11 Homotopy groups of CW-complexes

11/1. Changing homotopy groups by attaching a cell. Let X be a CW-complex, and $f : S^n \longrightarrow X$ be an attaching map for new cell. Let $Y = X \cup_f D^{n+1}$. We would like to understand how do the homotopy groups $\pi_* X$ and $\pi_* Y$ relate to each other.

Theorem 11.1. *Let X be a path-connected space (not necessarily a CW-complex) with a base point $x_0 \in X$, $f : S^n \longrightarrow X$ be a map such that $f(s_0) = x_0$, where s_0 is a base point of S^n. Let $Y = X \cup_f D^{n+1}$, and $i : X \longrightarrow Y$ be the inclusion. Then the induced homomorphism*

$$i_* : \pi_q(X, x_0) \longrightarrow \pi_q(Y, x_0) \tag{31}$$

(1) *is an isomorphism if $q < n$,*

(2) *is an epimorphism if $q = n$, and*

(3) *the kernel Ker $i_* : \pi_n(X, x_0) \longrightarrow \pi_n(Y, x_0)$ is generated by $\gamma^{-1}[f]\gamma \in \pi_n(X, x_0)$ where $\gamma \in \pi_1(X, x_0)$.*

Proof. First we prove a technical result. Let E^m be either D^m or S^m. In both cases we choose $\mathbf{R}^m$ to be a subspace of E^m:

$$S^m = \mathbf{R}^m \cup \{x_0\},$$

$$D^m = \overset{o}{D}{}^m \cup S^{m-1}, \quad \overset{o}{D}{}^m \cong \mathbf{R}^m.$$

Lemma 11.2. *Let $h : E^m \longrightarrow Y$ be a map, such that $h|_{E^m \setminus \mathbf{R}^m}$ sends $E^m \setminus \mathbf{R}^m$ to X. Then there exists a map $h_1 : E^m \longrightarrow Y$ homotopic to h so that:*

(a) $h_1|_{h^{-1}(X)} = h|_{h^{-1}(X)}$.

(b) *If $m \leq n$ then $h_1(E^m) \subset X$.*

(c) *If $m = n + 1$ there exist disks $d_1, \dots, d_r \subset E^m$ such that*

- **(c1)** $h_1(E^m \setminus \bigcup_{s=1}^{r} d_s) \subset X$;
- **(c2)** *the restriction $h_1|_{\overset{o}{d}_s} : \overset{o}{d}_s \longrightarrow D^m$ is a linear homeomorphism, $s = 1, \dots, r$.*

Proof. The proof goes down the line of arguments which we used several times starting with Free Point Lemma. We give here the outline only.

(1) Consider the disks (centered at the same point) $D_1^m \subset D_2^m \subset D_3^m \subset D_4^m \subset D_5^m \subset D^m$ of radii $i\rho/5$, $i = 1, \dots, 5$.

(2) The set $h^{-1}(D_5^m) \subset E^m$ is compact, furthermore, $h^{-1}(D_5^m) \subset \mathbf{R}^m \subset E^m$. Choose a simplex $\Delta^m \supset h^{-1}(D_5^m)$ and a triangulation $\{\Delta_\alpha\}$ of Δ such that if $f(\Delta_\alpha) \cap D_i^m \neq \emptyset$ then $f(\Delta_\alpha) \subset D_{i+1}^m$ for $i = 1, 2, 3, 4$, and $\operatorname{diam}(f(\Delta_\alpha)) < \rho/5$.

(3) Let $K = \bigcup\limits_{f(\Delta_\alpha)\cap D_4^m \neq \emptyset} \Delta_\alpha$. Then we construct a map $h' : K \longrightarrow D^m$ by extending linearly h restricted on the vertices of K.

(4) We assume that the center y_0 of the disks D_i^m is not in the image of any face of the simplices Δ_α. If it happens to be in such image, we choose a homotopy which moves a poit y_0 away from those images. Thus there is a small disk d_0^m centered at y_0 so that the points of d_0 are not in the image of any face of the simplices Δ_α as well.

(5) Now we use the same formula as before to construct a map $h'' : E^m \longrightarrow Y$ so that h'' coincides with h outside of K and h'' coincides with h' on $h^{-1}(D_2^m)$.

