12 Homology groups: basic constructions

The homotopy groups $\pi_q(X)$ are very important invariants. They are defined in the most natural way, and capture the important information about topological spaces. However it is very difficult to compute the homotopy groups, as we have seen. There are just few finite CW-complexes for which all homotopy groups are known. Even for the sphere S^n the problem to compute the homotopy groups is far from to be solved. Here we define different invariants: homology groups $H_n(X)$ and cohomology groups $H^n(X)$. These groups are much easier to compute: we will be able to compute the homology groups for all basic examples. However, their definition requires more work.

12/1. Singular homology. We alredy defined the *standard q-simplex*:

$$\Delta^q = \left\{ (t_0,\ldots,t_q) \mid t_0 \geq 0,\ldots,t_q \geq 0,\ \sum_{i=0}^{q} t_i = 1 \right\} \subset \mathbf{R}^{q+1}.$$

Remark. Note that the standard simplex Δ^q has vertices $A_0 = (1,0,\ldots,0)$, $A_1 = (0,1,0,\ldots,0),\ldots,A_q = (0,0,\ldots,0,1)$ in the space $\mathbf{R}^{q+1}$. In particular it defines the *orientation* of Δ^q. The simplex Δ^q has the i-th face $(i = 0,\ldots,q)$

$$\Delta^{q-1}(i) = \{(t_0,\ldots,t_q) \mid t_i = 0\ \}$$

which is a standard $(q-1)$-simplex in the space

$$\mathbf{R}^q = \{(t_0,\ldots,t_q \mid t_i = 0\ \} \subset \mathbf{R}^{q+1}$$

with the *induced orientation.*

A *singular q-simplex of the space* X is a continuous map $f : \Delta^q \longrightarrow X$. A *singular q-chain* is a finite linear combination $\sum k_i f_i$, where each $f_i : \Delta^q \longrightarrow X$ is a singular q-simplex, $k \in \mathbf{Z}$. The group q-chains $C_q(X)$ is a free abelian group generated by all singular q-simplices of the space X.

Now we define the "boundary homomorphism" $\partial_q : C_q(X) \longrightarrow C_{q-1}(X)$ as follows. Let $f : \Delta^q \longrightarrow X$ be a singular simplex, then we denote $\Gamma_i(f) = f|_{\Delta^{q-1}(i)}$ its restriction on the i-th face $\Delta^{q-1}(i)$. We define:

$$\partial_q f = \sum_{i=0}^{q} (-1)^i \Gamma_i(f).$$

Lemma 12.1. *The composition*

$$C_{q+1}(X) \xrightarrow{\partial_{q+1}} C_q(X) \xrightarrow{\partial_q} C_{q-1}(X)$$

is trivial, i.e. Im $\partial_{q+1} \subset$ Ker ∂_q.

Proof. It follows from the definition and the identity:

$$\Gamma_i(\Gamma_j(f)) = \begin{cases} \Gamma_{j-1}(\Gamma_i(f)) & \text{for } j > i, \\ \Gamma_j(\Gamma_{i+1}(f)) & \text{for } j \le i. \end{cases} \tag{41}$$

Exercise 12.1. Check the identity (41) and complete the proof of Lemma 12.1.

Main Definition: The group $H_q(X) = \text{Ker } \partial_q / \text{Im } \partial_{q+1}$ is a *q-th homology group* of the space X. (The group $H_0(X) = C_0(x)/\text{Im } \partial_1$, and $H_q(X) = 0$ for $q < 0$.)

The group $Z_q(X) = \text{Ker } \partial_q$ is called the *group of cycles*, and the group $B_q(X) = \text{Im } \partial_{q+1}$ the *group of boundaries*. Thus $H_q(X) = Z_q(X)/B_q(X)$. If $c_1, c_2 \in C_q(X)$ are such elements that $c_1 - c_2 = \partial_{q+1}(d)$, then we say that the chain c_1 is *homologic to* c_2. We call a class $[c] \in H_q(X)$ a *homological class of the cycle* c.

The group $H_q(X)$ is an abelian group; if it is finitely-generated, then $H_q(X) \equiv \mathbf{Z} \oplus \ldots \oplus \mathbf{Z} \oplus \mathbf{Z}_{k_1} \oplus \ldots \oplus \mathbf{Z}_{k_m}$; the rank of this group (i.e. the number of $\mathbf{Z}$'s in this decomposition) is the *Betti* number of the space X.

12/2. Chain complexes, chain maps and homotopies. A *chain complex* $\mathcal{C}$ is a sequence of abelian groups and homomorphisms

$$\ldots \longrightarrow C_{q+1} \xrightarrow{\partial_{q+1}} C_q \xrightarrow{\partial_q} C_{q-1} \longrightarrow \ldots \longrightarrow C_1 \xrightarrow{\partial_1} C_0 \longrightarrow 0\ , \tag{42}$$

such that $\partial_q \circ \partial_{q+1} = 0$ for all $q \ge 1$. For a given chain complex $\mathcal{C}$ the group $H_q(\mathcal{C}) = \text{Ker } \partial_q / \text{Im } \partial_{q+1}$ is the q-th homology group of $\mathcal{C}$. The chain complex

$$\ldots \longrightarrow C_{q+1}(X) \xrightarrow{\partial_{q+1}} C_q(X) \xrightarrow{\partial_q} C_{q-1}(X) \longrightarrow \ldots \longrightarrow C_1(X) \xrightarrow{\partial_1} C_0(X) \longrightarrow 0\ , \tag{43}$$

will be denoted as $\mathcal{C}(X)$. Thus $H_q(X) = H_q(\mathcal{C}(X))$.

