13 Homology groups of CW-complexes

The main goal of this section is to develop a technique to compute homology groups of CW-complexes. The singular chain complex $\mathcal{C}(X)$ is far too big to peform computations. We will construct here a *cellular chain complex* $\mathcal{E}(X)$ which is much smaller than $\mathcal{C}(X)$. We start with computations of homology groups of spheres and wedges of spheres.

13/1. Homology groups of spheres.

Theorem 13.1.

$$\widetilde{H}_q(S^n) \cong \begin{cases} \mathbf{Z} & \text{if } q = n, \\ 0 & \text{else.} \end{cases}$$

Remark. We use here the reduced homology groups to unify the formula for $n = 0$ and $n \geq 1$. We already know that $H_0(S^0) = \mathbf{Z} \oplus \mathbf{Z}$, hence $\widetilde{H}_0(S^0) = \mathbf{Z}$.

Proof. Consider a long exact sequence for the pair (D^n, S^{n-1}):

$$\widetilde{H}_q(D^n) \longrightarrow H_q(D^n, S^{n-1}) \longrightarrow \widetilde{H}_q(S^{n-1}) \longrightarrow \widetilde{H}_{q-1}(D^n)$$

We have $\widetilde{H}_q(D^n) = 0$, $\widetilde{H}_{q-1}(D^n) = 0$. Thus $H_q(D^n, S^{n-1}) \cong \widetilde{H}_q(S^{n-1})$. Induction on n concludes the proof. □

Theorem 13.2. *Let X be a space. Then $\widetilde{H}_{q+1}(\Sigma X) \cong \widetilde{H}_q(X)$ for each q.*

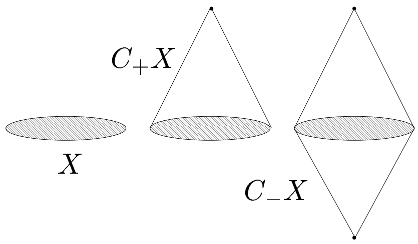

Fig. 13.1.

Proof. We notice that $\Sigma X = C_+X \cup C_-X$, see Fig. 13.1. Consider a long exact sequnce in homology for the pair (C_+X, X):

$$\cdots \to \widetilde{H}_q(C_+X) \longrightarrow H_q(C_+X, X) \longrightarrow \widetilde{H}_{q-1}(X)$$
$$\longrightarrow \widetilde{H}_{q-1}(C_+X) \to \cdots$$

Clearly we have that $\widetilde{H}_*(C_+X) = 0$ since the cone C_+X is contractible.

Thus $H_q(C_+X, X) \cong \widetilde{H}_{q-1}(X)$. Notice that the pair (C_+X, X) is always a Borsuk pair, thus

$$H_q(C_+X, X) \cong \widetilde{H}_q(C_+X/X) \cong \widetilde{H}_q(C_+X \cup C_-X) = \widetilde{H}_q(\Sigma X).$$

Theorem 13.2 is proved. □

Remark. The homeomorphism $\Delta^q \xrightarrow{\cong} D^q$ gives a particular repesentative for the generator $\bar{\iota}_q \in H_q(D^q, S^{q-1})$. The composition $\Delta^q \xrightarrow{\cong} D^q \xrightarrow{pr} D^q/S^{q-1}$ gives a particular repesentative for the generator $\iota_q \in H_q(S^q)$.

Theorem 13.2 leads to the following construction. Let $f : \Delta^q \longrightarrow X$ be a singular simplex. Consider the composition

$$\Sigma f : \Delta^{q+1} = C\Delta^q \xrightarrow{Cf} CX \xrightarrow{\text{projection}} CX/X \cong \Sigma X.$$

Thus we have the chain map $\Sigma : C_q(X) \longrightarrow C_{q+1}(\Sigma X)$.

Excercise 13.1. Show that the map $\Sigma : C_q(X) \longrightarrow C_{q+1}(\Sigma X)$ commutes with the boundary operator and induces the isomorphism $\Sigma : \widetilde{H}_q(X) \longrightarrow \widetilde{H}_{q+1}(\Sigma X)$.

13/2. Homology groups of a wedge.