(6) We notice that for each simplex Δ_α of the above triangulation the disk d_0^m is either in the image of the interior of $h''(\Delta_\alpha)$ or $d_0^m \cap h'(\Delta_\alpha) = \emptyset$. If the latter holds for all simplices Δ_α, then the map h is homotopic to a map $h_1 : E^m \longrightarrow Y$ so that $h_1(E^m) \subset X$ since we can just blow off the map h out of the "free" point y_0.

(7) Notice that if $d_0^m \cap h''(\Delta_\alpha) \neq \emptyset$, then $(h'')^{-1}(d_0)$ is an ellipsoid since h'' restricted on the simplex Δ_α is linear. Thus the inverse image of the disk d_j^m is a finite number of ellipsoids $d_1^m, \ldots, d_r^m \subset \mathbf{R}^m$.

(8) Now we stretch the disk d_0^m up to the disk D^m: it gives a a map h_1' (homotopic to h'') which sends each ellipsoid d_j^m linearly to the disk D^m.

Lemma 11.2 is proved. $\square$

Conclusion of proof of Theorem 11.1. Lemma 11.2 implies that the homomorphism (31) is epimorphism if $q < n$. (Actually the injectivity of i_* follows directly from the cell-aproximation arguments.)

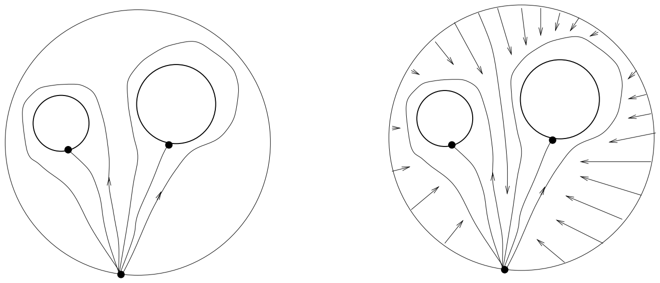

Fig. 11.1.

Now let $g : S^n \longrightarrow X$ be a map representing an element of $\operatorname{Ker} i_* : \pi_n(X, x_0) \longrightarrow \pi_n(Y, x_0)$, i.e. g extends to a map $h : D^{n+1} \longrightarrow Y$. We appply Lemma 11.2 to the map h to construct a map $h_1 : D^{n+1} \longrightarrow Y$ so that $h_1|_{S^n} = g = h_{S^n}$, and so that the map h_1 restricted to the boundary of each disk d_j^{n+1} coincides with the composition

$$\partial(d_j^{n+1}) \xrightarrow{\ell_j} S^n \xrightarrow{f} X$$

where ℓ_j is a linear map. Now we can use the argument simillar to the one we used to prove Theorem 6.5. We choose a path γ_j connecting the base point s_0 with some point $s_j \in \partial(d_j)$ in the same way as we did in Theorem 6.5, see Fig 11.1.

The rest of the proof is left to you. □

Corollary 11.3. *Let $A \subset X$ be a CW-pair, such that $X \setminus A$ does not contain cells of dimension $\leq n$. Then the homomorphism $i_* : \pi_q(A) \longrightarrow \pi_q(X)$ is isomorphism if $q < n$ and is epimorphism if $q = n$. In particular $\pi_n(X^{(n+1)}) \cong \pi_n(X)$, where $X^{(n+1)}$ is the $(n+1)$th skeleton of X.*

11/2. Homotopy groups of a wedge.

Theorem 11.4. *Let X be an n-connected CW-complex, and Y be a k-connected CW-complex. Then*

(1) *$\pi_q(X \vee Y) \cong \pi_q(X) \oplus \pi_q(Y)$ for $q \leq n+k$;*

(2) *for each $q \geq 1$ the group $\pi_q(X \vee Y)$ contains a direct summand isomorphic to $\pi_q(X) \oplus \pi_q(Y)$.*

Proof. By Theorem 5.11 the CW-complexes X and Y are homotopy equivalent to CW-complexes without cells in dimensions in between 0 and $n+1$ (for X) and in between 0 and $k+1$ (for Y). Thus we may assume that X and Y are such complexes. Consider the product $X \times Y$ with the product cell-structure. The wedge $X \vee Y$ is a subcomplex of $X \times Y$. Furthermore the difference $X \times Y \setminus X \vee Y$ has cells of dimension at least $n+k+2$. By Corollary 11.3 $\pi_q(X \vee Y) \cong \pi_q(X \times Y) \cong \pi_q(X) \oplus \pi_q(Y)$. To prove the second statement we notice that the composition

$$\pi_q(X) \oplus \pi_q(Y) \xrightarrow{i_*^X \oplus i_*^Y} \pi_q(X \vee Y) \longrightarrow \pi_q(X \times Y) \xrightarrow{\cong} \pi_q(X) \oplus \pi_q(Y),$$