Let $\mathcal{C}'$, $\mathcal{C}''$ be two chain complexes. A *chain map* $\varphi : \mathcal{C}' \longrightarrow \mathcal{C}''$ is a collection of homomorphisms $\varphi_q : C'_q \longrightarrow C''_q$ such that the diagram

$$\begin{tikzcd}
\cdots \arrow[r, "\partial'_{q+1}"] & C'_q \arrow[r, "\partial'_q"] \arrow[d, "\varphi_q"] & C'_{q-1} \arrow[r, "\partial'_{q-1}"] \arrow[d, "\varphi_{q-1}"] & \cdots \arrow[r] & C'_1 \arrow[r, "\partial'_1"] \arrow[d, "\varphi_1"] & C'_0 \arrow[r] \arrow[d, "\varphi_0"] & 0 \\
\cdots \arrow[r, "\partial''_{q+1}"] & C''_q \arrow[r, "\partial''_q"] & C''_{q-1} \arrow[r, "\partial''_{q-1}"] & \cdots \arrow[r] & C''_1 \arrow[r, "\partial''_1"] & C''_0 \arrow[r] & 0
\end{tikzcd} \tag{44}$$

commutes. It is clear that a chain map $\varphi : \mathcal{C}' \longrightarrow \mathcal{C}''$ induces the homomorphisms $\varphi_* : H_q(\mathcal{C}') \longrightarrow H_q(\mathcal{C}'')$. In particular, a map $g : X \longrightarrow Y$ induces the homomorphism $g_\# : C_q(X) \longrightarrow C_q(Y)$ (which maps a singular simplex $f : \Delta^q \longrightarrow X$ to a singular simplex $g \circ f : \Delta^q \longrightarrow Y$). It defines a chain map $g_\# : \mathcal{C}(X) \longrightarrow \mathcal{C}(Y)$ and homomorphisms $g_* : H_q(X) \longrightarrow H_q(Y)$.

Exercise 12.2. Prove the following statements:
1. Let $g : X \longrightarrow Y$, $h : Y \longrightarrow Z$ be two maps. Then $(h \circ g)_\# = h_\# \circ g_\#$, and $(h \circ g)_* = h_* \circ g_*$.
2. Let $i : X \longrightarrow X$ be the identity map. Then $i_* = Id$.

Let $\varphi, \psi : \mathcal{C}' \longrightarrow \mathcal{C}''$ be two chain maps. We say the φ, ψ are *chain homotopic* if there are homomorphisms $D_q : C'_q \longrightarrow C''_{q-1}$ such that for each q

$$D_{q-1} \circ \partial'_q + \partial''_{q+1} \circ D_q = \varphi_q - \psi_q,$$

(here $D_{-1} = 0$). In that case we will write down $\varphi \sim \psi$.

Theorem 12.2. *Let $\varphi, \psi : \mathcal{C}' \longrightarrow \mathcal{C}''$ be two chain maps, and $\varphi \sim \psi$. Then $\varphi_* = \psi_* : H_q(\mathcal{C}') \longrightarrow H_q(\mathcal{C}'')$.*

Exercise 12.3. Prove Theorem 12.2.

Theorem 12.3. *Let $g, h : X \longrightarrow Y$ be homotopic maps, then $g_* = h_* : H_q(X) \longrightarrow H_q(Y)$. In other words, homotopic maps induce the same homomorphism in homology groups.*

Proof. By definition we have a homotopy $H : X \times I \longrightarrow Y$, such that $H|_{X \times \{0\}} = g$, $H|_{X \times \{1\}} = h$. Then for any singular simplex $f : \Delta^q \longrightarrow X$ we have a map $H \circ (f \times I) : \Delta^q \times I \longrightarrow Y$. The cylinder $\Delta^q \times I$ has a canonical simplicial structure: we subdivide $\Delta^q \times I$ into $(q+1)$-simplices $\overline{\Delta}^{q+1}(i)$, $i = 0, \ldots, q$, as follows:

$$\overline{\Delta}^{q+1}(i) = \{(t_0, \ldots, t_q, \tau) \in \Delta^q \times I \mid t_0 + \ldots + t_{i-1} \le \tau \le t_0 + \ldots + t_i\},$$

see Fig. 12.1. for $q = 1, 2$:

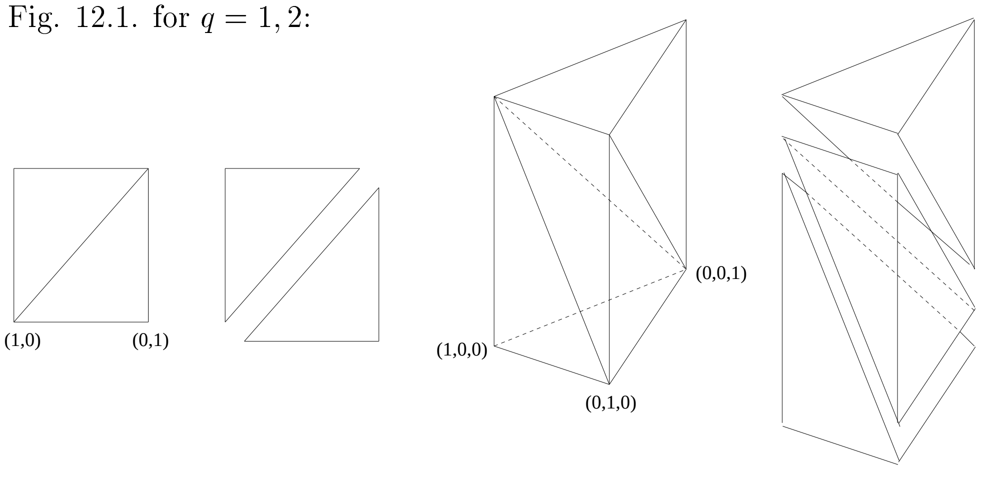

Fig. 12.1.

The map $G = H \circ (f \times I) : \Delta^q \times I \longrightarrow Y$ defines $(q+1)$ singular simplices of dimension $(q+1)$. We define

$$D(f) = \sum_{i=0}^{q} (-1)^i G|_{\overline{\Delta}^{q+1}(i)}.$$

It is easy to check that the homomorphisms

$$D_q : C_q(X) \longrightarrow C_{q+1}(Y), \quad D_q\left(\sum k_i f_i\right) = \sum k_i D_q(f_i)$$

define a chain homotopy $D : \mathcal{C}(X) \longrightarrow \mathcal{C}(Y)$.