Theorem 13.3. *Let A be a set of indices, and S^n_α be a copy of the n-th sphere, $\alpha \in A$. Then*

$$\widetilde{H}_q\left(\bigvee_{\alpha\in A} S^n_\alpha\right) = \begin{cases} \bigoplus_{\alpha\in A} \mathbf{Z}(\alpha), & \text{if } q = n, \\ 0, & \text{else.} \end{cases}$$

Here $\bigoplus_{\alpha\in A} \mathbf{Z}(\alpha)$ is a free abelian group with generators $\alpha \in A$.

This result follows from Theorem 13.2 because of the homotopy equivalence

$$\Sigma\left(\bigvee_{\alpha\in A} S^n_\alpha\right) \sim \bigvee_{\alpha\in A} \Sigma S^n_\alpha = \bigvee_{\alpha\in A} S^{n+1}_\alpha.$$

On the other hand Theorem 13.3 is a particular case in the following result.

Theorem 13.4. *Let (X_α, x_α) be based spaces, $\alpha \in A$. Assume that the pair (X_α, x_α) is Borsuk pair for each $\alpha \in A$. Then*

$$\widetilde{H}_q\left(\bigvee_{\alpha\in A} X_\alpha\right) = \bigoplus_{\alpha\in A} \widetilde{H}_q(X_\alpha).$$

Excercise 13.2. Prove Theorem 13.4. **Hint:** The wedge $\bigvee_{\alpha\in A} X_\alpha$ is a factor-space of the disjoint union $\bigsqcup_{\alpha\in A} X_\alpha$ by the union of the base points.

13/3. Maps $\bigvee_{\alpha\in A} S^n_\alpha \xrightarrow{g} \bigvee_{\beta\in B} S^n_\beta$. Let $f : S^n \longrightarrow S^n$ be a map. Then the homotopy class $[f] = d\iota_n$, where $d \in \mathbf{Z}$, and $\iota_n \in \pi_n(S^n)$ is a generator represented by the identity map $S^n \longrightarrow S^n$. Recall that $d = \deg f$.

Claim 13.1. *Let $f : S^n \longrightarrow S^n$ be a map of degree $d = \deg f$. Then the induced homomorphism $f_* : H_n(S^n) \longrightarrow H_n(S^n)$ is the multiplication by d.*

Proof. We constructed earlier a map $g(d) : S^n \longrightarrow S^n$ of the degree d:

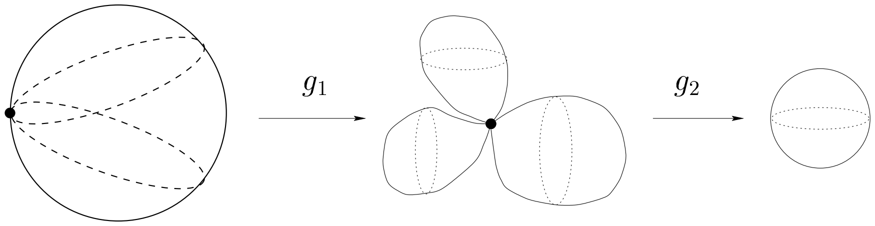

Fig. 13.2.

where the map

$$g_1 : S^n \longrightarrow \bigvee_j^d S_j^n$$

collapses $(d-1)$ spheres S^{n-1} as it is shown at Fig. 13.2, and

$$g_2 : \bigvee_j^d S_j^n \longrightarrow S^n$$

is the folding map. The map $g(d) = g_2 \circ g_1$ is of degree d since this is a representative of $d\iota_n$. Thus $g(d) \sim f$. The composition of the map g_1 with a projection $p_j : \bigvee\limits_j^d S_j^n \longrightarrow S_j^n$ is homotopic to the identinty. Thus $(g_1)_* : H_n(S^n) \longrightarrow H_n(\bigvee_j^d S_j^n)$ gives $(g_1)_*(1) = 1 \oplus \cdots \oplus 1$. We notice that the map

$$S_j^n \xrightarrow{i_j} \bigvee_j^d S_j^n \xrightarrow{g_2} S^n$$

is homotopic to the identity map. Thus $(g_2)_*(1 \oplus \cdots \oplus 1) = 1 + \cdots + 1 = d$. This implies that the homomorphism $g(d)_* = (g_2)_* \circ (g_1)_* : H_n(S^n) \longrightarrow H_n(S^n)$ is the multiplication by d. This completes the proof since the map f is homotopic to the map $g(d)$. $\square$