(where $i^X : X \longrightarrow X \vee Y$, $i^Y : Y \longrightarrow X \vee Y$ are embeddings) is the identity homomorphism. □

Corollary 11.5. *There is an isomorphism $\pi_n(S^n \vee \cdots \vee S^n) \cong \mathbf{Z} \oplus \cdots \oplus \mathbf{Z}$ with generators induced by the embeddings $S^n \longrightarrow S^n \vee \cdots \vee S^n$.*

Exercise 11.1. Let X be an n-connected CW-complex, and Y be a k-connected CW-complex. Prove the isomorphism:

$$\pi_{n+k+1}(X \vee Y) \cong \pi_{n+k+1}(X) \oplus \pi_{n+k+1}(Y) \oplus [\pi_{n+1}(X), \pi_{k+1}(Y)].$$

In particular it follows that $\pi_3(S^2 \vee S^2) \cong \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}$.

11/3. The first nontrivial homotopy group of a CW-complex. Let X be $(n-1)$-connected CW-complex. We know very well that the homotopy groups $\pi_q(X) = 0$ if $q \leq n-1$. Our goal is to describe the group $\pi_n(X)$. Again we can assume that X does not contain cells of dimension in between 0 and n. Then the n-skeleton $X^{(n)}$ is a wedge of spheres: $X^{(n)} = S^n_1 \vee \cdots \vee S^n_s$. Let $g_i : S^n \longrightarrow S^n_1 \vee \cdots \vee S^n_s$ be the embedding of the i-th sphere, and let $r_j : S^n \longrightarrow S^n_1 \vee \cdots \vee S^n_s$ be the attaching maps of the $n+1$ cells $e^{n+1}_1, \ldots, e^{n+1}_\beta$. The maps g_i determine the generators of the group $\pi_n(X^{(n)})$, and let $\rho_j \in \pi_n(X^{(n)})$ be the elements determined by the maps r_j. The following theorem is a straightforward corollary of Theorem 11.1.

Theorem 11.6. *The homotopy group $\pi_n(X)$ is isomorphic to the factor group of the homotopy group $\pi_n(X^{(n)}) \cong \mathbf{Z} \oplus \cdots \oplus \mathbf{Z}$ by the subgroup generated by ρ_j, $j = 1, \ldots, \beta$.*

Remark. Theorem 11.6 is analogous to Theorem 6.5 about the fundamental group. This result gives an impression that we can calculate the first nontrivial homotopy group of any CW-complex without any problems. However, we do not offer here an efficient algorithm to do this calculation. The difficulty shows up when we start with any CW-complex X and construct new CW-complex X' homotopy equivalent to X and without cells in dimensions $\leq n-1$. The process we described in Theorem 5.11 is not really algoriphmic. Thus Theorem 11.6 should not be considered as a computational tool, but rather as a "theoretical device" which allows to prove general facts about homotopy groups.

Exercise 11.2. Let (X, A) be a CW-pair with connected subcomplex A, and such that $X \setminus A$ contains cells of dimension $\geq n$, where $n \geq 3$. Let $\pi = \pi_1(A)$ acting on $\pi_n(X, A)$ by changing a base point. This action gives $\pi_n(X, A)$ a structure of $\mathbf{Z}[\pi]$-module. Prove that the $\mathbf{Z}[\pi]$-module $\pi_n(X, A)$ is generated by the n-cells of $X \setminus A$ with relations corresponding the $(n+1)$-cells of $X \setminus A$.

Exercise 11.3. Let (X, A) be a CW-pair with simply connected subcomplex A, and such that $X \setminus A$ contains cells of dimension $\geq n \geq 2$. Prove that the natural map $j : (X, A) \longrightarrow (X/A, *)$ induces isomomorphism $j_* : \pi_n(X, A) \longrightarrow \pi_n(X/A)$.

