isomorphism for $q = 2, 3, \ldots, n-1$, and epimorphism if $q = n$.*

(b) *If the induced homomorphism in homology groups $f_* : H_q(X) \longrightarrow H_q(Y)$ is isomorphism for $q = 2, 3, \ldots, n-1$, and epimorphism if $q = n$, then the homomorphism in homotopy groups $f_* : \pi_q(X, x_0) \longrightarrow \pi_q(Y, f(x_0))$ is isomorphism for $q = 2, 3, \ldots, n-1$, and epimorphism if $q = n$.*

Proof. We can assume that $f : X \longrightarrow Y$ is an embedding, see Claim 9.1. Then the long exact sequence in homotopy

$$\cdots \to \pi_q(X, x_0) \xrightarrow{f_*} \pi_q(Y, x_0) \longrightarrow \pi_q(Y, X, x_0) \longrightarrow \pi_{q-1}(X, x_0) \to \cdots$$

gives that if $f_* : \pi_q(X, x_0) \longrightarrow \pi_q(Y, f(x_0))$ is isomorphism for $q = 2, 3, \ldots, n-1$, and epimorphism if $q = n$, then $\pi_q(Y, X, x_0) = 0$ if $q \leq n$. Analogously the long exact sequence in homology

$$\cdots \to H_q(X) \xrightarrow{f_*} H_q(Y) \longrightarrow H_q(Y, X) \longrightarrow H_{q-1}(X) \to \cdots$$

gives that if $f_* : H_q(X) \longrightarrow H_q(Y)$ is isomorphism for $q = 2, 3, \ldots, n-1$, and epimorphism if $q = n$, then $H_q(Y, X) = 0$ for $q \leq n$. Thus the relative Hurewicz Theorem implies the desired result. $\square$

Corollary 14.11. *Let X, Y be simply connected spaces and $f : X \longrightarrow Y$ be a map which induces isomorphism $f_* : H_q(X) \longrightarrow H_q(Y)$ for all $q \geq 0$. Then f is weak homotopy equivalence. (In particular, if X, Y are CW-complexes, then f is homotopy equivalence.)*

Exercise 14.16. Let X be a connected, simply connected CW-complex with $\widetilde{H}_n(X) = \mathbf{Z}$, $n \geq 2$, and $\widetilde{H}_q(X) = 0$ if $q \neq n$. Prove that X is homotopy equivalent to S^n.