Now we consider a map $g : \bigvee\limits_{\alpha\in A} S_\alpha^n \xrightarrow{g} \bigvee\limits_{\beta\in B} S_\beta^n$. Let $i_\alpha : S_\alpha^n \longrightarrow \bigvee\limits_{\alpha\in A}$ be the canonical inclusion, and $p_\beta : \bigvee\limits_{\beta\in B} S_\beta^n \longrightarrow S_\beta^n$ be the projection on the β-th summand. We have the commutative diagram:

$$\begin{tikzcd}
\bigvee\limits_{\alpha\in A} S_\alpha^n \arrow[r, "g"] & \bigvee\limits_{\beta\in B} S_\beta^n \arrow[d, "p_\beta"] \\
S_\alpha^n \arrow[u, "i_\alpha"] \arrow[r, "g_{\alpha\beta}"] & S_\beta^n
\end{tikzcd} \tag{60}$$

Let the map $g_{\alpha\beta} : S^n_\alpha \longrightarrow S^n_\beta$ have degree $d_{\alpha\beta}$, and let $\{d_{\alpha\beta}\}_{\alpha\in A,\beta\in B}$ be the matrix of those degrees.

Theorem 13.5. *Let* $g : \bigvee_{\alpha\in A} S^n_\alpha \xrightarrow{g} \bigvee_{\beta\in B} S^n_\beta$ *be a map. Then the homomorphism*

$$\bigoplus_{\alpha\in A}\mathbf{Z}(\alpha) = H_n\left(\bigvee_{\alpha\in A} S^n_\alpha\right) \xrightarrow{g_*} H_n\left(\bigvee_{\beta\in B} S^n_\beta\right) = \bigoplus_{\beta\in B}\mathbf{Z}(\beta)$$

is given by multiplication with the matrix $\{d_{\alpha\beta}\}_{\alpha\in A,\beta\in B}$, *where* $d_{\alpha\beta} = \deg g_{\alpha\beta}$.

Excercise 13.3. Use Claim 13.1 to prove Theorem 13.5.

13/4. Cellular chain complex. Let X be a CW-complex, and $X^{(q)}$ be its q-th skeleton. The factor-space $X^{(q)}/X^{(q-1)}$ is homeomorphic to the wedge $\bigvee_{i\in E_q} S^q_i$, where E_q is the set of q-cells of X. It implies that

$$H_j(X^{(q)}, X^{(q-1)}) = \begin{cases} \bigoplus_{i\in E_n}\mathbf{Z}(i) & \text{if } j = q, \\ 0 & \text{else.} \end{cases} \tag{61}$$

We define the cellular chain complex $\mathcal{E}(X)$ as follows. Let $\mathcal{E}_q(X) = H_q(X^{(q)}, X^{(q-1)}) = \bigoplus_{i\in E_n}\mathbf{Z}(i)$. The boundary operator $\partial_q : \mathcal{E}_q(X) \longrightarrow \mathcal{E}_{q-1}(X)$ is the boundary homomorphism in the long exact sequnce for the triple $(X^{(q)}, X^{(q-1)}, X^{(q-2)})$:

$$\begin{tikzcd}
\cdots \arrow[r] & H_q(X^{(q)}, X^{(q-2)}) \arrow[r] & H_q(X^{(q)}, X^{(q-1)}) \arrow[r, "\partial_*"] & H_q(X^{(q-1)}, X^{(q-2)}) \arrow[r] & \cdots \\
& & \mathcal{E}_q(X) \arrow[u, "\cong"] \arrow[r, "\partial_q"] & \mathcal{E}_{q-1}(X) \arrow[u, "\cong"] &
\end{tikzcd}$$

The following result implies that $\mathcal{E}(X)$ is a chain complex.

Claim 13.2. *The composition* $\mathcal{E}_{q+1}(X) \xrightarrow{\partial_{q+1}} \mathcal{E}_q(X) \xrightarrow{\partial_q} \mathcal{E}_{q-1}(X)$ *is zero.*

Proof. We have the following commutative diagram of pairs:

$$\begin{tikzcd}
(X^{(q)}, X^{(q-2)}) \arrow[r] \arrow[d, "\alpha"] & (X^{(q+1)}, X^{(q-2)}) \arrow[r] \arrow[d, "Id"] & (X^{(q+1)}, X^{(q)}) \arrow[d, "\gamma"] \\
(X^{(q)}, X^{(q-1)}) \arrow[r] & (X^{(q+1)}, X^{(q-1)}) \arrow[r] & (X^{(q+1)}, X^{(q)})
\end{tikzcd}$$