Corollary 12.4. *Let X and Y be homotopy equivalent spaces. Then $H_q(X) \cong H_q(Y)$ for all q.*

Remark. There is a natural question: what happens if X and Y are weak homotopy equivalent? We will find the answer on this question in the next section.

12/3. First computations. The groups $C_q(X)$ are really huge by definition, and it is hard to compute homology directly. We will learn how to do this in a while, however even now we can prove several important facts.

Let $*$ be a space consisting of a single point. Clearly there is a unique map $f_q : \Delta^q \longrightarrow *$ for any q. We have that $C_q(*) = \mathbf{Z}$ for all $q \geq 0$. By definition, $\partial_q(f_q) = \sum(-1)^i \Gamma_i(f_q) = \sum(-1)^i f_{q-1}$. It follows then that

$$\partial_q(f_q) = \begin{cases} 0, & \text{for odd } q, \\ 1, & \text{for even } q. \end{cases}$$

The complex $\mathcal{C}(*)$ is the following:

$$\ldots \mathbf{Z} \xrightarrow{Id} \mathbf{Z} \xrightarrow{0} \mathbf{Z} \xrightarrow{Id} \mathbf{Z} \xrightarrow{0} \mathbf{Z} \longrightarrow 0.$$

The argument above proves the following statement.

Claim 12.1. $H_q(*) = \begin{cases} \mathbf{Z}, & \textit{if } q = 0 \\ 0, & \textit{else.} \end{cases}$

A space X with the same homology groups as of the point is called an *acyclic space*.

Corollary 12.5. *Let X be a contractible space, then it is acyclic.*

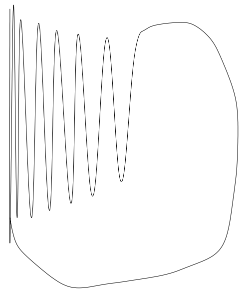

Fig. 12.2.

Remark. The opposite statement does not hold. The simplest example may be constructed out of the function $\sin \frac{1}{x}$, see Fig. 12.2.

Exercise 12.4. Prove that $H_0(X) \cong \mathbf{Z}$ if X is a path-connected space.

Exercise 12.5. Prove that $H_0(X) \cong \mathbf{Z} \oplus \ldots \oplus \mathbf{Z}$, where the number of $\mathbf{Z}$'s is the same as the number of path-connected components of X.

Exercise 12.6. Prove that if $f : X \longrightarrow Y$ is a map of path-connected spaces, then $f_* : H_0(X) \longrightarrow H_0(Y)$ is an isomorphism.

12/4. Relative homology groups. Let A be a subspace of X. Then $C_q(A) \subset C_q(X)$, and $\partial_q(C_q(A)) \subset C_{q-1}(A)$ by definition. Notice that each generator of the group $C_q(A)$ maps to a generator of the group $C_q(X)$. The group $C_q(X, A) = C_q(X)/C_q(A)$ is a group of *relative q-chains* of the space X modulo subspace A. Note that $C_q(X, A)$ is a free abelian group. Alternatively the group $C_q(X, A)$ may be defined as a free abelian group with generators

$$f : \Delta^q \longrightarrow X, \quad f(\Delta^q) \cap (X \setminus A) \neq \emptyset.$$

The boundary operator $\partial_q : C_q(X) \longrightarrow C_{q-1}(X)$ induces the operator $\partial_q : C_q(X, A) \longrightarrow C_{q-1}(X, A)$, and we obtain the complex $\mathcal{C}(X, A)$:

$$\cdots \longrightarrow C_q(X, A) \xrightarrow{\partial_q} C_{q-1}(X, A) \xrightarrow{\partial_{q-1}} \cdots \xrightarrow{\partial_2} C_1(X, A) \xrightarrow{\partial_1} C_0(X, A) \longrightarrow 0. \tag{45}$$

It is easy to check that we have a *short exact sequence of complexes*:

$$0 \longrightarrow \mathcal{C}(A) \xrightarrow{i_\#} \mathcal{C}(X) \xrightarrow{j_\#} \mathcal{C}(X, A) \longrightarrow 0. \tag{46}$$

It is very common situation in the homological algebra to work with a short exact sequence of complexes. The nature of the complexes is not important for the following statement: the complexes below may be over any abelian category.

Lemma 12.6. (LES-Lemma) *Let $0 \longrightarrow \mathcal{C}' \xrightarrow{i} \mathcal{C} \longrightarrow \mathcal{C}'' \longrightarrow 0$ be a short exact sequence of complexes. Then there is a long exact sequence of homology groups*

$$\cdots \longrightarrow H_q(\mathcal{C}') \xrightarrow{i_*} H_q(\mathcal{C}) \xrightarrow{j_*} H_q(\mathcal{C}'') \xrightarrow{\partial} H_{q-1}(\mathcal{C}') \xrightarrow{i_*} \cdots \tag{47}$$

where the homomorphisms i_* and j_* are induced by i and j respectively, and ∂ is the boundary homorphism to be defined.

Proof. First we define the boundary homomorphism $\partial : H_q(\mathcal{C}'') \longrightarrow H_{q-1}(\mathcal{C}')$. We have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & C'_{q+1} \arrow[r, "i_{q+1}"] \arrow[d, "\partial'_{q+1}"'] & C_{q+1} \arrow[r, "j_{q+1}"] \arrow[d, "\partial_{q+1}"'] & C''_{q+1} \arrow[r] \arrow[d, "\partial''_{q+1}"'] & 0 \\
0 \arrow[r] & C'_q \arrow[r, "i_q"] \arrow[d, "\partial'_q"'] & C_q \arrow[r, "j_q"] \arrow[d, "\partial_q"'] & C''_q \arrow[r] \arrow[d, "\partial''_q"'] & 0 \\
0 \arrow[r] & C'_{q-1} \arrow[r, "i_{q-1}"] & C_{q-1} \arrow[r, "j_{q-1}"] & C''_{q-1} \arrow[r] & 0
\end{tikzcd} \tag{48}$$

Let $\alpha \in H_q(\mathcal{C}'')$, and $c'' \in \text{Ker } \partial''_q$ such that $\alpha = [c'']$. Choose an element $\widetilde{c} \in C_q$ such that $j_q(\widetilde{c}) = c''$, then the element $c = \partial_q(\widetilde{c}) \in C_{q-1}$ is such that $j_{q-1}(c) = 0$ by commutativity of (48). The exactness of the bottom row gives that there exists an element $c' \in C'_{q-1}$ such that $i_q(c') = c$.