11/4. Weak homotopy equivalence. Recall that spaces X and Y are weak homotopy equivalent if there is a natural bijection $\varphi_Z : [Z, X] \longrightarrow [Z, Y]$ for any CW-complex Z (natural with respect to maps $Z \longrightarrow Z'$. We have seen that the fibers of a Serre fiber bundle are weak homotopy equivalent. The definition of weak homotopy equivalence does not offer any hint how to construct the bijection φ_Z. The best possible case is when the bijection φ_Z is induced by a map $f : X \longrightarrow Y$.

A map $f : X \longrightarrow Y$ is a *weak homotopy equivalence* if for any CW-complex Z the induced map $f_* : [Z, X] \longrightarrow [Z, Y]$ is a bijection.

Remark. Clearly if $f : X \longrightarrow Y$ is a weak homotopy equivalence, then X is weak homotopy equivalent to Y. The opposite statement fails. Indeed, let $X = \mathbf{Z} \subset \mathbf{R}$, and $Y = \mathbf{Q} \subset \mathbf{R}$ with induced topology. It is easy to check that $\mathbf{Z} \overset{w}{\sim} \mathbf{Q}$, however there is no continiuos bijection $f : \mathbf{Q} \longrightarrow \mathbf{Z}$. Thus there is no bijection $[pt, \mathbf{Q}] \longrightarrow [pt, \mathbf{Z}]$ induced by any continuous map f. However, if any two (reasonably good spaces, like Hausdorff) X, Y are weak homotopy equivalent, then we will prove soon that there exist a CW-complex W and weak homotopy equivalences $f : W \longrightarrow X$, $g : W \longrightarrow Y$. Also we are about to prove that weak homotopy equivalence coincides with homotopy equivalence on the category of CW-complexes.

Theorem 11.7. *Let $f : X \longrightarrow Y$ be a continuous map. Then the following statements are equivalent.*

(1) *The map $f : X \longrightarrow Y$ is weak homotopy equivalence.*

(2) *The induced homomorphism $f_* : \pi_n(X, x_0) \longrightarrow \pi_n(Y, f(x_0))$ is isomorphism for all n and $x_0 \in X$.*

(3) *Let (W, A) be a CW-pair, and $h : A \longrightarrow X$, $g : W \longrightarrow Y$ be such maps that the following diagram commutes up to homotopy*

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ A \arrow[u, "h"] \arrow[ur, "f \circ h"] \arrow[r, "i"] & W \arrow[u, "g"'] \end{tikzcd} \tag{32}$$

i.e. $f \circ h \sim g|_A = i \circ g$. Then there exists a map $\tilde{h} : W \longrightarrow X$ such that in the diagram

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ A \arrow[u, "h"] \arrow[r, "i"] & W \arrow[u, "g"'] \arrow[ul, "\tilde{h}"'] \end{tikzcd} \tag{33}$$

$\tilde{h}|_A = \tilde{h} \circ i = h$ and $\tilde{h} \circ f \sim g$.

Proof. The implication **(1)** $\Longrightarrow$ **(2)** is obvious.

(3) $\Longrightarrow$ **(1)**. Let $(W, A) = (Z, \emptyset)$. Then we have that for any map $g : Z \longrightarrow Y$ there exists a map $\tilde{h} : Z \longrightarrow X$ so that the triangle

$$\begin{tikzcd} X \arrow[rr, "f"] & & Y \\ & & W \arrow[u, "g"] \arrow[ull, "\tilde{h}"] \end{tikzcd} \tag{34}$$

commutes up to homotopy. It implies that the map $f_* : [Z, X] \longrightarrow [Z, Y]$ is epimorphism. To prove that f_* is injective, consider the pair $(W, A) = (Z \times I, Z \times \{0\} \cup Z \times \{1\})$. Let $h_0 : Z \longrightarrow X$, $h_1 : Z \longrightarrow X$ be two maps so that the compositions $h_0 \circ f : Z \longrightarrow Y$, $h_1 \circ f : Z \longrightarrow Y$ are homotopic. Let $G : Z \times I \longrightarrow Y$ be a homotopy between the maps $h_0 \circ f$and $h_1 \circ f$. The statement (3) implies that there exists a homotopy $\tilde{H} : Z \times I \longrightarrow X$ so that the diagram

$$\begin{tikzcd} X \arrow[rr, "f"] & & Y \\ Z \times \{0\} \cup Z \times \{1\} \arrow[u, "h_0 \cup h_1"] \arrow[rr, "i"] & & Z \times I \arrow[u, "G"] \arrow[ull, "\tilde{H}"] \end{tikzcd} \tag{35}$$

commutes up to homotopy. In particular, it means that the maps $h_0, h_1 : Z \longrightarrow X$ were homotopic in the first place.