This diagram gives the following commutative diagram in homology groups:

$$
\begin{tikzcd}
 & & \vdots \arrow[d] & \\
H_{q+1}(X^{(q+1)}, X^{(q-2)}) \arrow[r] \arrow[d] & H_{q+1}(X^{(q+1)}, X^{(q)}) \arrow[r, "\partial_*"] \arrow[d, "Id"] \arrow[dr, "\eth_{q+1}"] & H_q(X^{(q)}, X^{(q-2)}) \arrow[r] \arrow[d, "\alpha_*"] & \cdots \\
H_{q+1}(X^{(q+1)}, X^{(q-1)}) \arrow[r] & H_{q+1}(X^{(q+1)}, X^{(q)}) \arrow[r, "\partial_*"] & H_q(X^{(q)}, X^{(q-1)}) \arrow[r] \arrow[d, "\eth_q"] & \cdots \\
 & & H_{q-1}(X^{(q-1)}, X^{(q-2)}) \arrow[d] & \\
 & & \vdots &
\end{tikzcd}
$$

Here the right column is the long exact sequence for the triple $(X^{(q)}, X^{(q-1)}, X^{(q-2)})$. The boundary operator $\eth_{q+1} = \alpha_* \circ \partial_*$ by commutativity. Thus $\eth_q \circ \eth_{q+1} = \eth_q \circ (\alpha_* \circ \partial_*) = (\eth_q \circ \alpha_*) \circ \partial_* = 0$ since $\eth_q \circ \alpha_* = 0$ by the exactness of the column. $\square$

The chain complex $\mathcal{E}(X)$

$$\cdots \longrightarrow \mathcal{E}_q(X) \xrightarrow{\eth_q} \mathcal{E}_{q-1}(X) \longrightarrow \cdots \longrightarrow \mathcal{E}_1(X) \xrightarrow{\eth_1} \mathcal{E}_0(X) \longrightarrow 0$$

is the *cellular chain complex of* X.

Theorem 13.6. *There is an isomorphism* $H_q(\mathcal{E}(X)) \cong H_q(X)$ *for each* q *and any CW-complex* X.

Proof. We prove the following three isomorphisms:

(a) $H_q(\mathcal{E}(X)) \cong H_q(X^{(q+1)}, X^{(q-2)})$,

(b) $H_q(X^{(q+1)}, X^{(q-2)}) \cong \widetilde{H}_q(X^{(q+1)})$,

(c) $H_q(X^{(q+1)}) \cong H_q(X)$.

(a) Consider the following commutative diagram

$$
\begin{tikzcd}
 & H_q(X^{(q-1)}, X^{(q-2)}) = 0 \arrow[d] & & \\
H_{q+1}(X^{(q+1)}, X^{(q)}) \arrow[r, "\partial_*"] \arrow[dr, "\flat\partial_{q+1}"'] & H_q(X^{(q)}, X^{(q-2)}) \arrow[r, "\alpha"] \arrow[d, "\beta"'] & H_q(X^{(q+1)}, X^{(q-2)}) \arrow[r] & H_q(X^{(q+1)}, X^{(q)}) = 0 \\
 & H_q(X^{(q)}, X^{(q-1)}) = \mathcal{E}_q(X) \arrow[d, "\flat\partial_q"'] & & \\
 & H_{q-1}(X^{(q-1)}, X^{(q-2)}) = \mathcal{E}_{q-1}(X) & &
\end{tikzcd}
$$

The exactness at the term $H_q(X^{(q)}, X^{(q-2)})$ implies

$$H_q(X^{(q+1)}, X^{(q-2)}) \cong H_q(X^{(q)}, X^{(q-2)})/\text{Ker } \alpha \cong H_q(X^{(q)}, X^{(q-2)})/\text{Im } \partial_*.$$

The homomorphism β is monomorphism since $H_q(X^{(q-1)}, X^{(q-2)}) = 0$. Thus

$$H_q(X^{(q)}, X^{(q-2)})/\text{Im } \partial_* \cong \beta(H_q(X^{(q)}, X^{(q-2)}))/\beta(\text{Im } \partial_*) \cong$$

$$\text{Im } \beta/\text{Im } (\beta \circ \partial_*) \cong \text{Ker } \flat\partial_q/\text{Im } \flat\partial_{q+1} = H_q(\mathcal{E}(X)).$$