Now we notice that $c' \in \text{Ker } \partial'_{q-1}$: it follows from the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & C'_{q-1} \arrow[r, "i_{q-1}"] \arrow[d, "\partial'_{q-1}"'] & C_{q-1} \arrow[r, "j_{q-1}"] \arrow[d, "\partial_{q-1}"'] & C''_{q-1} \arrow[r] \arrow[d, "\partial''_{q-1}"'] & 0 \\ 0 \arrow[r] & C'_{q-2} \arrow[r, "i_{q-2}"] & C_{q-2} \arrow[r, "j_{q-2}"] & C''_q \arrow[r] & 0 \end{tikzcd} \tag{49}$$

since i_{q-1} is monomorphism, $c = \partial_q(\widetilde{c})$, and

$$i_{q-1} \circ \partial'_{q-1}(c') = \partial_{q-1} \circ i_q(c) = \partial_{q-1} \circ \partial_q(\widetilde{c}) = 0.$$

Thus $c' \in \text{Ker } \partial_{q-1}$, and we define $\partial(\alpha) = [c'] \in H_{q-1}(\mathcal{C}')$.

Exercise 12.7. Prove that the homomorphism $\partial : H_q(\mathcal{C}'') \longrightarrow H_{q-1}(\mathcal{C}')$ is well-defined.

The proof that the sequence (47) is exact is rather routine exercise. We will prove only the exactness at the term $H_q(\mathcal{C}'')$, i.e. that $\text{Im } j_* = \text{Ker } \partial$.

The inclusion $\text{Im } j_* \subset \text{Ker } \partial$ follows immediately from the definition. Now we prove that $\text{Ker } \partial \subset \text{Im } j_*$. Let $\alpha \in \text{Ker } \partial$. As above (in the definition of ∂) we consider a cycle $c'' \in C''_q$, an element $\widetilde{c}$, such that $j_q(\widetilde{c}) = c''$, then the element $c = \partial_q(\widetilde{c})$, and, finally, the element c' such that $i_{q-1}(c') = c$. We know that $[c'] = 0$, i.e. $c' \in \text{Im } \partial_q$. Let $b' \in C'_q$ be such that $\partial_q(b') = c'$. Let $\widetilde{c}_1 = i_q(b')$. By commutativity of (48) $\partial_q(\widetilde{c} - \widetilde{c}_1) = 0$, and by exatness of the second row of (48) $j_q(\widetilde{c} - \widetilde{c}_1) = c''$. Thus the element $d = \widetilde{c} - \widetilde{c}_1 \in C_q$ is a cycle, and $j_q(d) = c''$, and $j_*([d]) = \alpha$.

Exercise 12.8. Prove the exactness of (47) at the term $H_q(\mathcal{C}')$.

The exactness of (47) at the term $H_q(\mathcal{C})$ is an easy exercise. $\square$

Now we specify Lemma 12.6 in the case of the exact sequence of complexes

$$0 \longrightarrow \mathcal{C}(A) \xrightarrow{i_\#} \mathcal{C}(X) \xrightarrow{j_\#} \mathcal{C}(X, A) \longrightarrow 0.$$

The boundary operator $\partial_q : C_q(X, A) \longrightarrow C_{q-1}(X, A)$ is induced by the boundary operator $\partial_q : C_q(X) \longrightarrow C_{q-1}(X)$, and clearly $\partial_q(c) \in C_{q-1}(A)$ if $c \in C_{q-1}(X, A)$ is a cycle.

Corollary 12.7. *Let (X, A) be a pair of spaces. Then there is an exact sequence of homology groups:*

$$\cdots \longrightarrow H_q(A) \xrightarrow{i_*} \cdots \xrightarrow{\partial} H_q(X) \xrightarrow{j_*} H_q(X, A) \xrightarrow{\partial} H_{q-1}(A) \xrightarrow{i_*} \cdots . \tag{50}$$

Let $B \subset A \subset X$ be a triple of spaces. We have the following maps of pairs:

$$(A, B) \xrightarrow{i} (X, B) \xrightarrow{j} (X, A) \tag{51}$$

which induce the homomorphisms $\mathcal{C}(A, B) \xrightarrow{i_\#} \mathcal{C}(X, B) \xrightarrow{j_\#} \mathcal{C}(X, A)$.

Exercise 12.9. Prove that the sequence of complexes

$$0 \longrightarrow \mathcal{C}(A, B) \xrightarrow{i_\#} \mathcal{C}(X, B) \xrightarrow{j_\#} \mathcal{C}(X, A) \longrightarrow 0 \tag{52}$$

is exact.

Exercise 12.9 and the LES-Lemma imply the following result.

Corollary 12.8. *Let $B \subset A \subset X$ be a triple of spaces. Then there is a long exact sequence in homology:*

$$\cdots \longrightarrow H_q(A, B) \xrightarrow{i_*} H_q(X, B) \xrightarrow{j_*} H_q(X, A) \xrightarrow{\partial} H_{q-1}(A, B) \xrightarrow{i_*} \cdots . \tag{53}$$

The relative homology groups are natural invariant.