(b) Consider long exact sequence for the triple $(X^{(q+1)}, X^{(i)}, X^{(i-1)})$ where $i = q-2$, $q-3, \ldots, 1, 0$:

$$0 = H_q(X^{(i)}, X^{(i-1)}) \longrightarrow H_q(X^{(q+1)}, X^{(i-1)}) \xrightarrow{\cong} H_q(X^{(q+1)}, X^{(i)}) \longrightarrow H_{q-1}(X^{(i)}, X^{(i-1)}) = 0$$

Thus we obtain the isomorphisms:

$$H_q(X^{(q+1)}, X^{(q-2)}) \cong H_q(X^{(q+1)}, X^{(q-3)}) \cong \cdots \cong H_q(X^{(q+1)}, X^{(0)}) \cong H_q(X^{(q+1)}).$$

(c) Consider long eaxct sequence in homology for the pair $(X^{(j)}, X^{(q+2)})$ for $j = q+2$, $q+3, \ldots$:

$$0 = H_{q+1}(X^{(j)}, X^{(q+2)}) \longrightarrow H_q(X^{(q+2)}) \xrightarrow{\cong} H_q(X^{(j)}) \longrightarrow H_q(X^{(j)}, X^{(q+2)}) = 0.$$

Thus $H_q(X^{(q+2)}) \cong H_q(X^{(q+3)}) \cong \cdots \cong H_q(X)$. □

13/5. Geometric meaning of the boundary homomorphism $\overset{b}{\partial}_q$. Consider closely the groups $\mathcal{E}_q(X)$ and the boundary operator $\overset{b}{\partial}_q : \mathcal{E}_q(X) \longrightarrow \mathcal{E}_{q-1}(X)$. First we recall that

$$\mathcal{E}_q(X) = \left\{ \sum_{i \in E_q} \lambda_i e_i^q \right\}$$

where e_i^q are the q-th cells of the CW-complex X. The isomorphism of the group $\mathcal{E}_q(X)$ with a free abelian group is not unique: it depends on the choice of the homeomorphism $X^{(q)}/X^{(q-1)} \cong \bigvee S^n$. The choice of this homeomorphism is detemined by the characteristic maps

$$\begin{tikzcd} D^q \arrow[r, "\Phi_i"] & X^{(q)} \\ S^{q-1} \arrow[u] \arrow[r, "\varphi_i"] & X^{(q-1)} \arrow[u] \end{tikzcd}$$

The map Φ_i induces the homeomorphism

$$\bar{\Phi}_i : S^q = D^q/D^{q-1} \longrightarrow \bar{e}^q/\partial e^q \subset X^{(q)}/X^{(q-1)}.$$

Two characteristic maps $\Phi_i, \Phi_i' : (D^q, S^{q-1}) \longrightarrow (X^{(q)}, X^{(q-1)})$ are *of the same orientation* if the composition (which is a homeomorphism)

$$S^q \xrightarrow{\bar{\Phi}_i} \bar{e}^q/\partial e^q \xrightarrow{(\bar{\Phi}_i')^{-1}} S^q$$

has degree one. It means that the map $\bar{\Phi}_i \circ (\bar{\Phi}_i')^{-1}$ is homotopic to the identity map. If the degree of the map $\bar{\Phi}_i \circ (\bar{\Phi}_i')^{-1}$ is -1, the characteristic maps Φ_i, Φ_i' have the *opposite orientation.* Thus the group

$$\mathcal{E}_q(X) = \left\{ \sum_{i \in E_q} \lambda_i e_i^q \right\}$$

should be thought as a free abelian group with generators the *oriented q-cells.*

Let e_i^q be a q-cell of X, and σ_j^{q-1} be a $(q-1)$-cell. The attaching map $\varphi_i : S^{q-1} \longrightarrow X^{(q-1)}$ defines the map

$$\psi : S^{q-1} \xrightarrow{\pi_i} X^{(q-1)} \xrightarrow{\text{projection}} X^{(q-1)}/(X^{(q-2)} \cup \{\text{all } (q-1)\text{-cells except } \sigma_j^{q-1}\}) =$$

$$\bar{\sigma}_j^{q-1}/\partial\sigma_j^{q-1} \xrightarrow{(\bar{\Phi_j})^{-1}} S^{q-1}.$$

Let $\deg \psi = [e_i^q : \sigma_j^{q-1}]$.