Exercise 12.10. Let $B \subset A \subset X$ and $B' \subset A' \subset X'$ be two triples of spaces, and $f : X \longrightarrow X'$ be such a map that $f(B) \subset B'$, and $f(A) \subset A'$. Prove that the following diagram commutes:

$$\begin{tikzcd}
\cdots \arrow[r] & H_q(A,B) \arrow[r, "i_*"] \arrow[d, "f_*"'] & H_q(X,B) \arrow[r, "j_*"] \arrow[d, "f_*"'] & H_q(X,A) \arrow[r, "\partial"] \arrow[d, "f_*"'] & H_{q-1}(A,B) \arrow[r] \arrow[d, "f_*"'] & \cdots \\
\cdots \arrow[r] & H_q(A',B') \arrow[r, "i_*"] & H_q(X',B') \arrow[r, "j_*"] & H_q(X',A') \arrow[r, "\partial"] & H_{q-1}(A',B') \arrow[r] & \cdots
\end{tikzcd}$$

Exercise 12.11. Let $f : (X, A) \longrightarrow (X', A')$ be such map of pairs that the induced maps $f : X \longrightarrow X'$ and $f|_A : A \longrightarrow A'$ are homotopy equivalences. Prove that $f_* : H_q(X, A) \longrightarrow H_q(X', A')$ is an isomorphism for each q.

Remark. One may expect that there is a long exact sequence in homology groups for a Serre fiber bundle $E \longrightarrow B$. However there is no such exact sequence in general case: here there is a *spectral sequence* which relates the homology groups of the base, the total space and the fiber. Again, we are not ready even to discuss this yet.

12/5. Relative homology groups and regular homology groups. Let (X, A) be a pair of spaces. The space X/A has a base point a (the image of A under the projection $X \longrightarrow A$. There is a map of pairs $p : (X, A) \longrightarrow (A, a)$ induced by the projection $X \longrightarrow X/A$. Besides, the is the inclusion map $i : X \longrightarrow X \cup C(A)$, and thus the map of pairs $i : (X, A) \longrightarrow (X \cup C(A), C(A))$. Let v be the vertex of the cone $C(A)$.

Theorem 12.9. *Let (X, A) be a pair of spaces. Then the inclusion*

$$i : (X, A) \longrightarrow (X \cup C(A), C(A))$$

induces the isomorphism $H_q(X, A) \cong H_q(X \cup C(A), C(A)) = H_q(X \cup C(A), v)$.

We have to get ready to prove Theorem 12.9. Recall that for each simplex Δ^q there is the barycentric subdivision of Δ^q. First we examine the barycentric subdivision one more time. Let Δ^q be given by the vertices $A_0, \ldots, A_q$. Let $f : \Delta^q \longrightarrow X$ be a singular simplex. We would like to give a natural description of all q-simplices of the barycentric subdivision of Δ^q in terms of the symmetric group Σ_{q+1} acting on the vertices $(A_0, \ldots, A_q)$ of Δ^q.

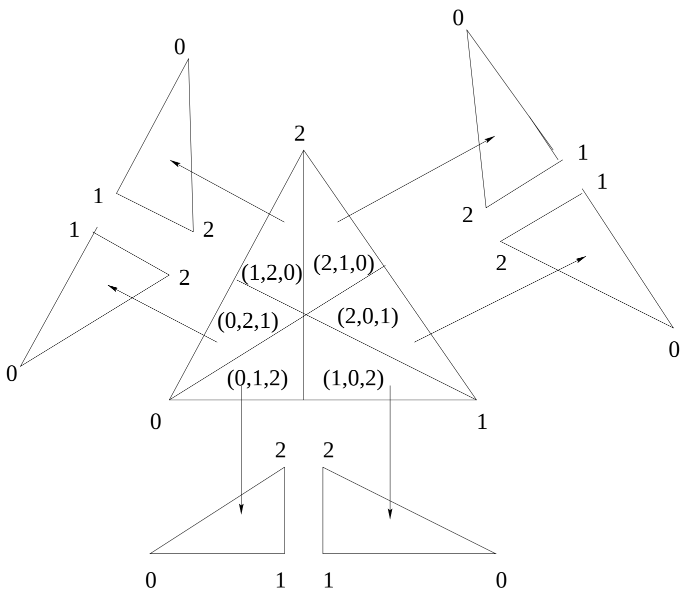

Fig. 12.3.

First, let $q = 1$, then Δ^1 is given by vertices (A_0, A_1). Let B_0 is the barycenter of Δ^1. Then we let $\overline{\Delta}^1(0,1) := (A_0, B_0)$ and $\overline{\Delta}^1(0,1) := (A_1, B_0)$. Here the simplex $\overline{\Delta}^1(0,1)$ is obtained from $\overline{\Delta}^1(0,1)$ by permutation $(0,1)$ which acts on the vertices (A_0, A_1). By induction, let $\overline{\Delta}^q(0, \ldots, q)$ be the simplex which has the same first q vertices as the simplex $\overline{\Delta}^{q-1}(0, \ldots, q-1)$ and the last one being the barycenter of the simplex Δ^q. The symmetric group Σ_{q+1} acts on the vertices $(A_0, \ldots, A_q)$ of Δ^q, and each permutation $\sigma \in \Sigma_{q+1}$ gives a linear map $\sigma : \Delta^q \to \Delta^q$ leaving the barycenter B_0 of Δ^q fixed. Then the simplex $\overline{\Delta}^q(\sigma)$ is defined as the image $\sigma(\overline{\Delta}^q(0, \ldots, q))$. Thus we can list all simplices $\overline{\Delta}^q(\sigma)$ of the barycentric subdivision of Δ^q by the elements $\sigma \in \Sigma_{q+1}$. Let $(-1)^\sigma$ be the sign of the permutation $\sigma \in \Sigma_{q+1}$, see Fig. 12.4.

Now we define a chain map $\beta : \mathcal{C}(X) \longrightarrow \mathcal{C}(X)$ as follows. Let $f : \Delta^q \longrightarrow X$ be a generator, and $f_\sigma = f|_{\overline{\Delta}^q(\sigma)}$. Then

$$\beta(f : \Delta^q \longrightarrow X) = \sum_{\sigma \in \Sigma_{q+1}} (-1)^\sigma (f_\sigma : \overline{\Delta}^q(\sigma) \longrightarrow X).$$

Then we define $\beta(\sum_i \lambda_i f_i) = \sum_i \lambda_i \beta(f_i)$. It is easy to check that $\beta\partial_q = \partial_q\beta$. (Here the choice of the above sign $(-1)^\sigma$ is important.)

Lemma 12.10. *The chain map $\beta : \mathcal{C}(X) \longrightarrow \mathcal{C}(X)$ induces the identity homomorphism in homology:*

$$Id = \beta_* : H_q(\mathcal{C}(X)) \longrightarrow H_q(\mathcal{C}(X)) \quad \textit{for each } q \geq 0.$$

Proof. It is enough to construct a chain homotopy $D_q : C_q(X) \longrightarrow C_{q+1}(X)$ so that $\beta - Id = D_{q-1} \circ \partial'_q + \partial''_{q+1} \circ D_q$. We construct the triangulation of $\Delta^q \times I$ as follows. The cases $q = 0, 1, 2$ are shown at Fig. 12.4. Now the bottom simplex $\Delta^q \times \{0\}$ is given the standard triangulation (just one simplex), and the top simplex $\Delta^q \times \{1\}$ is given the barycentric subdivision. The side $\partial\Delta^q \times I$ is given the subdivision by induction. Now consider the center v of the simplex $\Delta^q \times \{1\}$, and consider the cones with the vertex v over each q-simplex $\overline{\Delta}^q$, where

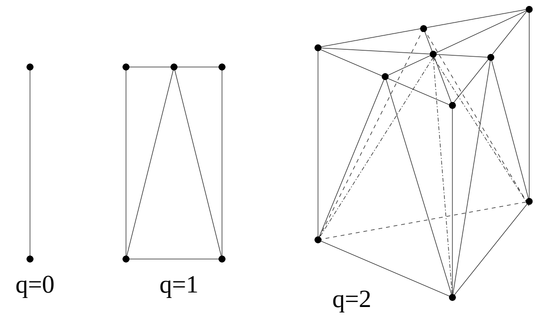

Fig. 12.4.

$$\overline{\Delta}^q \subset \Delta^q \times \{0\} \cup \partial\Delta^q \times I \cup \Delta^q \times \{1\}.$$

This triangulation gives the chain $D_q(f)$, where $f : \Delta^q \longrightarrow X$ is a singular simplex. We notice that $D_q(f)$ is defined as via the map

$$G : \Delta^q \times I \xrightarrow{\text{projection}} \Delta^q \times \{0\} \xrightarrow{f} X$$

by restricting G on the corresponding simplices. Lemma 12.9 follows. $\square$

Let $\mathfrak{U} = \{U_i\}$ be a finite open covering of a space X. We define the group

$$C_q^{\mathfrak{U}}(X) = \{\text{free abelian group}\}\,(f : \Delta^q \longrightarrow X \mid f(\Delta^q) \subset U_i \text{ for some } U_i \in \mathfrak{U}).$$

Clearly $C_q^{\mathfrak{U}}(X) \subset C_q(X)$ and the restriction of the boundary operator $\partial_q : C_q(X) \longrightarrow C_{q-1}(X)$ defines the operator $\partial_q : C_q^{\mathfrak{U}}(X) \longrightarrow C_{q-1}^{\mathfrak{U}}(X)$. Thus we have the complex $\mathcal{C}^{\mathfrak{U}}(X)$.

Lemma 12.11. *The chain map (inclusion) $i : \mathcal{C}^{\mathfrak{U}}(X) \longrightarrow \mathcal{C}(X)$ induces isomorphism in the homology groups*

$$i_* : H_q(\mathcal{C}^{\mathfrak{U}}(X)) \xrightarrow{\cong} H_q(\mathcal{C}(X)). \tag{54}$$

Proof. Let $\alpha \in H_q(X)$, and $\alpha = [c]$, where $c \in Z_q(X)$ is a cycle. To prove that α is epimorphism, we have to prove that

(i) *there is $c' \in Z_q^{\mathfrak{U}}(X)$ and $d \in C_{q+1}$ so that $\partial_{q+1}(d) = c - c'$.*

Let $c' \in Z_q^{\mathfrak{U}}(X)$, and $c' = \partial_{q+1} d$ where $d \in C_{q+1}(X)$. To prove that i_* is monomorphism, we have to show that

(i) *there is $d' \in C_{q+1}^{\mathfrak{U}}(X)$ such that $\partial_{q+1}(d') = c'$.*

The above statements follow from the following three observations:

(1) For any $c \in C_q(X)$ there is $n \geq 1$ so that $\beta^n c \in C_q^{\mathfrak{U}}(X)$.

(2) For any $c \in C_q(X)$ and $n \geq 1$ there is $d \in C_q(X)$ such that $\partial_{q+1}(d) = c - \beta^n c$. (Lemma 12.10.)

(3) Let $c' \in Z_q^{\mathfrak{U}}(X)$, then for any $n \geq 1$ there is $d' \in C_{q+1}^{\mathfrak{U}}(X)$ such that $\partial_{q+1} d = c' - \beta^n c'$.

Exercise 12.12. Prove the properties (1) and (3).

Exercise 12.13. Show that the above statements (i), (ii) follow from (1), (2), (3).

This concludes the proof. □

Remark. Let $\mathfrak{V} = \{V_j\}$ be a finite covering of X, such that $X = \bigcup_j \overset{o}{V}_j$, where $\overset{o}{V}$ is the interior of V. Then the chain map $\mathcal{C}^{\mathfrak{V}}(X) \longrightarrow \mathcal{C}(X)$ also induces isomorphism in the homology groups.