Remark. The number $[e_i^q : \sigma_j^{q-1}]$ depends on the choice of characteristic maps for the cells e_i^q, σ_j^{q-1} only through the orientation. It is easy to see that the number will change the sign if the orientation of either cell e_i^q or σ_j^{q-1} would be different. It is important to notice that $[e_i^q : \sigma_j^{q-1}] = 0$ if the cells e_i^q, σ_j^{q-1} do not intersect. Thus the number $[e_i^q : \sigma_j^{q-1}] \neq 0$ only for finite number of cells $\sigma_j^{(q-1)}$.

Theorem 13.7. *The boundary operator $\eth_q : \mathcal{E}_q(X) \longrightarrow \mathcal{E}_{q-1}(X)$ is given by the formula:*

$$\eth_q(e^q) = \sum_{j \in E_{q-1}} [e^q : \sigma_j^{q-1}]\sigma_j^{q-1}. \tag{62}$$

Proof. Let $\Phi : (D^q, S^{q-1}, \emptyset) \longrightarrow (X^{(q)}, X^{(q-1)}, X^{(q-2)})$ be the map determined by the characteristic map of the cell e^q. We have the following commutative diagram:

$$\begin{tikzcd}
D^q \arrow[r, "\Phi"] & X^{(q)} \\
S^{q-1} \arrow[u] \arrow[r, "\varphi"] & X^{(q-1)} \arrow[u]
\end{tikzcd}$$

The maps Φ, φ determine the map of the triples

$$\Phi : (D^q, S^{q-1}, \emptyset) \longrightarrow (X^{(q)}, X^{(q-1)}, X^{(q-2)}).$$

We have the following diagram in homology groups:

$$\begin{tikzcd}
\mathbf{Z} \arrow[r, "\cong"] & H_q(D^q, S^{q-1}) \arrow[r, "\partial_*"] \arrow[d, "\alpha"] & H_{q-1}(S^{q-1}) \arrow[d, "\beta"] & \mathbf{Z} \arrow[l, "\cong"'] \\
\widetilde{H}_q(X^{(q)}/X^{(q-1)}) \cong & H_q(X^{(q)}, X^{(q-1)}) \arrow[r, "\partial"] \arrow[d, "\cong"] & H_{q-1}(X^{(q-1)}, X^{(q-2)}) \arrow[d, "\cong"] & \cong \widetilde{H}_{q-1}(X^{(q-1)}/X^{(q-2)}) \\
& \mathcal{E}_q(X) \arrow[r, "\eth"] & \mathcal{E}_{q-1}(X) &
\end{tikzcd}$$

Here $\alpha(1) = e^q$, and $\beta(1) = \eth\, e^q$. We have the composition

$$\gamma : H_{q-1}(S^{q-1}) \xrightarrow{\beta} H_{q-1}(X^{(q-1)}, X^{(q-2)}) = \widetilde{H}_{q-1}(X^{(q-1)}/X^{(q-2)}) = H_q(\bigvee_j S_j^{q-1})$$

$$= \bigoplus_j H_{q-1}(S_j^{q-1}) \xrightarrow{\text{projection}} H_{q-1}(S^{q-1}_{\sigma_j^{q-1}}).$$

By definition the degree of the homomorphism γ the coefficient with σ_j^{q-1} in $\eth\, e^q$ is equal to $[e^q : \sigma_j^{q-1}]$. Thus

$$\eth\, e^q = \sum_j [e^q : \sigma_j^{q-1}]\sigma_j^{q-1}. \quad \square$$

13/6. Some computations. First we compute again the homology groups S^n. We choose the standard cell decomposition: $S^n = e^n \cup e^0$.

$n = 1$. Here we have $\mathcal{E}_0 = \mathbf{Z}$ with the generator e^0, $\mathcal{E}_1 = \mathbf{Z}$ with the generator e^1, $\mathcal{E}_q = 0$ if $q \neq 0, 1$. Clearly $\partial e^1 = e_0 - e_0 = 0$.

$n > 1$. Here we have $\mathcal{E}_0 = \mathbf{Z}$ with the generator e^0, $\mathcal{E}_n = \mathbf{Z}$ with the generator e^n, $\mathcal{E}_q = 0$ if $q \neq 0, n$. Clearly $\partial e^n = 0$.