Remark. Let (X, A) be a pair of spaces. Then a covering $\mathfrak{U} = \{U_j\}$ induces a covering $\{U_j \cap A\}$. We denote a corresponding chain complex by $\mathcal{C}^{\mathfrak{V}}(A)$. Then for each q we have a short exact sequence

$$0 \to C_q^{\mathfrak{V}}(A) \longrightarrow C_q^{\mathfrak{V}}(X) \longrightarrow C_q^{\mathfrak{V}}(X, A) \to 0$$

which determines the relative chain complex $\mathcal{C}^{\mathfrak{V}}(X, A)$. It easy to modify the proof of Lemma 12.11 (and use five-lemma) to show that the natural chain map $\mathcal{C}^{\mathfrak{V}}(X, A) \longrightarrow \mathcal{C}(X, A)$ induces an isomorphism in the homology groups

$$H_q(\mathcal{C}^{\mathfrak{V}}(X, A)) \xrightarrow{\cong} H_q(\mathcal{C}(X, A)) = H_q(X, A).$$

Proof of Theorem 12.9. Consider the following covering of the space $X \cup C(A)$. Let $U_1 = (X \cup C(A)) \setminus X$ and $U_2 = X \cup \overline{C}(A)$, where $\overline{C}(A)$ is the half-cone over A, i.e. $\overline{C}(A) = \{(a, t) \in C(A) \mid 0 \leq t < 1/2\}$. The relative version of Lemma 12.11 (see the above remark) implies that the embedding

$$\mathcal{C}^{\mathfrak{U}}(X \cup C(A), C(A)) \longrightarrow \mathcal{C}(X \cup C(A), C(A))$$

induces an isomorphism in the homology groups. By definition of a relative chain complex, we have the isomorphism:

$$C_q^{\mathfrak{U}}(X \cup C(A), C(A)) \cong C_q^{\mathfrak{U}}(X \cup C(A)) / C_q^{\mathfrak{U}}(C(A)).$$

Then we observe that there is an isomorphism

$$C_q^{\mathfrak{U}}(X \cup C(A))/C_q^{\mathfrak{U}}(C(A)) \cong C_q(X \cup \overline{C}(A))/C_q(\overline{C}(A)) = C_q(X \cup \overline{C}(A), \overline{C}(A)).$$

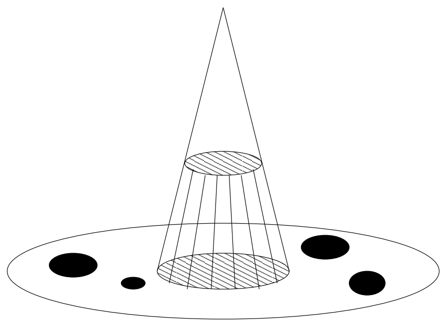

Fig. 12.5.

Indeed, let $f : \Delta^q \to X \cup C(A)$ be a generator in $C_q^{\mathfrak{U}}(X \cup C(A), C(A))$, i.e. $f(\Delta^q) \cap C(A) = \emptyset$ and $f(\Delta^q) \subset U_1$ or $f(\Delta^q) \subset U_2$. Since $U_1 = (X \cup C(A)) \setminus X$, $f(\Delta^q) \subset U_2 = X \cup \overline{C}(A)$ and $f(\Delta^q) \cap \overline{C}(A) = \emptyset$, the map $f : \Delta^q \to X \cup C(A)$ is a generator of the free abelian group

$$C_q(X \cup \overline{C}(A), \overline{C}(A)) = C_q(X \cup \overline{C}(A))/C_q(\overline{C}(A)).$$

It is also easy to check that any generator in $C_q(X \cup \overline{C}(A))/C_q(\overline{C}(A))$ gives a generator in the group $C_q^{\mathfrak{U}}(X \cup C(A), C(A))$.

Since $X \cup \overline{C}(A)$ is homotopy equivalent to X, and $\overline{C}(A) \sim A$, we obtain the isomorphisms

$$H_q(X \cup C(A), C(A)) \cong H_q(X \cup \overline{C}(A), \overline{C}(A)) \cong H_q(X, A).$$

This conclucles the proof of Theorem 12.9. $\square$

Corollary 12.12. *Let (X, A) be a Borsuk pair. Then the projection $p : (X, A) \longrightarrow (A, a)$ induces the isomorphism $p_* : H_q(X, A) \longrightarrow H_q(X/A, a)$ for each q.*

Exercise 12.14. Prove Corollary 12.12.

12/6. Excision Theorem. Let (X, A) be a pair of spaces, and $B \subset A$. The map of pairs $e : (X \setminus B, A \setminus B) \longrightarrow (A, B)$ induces the *excision homomorphism*:

$$e_* : H_q(X \setminus B, A \setminus B) \longrightarrow H_q(X, A) \qquad (55)$$

The following result is known as the *Excision Theorem.*

Theorem 12.13. *Let (X, A) be a pair of Hausdorff spaces, and $B \subset A$ so that $\overline{B} \subset \overset{o}{A}$. Then the homomorphism $e_* : H_q(X \setminus B, A \setminus B) \longrightarrow H_q(X, A)$ is an isomorphism.*

Proof. We use the condition $\overline{B} \subset \overset{o}{A}$ to notice that

$$(X \overset{o}{\setminus} B) \supset X \setminus \overline{B} \supset X \setminus \overset{o}{A}.$$

Thus $\overset{o}{A} \cup (X \overset{o}{\setminus} B) = X$. We consider the covering $\mathfrak{V} = \{V_1, V_2\}$ of X, where $V_1 = A$, $V_2 = X \setminus B$. The chain complex $\mathcal{C}^{\mathfrak{V}}(X)$ (see the remark following Exercise 12.13) gives the chain map $i : \mathcal{C}^{\mathfrak{V}}(X) \longrightarrow \mathcal{C}(X)$. Note that for each q

$$C_q^{\mathfrak{V}}(X) \cong C_q(A) + C_q(X \setminus A) \subset C_q(X).$$

Consider the relative chain complex $\mathcal{C}^{\mathfrak{V}}(X,A)$.To prove the excision property we consider the following commutative diagram of chain complexes:

$$\begin{tikzcd}
\mathcal{C}(X \setminus B, A \setminus B) \arrow[r, "j_1"] \arrow[dr, "j_2"'] & \mathcal{C}(X,A) \\
& \mathcal{C}^{\mathfrak{V}}(X,A) \arrow[u, "j_3"']
\end{tikzcd} \tag{56}$$

Here the chain maps j_1 and j_3 are induced by natural inclusions.