Thus in both cases we have that $\widetilde{H}_n(S^n) = \mathbf{Z}$, and $\widetilde{H}_q(S^n) = 0$ if $q \neq n$.

Homology groups of $\mathbf{RP}^n$. Here we have to work a bit harder. We need the following geometric fact. Let the sphere $S^n \subset \mathbf{R}^{n+1}$ is given by the equation $x_1^2 + \cdots + x_{n+1}^2 = 1$.

Lemma 13.8. *Let $A : S^n \longrightarrow S^n$ be the antipodal map, $A : x \mapsto -x$, and $\iota_n \in \pi_n(S^n)$ be the generator represented by the identity map $S^n \longrightarrow S^n$. Then the homotopy class $[A] \in \pi_n(S^n)$ is equal to*

$$[A] = \begin{cases} \iota_n, & \text{if } n \text{ is odd,} \\ -\iota_n, & \text{if } n \text{ is even.} \end{cases}$$

Excercise 13.4. Prove Lemma 13.8.

Let $e^0, \ldots, e^n$ be the cells in the standard cell decomposition of $\mathbf{RP}^n$. Recall that $(\mathbf{RP}^n)^{(q)} = \mathbf{RP}^q$, and $e^q = \mathbf{RP}^q/\mathbf{RP}^{q-1}$, and that the Hopf map $S^{q-1} \longrightarrow \mathbf{RP}^{q-1}$ is the attaching map of the cell e^q.

Lemma 13.9. *Let $e^0, \ldots, e^n$ be the cells in the standard cell decomposition of $\mathbf{RP}^n$. Then*

$$[e^q : e^{q-1}] = \begin{cases} 2 & \text{if } q \text{ is odd,} \\ 0, & \text{if } q \text{ is even.} \end{cases}$$

Proof. Let $h : S^{q-1} \longrightarrow \mathbf{RP}^{q-1}$ be the Hopf map. We identify $\mathbf{RP}^{q-1}$ with the sphere S^{q-1} where the points $x, -x \in S^{q-1}$ are identified. The projective space $\mathbf{RP}^{q-2} \subset \mathbf{RP}^{q-1}$ is then the equator sphere $S^{q-2} \subset S^{q-1}$ with the intipodal points identified as well. Now the composition

$$S^{q-1} \xrightarrow{h} \mathbf{RP}^{q-1} \longrightarrow \mathbf{RP}^{q-1}/\mathbf{RP}^{q-2} = S^{q-1}$$

represents the element $[A] + \iota_{q-1} \in \pi_{q-1}(S^{q-1})$, see Fig. 13.3. Lemma 13.8 implies the desired formula. □

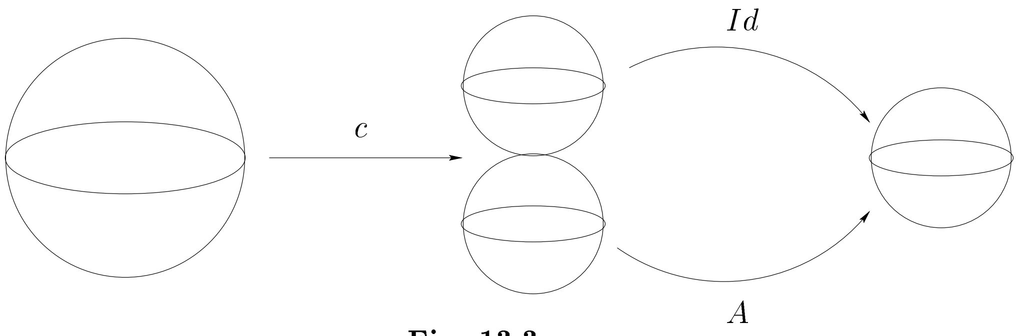

Fig. 13.3.