Now we construct the chain map j_2. By definition

$$C_q(X \setminus B, A \setminus B) = C_q(X \setminus B)/C_q(A \setminus B) \cong C_q(X \setminus B)/(C_q(X \setminus B) \cap C_q(A))$$

since $C_q(A \setminus B) = C_q(X \setminus B) \cap C_q(A)$. Similarly,

$$C_q^{\mathfrak{V}}(X,A) = C_q^{\mathfrak{V}}(X)/C_q^{\mathfrak{V}}(A) = (C_q(X \setminus B) + C_q(A))/C_q(A).$$

Now recall the following standard fact from the group theory.

Claim 12.2. *Let $G_1, G_2 \subset G$ be subgroups of an abelian group G. Then*

$$G_1/(G_1 \cap G_2) \cong (G_1 + G_2)/G_2.$$

If we let $G_1 := C_q(X \setminus B)$, $G_2 := C_q(A)$, then the isomorphism

$$j_2 : C_q(X \setminus B, A \setminus B) \longrightarrow C_q^{\mathfrak{V}}(X,A)$$

is given by Claim 12.2. We obtain the induced commutative diagram in homology groups

$$\begin{tikzcd}
H_q(X \setminus B, A \setminus B) \arrow[r, "(j_1)_*"] \arrow[dr, "(j_2)_*"'] & H_q(X,A) \\
& H_q(\mathcal{C}^{\mathfrak{V}}(X,A)) \arrow[u, "(j_3)_*"']
\end{tikzcd} \tag{57}$$

where $(j_1)_*$, $(j_2)_*$ and $(j_3)_*$ are isomorphisms. Thus $H_q(X \setminus B, A \setminus B) \cong H_q(X,A)$. $\square$

12/7. Mayer-Vietoris Theorem. Let $X = X_1 \cup X_2$. We notice that $\mathcal{C}(X_1 \cap X_2) = \mathcal{C}(X_1) \cap \mathcal{C}(X_2)$, and that $\mathcal{C}(X_1)$, $\mathcal{C}(X_2)$ are subcomplexes of $\mathcal{C}(X_1 \cup X_2)$. In particular, the complex $\mathcal{C}(X_1) + \mathcal{C}(X_2) \subset \mathcal{C}(X_1 \cup X_2)$ is well-defined. Let

$$j^{(1)} : \mathcal{C}(X_1 \cap X_2) \longrightarrow \mathcal{C}(X_1), \quad j^{(2)} : \mathcal{C}(X_1 \cap X_2) \longrightarrow \mathcal{C}(X_2),$$

$$i^{(1)} : \mathcal{C}(X_1) \longrightarrow \mathcal{C}(X_1 \cup X_2), \quad i^{(2)} : \mathcal{C}(X_2) \longrightarrow \mathcal{C}(X_1 \cup X_2)$$

be the inclusions. Consider the following sequence of complexes:

$$0 \longrightarrow \mathcal{C}(X_1 \cap X_2) \xrightarrow{\alpha} \mathcal{C}(X_1) \oplus \mathcal{C}(X_2) \xrightarrow{\beta} \mathcal{C}(X_1) + \mathcal{C}(X_2) \longrightarrow 0. \tag{58}$$

where $\alpha(c) = j^{(1)}(c) \oplus j^{(2)}(c)$, and $\beta(c_1 \oplus c_2) = i^{(1)}(c_1) - i^{(2)}(c_2) \in \mathcal{C}(X_1) + \mathcal{C}(X_2)$.

Claim 12.3. *The sequence (58) is a short exact sequence of chain complexes.*

Exercise 12.15. Prove Claim 12.3.

Lemma 12.14. *Let $X_1, X_2 \subset X$, and $X_1 \cup X_2 = X$, $\overset{o}{X}_1 \cup \overset{o}{X}_2 = X$. Then the chain map $\mathcal{C}(X_1) + \mathcal{C}(X_2) \longrightarrow \mathcal{C}(X_1 \cup X_2)$ induces isomorphism in the homology groups.*

Proof. Consider the covering $\mathfrak{V} = \{X_1, X_2\}$. Then by definition $\mathcal{C}^{\mathfrak{V}}(X_1 \cup X_2) = \mathcal{C}(X_1) + \mathcal{C}(X_2)$. Lemma 12.11 and the remark following Lemma 12.11 completes the proof. □

Theorem 12.15. (Mayer-Vietoris Theorem) *Let X be a space, and $X = X_1 \cup X_2$, and $X = \overset{o}{X}_1 \cup \overset{o}{X}_2$. Then there is a long exact sequence*

$$\cdots \to H_q(X_1 \cap X_2) \xrightarrow{\alpha_*} H_q(X_1) \oplus H_q(X_2) \xrightarrow{\beta_*} H_q(X_1 \cup X_2) \xrightarrow{\partial} H_{q-1}(X_1 \cap X_2) \to \cdots \tag{59}$$

This is the Mayer-Vietoris long exact sequence.

Proof. The short exact sequence of chain complexes (58) induces the long exact sequnce

$$\cdots \to H_q(\mathcal{C}(X_1 \cap X_2)) \xrightarrow{\alpha_*} H_q(\mathcal{C}(X_1)) \oplus H_q(\mathcal{C}(X_2)) \xrightarrow{\beta_*} H_q(\mathcal{C}(X_1) + \mathcal{C}(X_2))$$

$$\xrightarrow{\partial} H_{q-1}(\mathcal{C}(X_1 \cap X_2)) \xrightarrow{\alpha_*} H_{q-1}(\mathcal{C}(X_1)) \oplus H_{q-1}(\mathcal{C}(X_2)) \to \cdots$$

To complete the proof we replace the groups $H_q(\mathcal{C}(X_1) + \mathcal{C}(X_2))$ by $H_q(X_1 \cup X_2)$ using Lemma 12.14. □