Thus the chain complex $\mathcal{E}(\mathbf{RP}^{2k+1})$ is the following one

$$0 \longrightarrow \overset{2k+1}{\mathbf{Z}} \xrightarrow{0} \overset{2k}{\mathbf{Z}} \xrightarrow{\cdot 2} \overset{2k-1}{\mathbf{Z}} \xrightarrow{0} \cdots \xrightarrow{0} \overset{2}{\mathbf{Z}} \xrightarrow{\cdot 2} \overset{1}{\mathbf{Z}} \xrightarrow{0} \overset{0}{\mathbf{Z}}$$

Thus we have

$$\widetilde{H}_q(\mathbf{RP}^{2k+1}) = \begin{cases} \mathbf{Z}/2, & q = 1, 3, \ldots, 2k-1, \\ \mathbf{Z}, & q = 2k+1, \\ 0, & \text{else.} \end{cases}$$

The chain complex $\mathcal{E}(\mathbf{RP}^{2k+1})$ is the following one

$$0 \longrightarrow \overset{2k}{\mathbf{Z}} \xrightarrow{\cdot 2} \overset{2k-1}{\mathbf{Z}} \xrightarrow{0} \overset{2k-1}{\mathbf{Z}} \xrightarrow{0} \cdots \xrightarrow{0} \overset{2}{\mathbf{Z}} \xrightarrow{\cdot 2} \overset{1}{\mathbf{Z}} \xrightarrow{0} \overset{0}{\mathbf{Z}}$$

Thus we have

$$\widetilde{H}_q(\mathbf{RP}^{2k}) = \begin{cases} \mathbf{Z}/2, & q = 1, 3, \ldots, 2k-1, \\ 0, & \text{else.} \end{cases}$$

Homology groups of $\mathbf{CP}^n$, $\mathbf{HP}^n$. These groups are very easy to compute since $\mathcal{E}_{2q}(\mathbf{CP}^n) = \mathbf{Z}$, $q = 0, 2, \ldots, n$, and $\mathcal{E}_{2q+1}(\mathbf{CP}^n) = 0$. Similarly $\mathcal{E}_{4q}(\mathbf{HP}^n) = \mathbf{Z}$, $q = 0, 2, \ldots, n$, and $\mathcal{E}_q(\mathbf{HP}^n) = 0$ for all other q. Thus

$$H_q(\mathbf{CP}^n) = \begin{cases} \mathbf{Z}, & q = 0, 2, \ldots, 2n, \\ 0, & \text{else.} \end{cases}, \quad H_q(\mathbf{HP}^n) = \begin{cases} \mathbf{Z}, & q = 0, 4, \ldots, 4n, \\ 0, & \text{else.} \end{cases}$$

Exercise 13.5. Prove that there is no map $f : D^n \longrightarrow S^{n-1}$ so that the restriction

$$f|_{S^{n-1}} : S^{n-1} \longrightarrow S^{n-1}$$

has nonzero degree.

Theorem 13.10. (Brouwer Fixed Point Theorem) *Let $g : D^n \longrightarrow D^n$ be a continious map. Then there exists a fixed point of g, i.e. such $x \in D^n$ that $g(x) = x$.*

Exercise 13.6. Use Exercise 13.5 to prove Theorem 13.10.

Exercise 13.7. Let $M_g = T^2 \# \cdots \# T^2$ (g times). Compute the following homology groups:

(a) $H_q(M_g)$,

(b) $H_q(M_g \# \mathbf{RP}^2)$,

(c) $H_q(M_g \# Kl^2)$.

Exercise 13.8. Let $G(n,k)$ be the real Grassmannian manifold. Use the CW-decomposition of $G(n,k)$ given in Section 4 to compute the homology groups:

(a) $H_q(G(4,2))$,

(b) $H_q(G(5,3))$.

Exercise 13.9. Compute the homology groups:

(a) $H_q(\mathbf{RP}^2 \times \mathbf{RP}^3)$,

(b) $H_q(\mathbf{RP}^5 \times \mathbf{RP}^3)$,

(c) $H_q(\mathbf{RP}^2 \times \mathbf{RP}^4)$.

Exercise 13.10. Let $f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ be a complex polynomial with $a_n \neq 0$. Show that a polynomial $f(z)$, viewed as a map $f : \mathbf{C} \to \mathbf{C}$, can be always extended to a continious map $\hat{f} : S^2 \to S^2$. Prove that the degree of the map $\hat{f}$ equals to n.

Exercise 13.11. Let $f : S^{2n} \to S^{2n}$ be a map. Prove that there exists a point $x \in S^{2n}$ such that either $f(x) = x$ or $f(x) = -x$.

Exercise 13.12. Let $f : S^n \to S^n$ be a map of degree zero. Prove that there exist two points $x, y \in S^n$ with $f(x) = x$ and $f(y) = -y$.

Exercise 13.13. Construct a surjective map $f : S^n \to S^n$ of degree zero.