(2) $\Longrightarrow$ (3). Let $f : X \longrightarrow Y$ satisfy (2). We assume that $W = A \cup_\alpha D^{n+1}$, where $\alpha : S^n \longrightarrow A$ is the attaching map. Let $h : A \longrightarrow X$, $g : A \cup_\alpha D^{n+1} \longrightarrow Y$ be such maps that $f \circ h \sim g|_A$. Consider the diagram:

$$\begin{tikzcd} & X \arrow[r, "f"] & Y \\ S^n \arrow[r, "\alpha"] & A \arrow[u, "h"] \arrow[r, "i"] & A \cup_\alpha D^{n+1} \arrow[u, "g"] \end{tikzcd} \tag{36}$$

The composition $i \circ \alpha : S^n \longrightarrow A \cup_\alpha D^{n+1}$ is null-homotopic by construction, hence the map $g \circ i \circ \alpha : S^n \longrightarrow Y$ is null-homotopic as well. Thus $[g \circ i \circ \alpha] = 0$ in the group $\pi_n(Y)$. Notice that the map $f \circ h \circ \alpha : S^n \longrightarrow Y$ gives the same homotopy class as the map $g \circ i \circ \alpha$ since the diagram (36) is commutative up to homotopy by conditions of the statement (3). In particular, we have that $f_*([h \circ \alpha]) = [g \circ i \circ \alpha] = 0$. Hence $[h \circ \alpha] = 0$ in the group $\pi_n(X)$ since $f_* : \pi_n(X) \longrightarrow \pi_n(Y)$ is isomorphism. It implies that there exists a map $\beta : D^{n+1} \longrightarrow X$ extending the map $h \circ \alpha : S^n \longrightarrow X$. We have the following diagram:

$$\begin{tikzcd}
D^{n+1} \arrow[r, "\beta"] & X \arrow[rr, "f"] & & Y \\
S^n \arrow[u] \arrow[r, "\alpha"] & A \arrow[u, "h"] \arrow[rr, "i"] & & A \cup_\alpha D^{n+1} \arrow[u, "g"]
\end{tikzcd} \tag{37}$$

where the left square is commutative, and the right one is commutative up to homotopy. The left square gives us a map $\tilde{h}' : A \cup_\alpha D^{n+1} \longrightarrow X$, so that $f \circ \tilde{h}'|_A = f \circ h \sim g|_A$. We choose a homotopy $H : A \times I \longrightarrow Y$ so that

$$H|_{A\times\{0\}} = g|_A, \quad H|_{A\times\{1\}} = f \circ \tilde{h}'|_A = f \circ h.$$

Consider the cylinder $D^{n+1} \times I$ and its boundary $S^{n+1} = \partial(D^{n+1} \times I)$. No we construct a map $\gamma : S^{n+1} \longrightarrow Y$ as it is shown below, see Fig. 11.2.

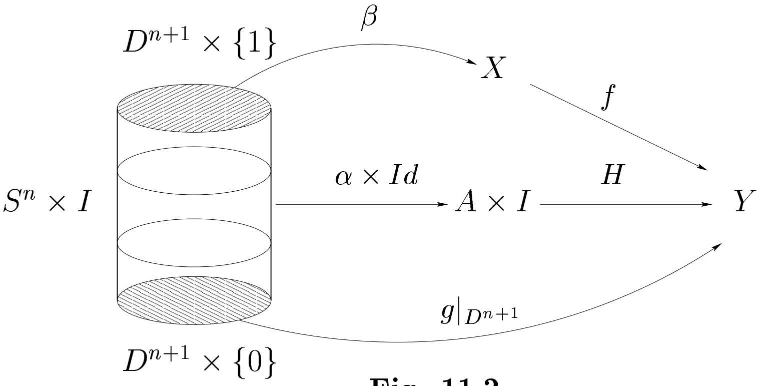

Fig. 11.2.

If the map $\gamma : S^{n+1} \longrightarrow Y$ is homotopic to zero, then we are done since we can extend γ to the interior of the cylinder $D^{n+1} \times I$, and it will give us a homotopy between $f \circ \tilde{h}'$ and g. However, there is no any reason to assume that $\gamma \sim 0$. To correct the construction we make the following observation.

Lemma 11.8. *Let $\xi \in \pi_q(Y, y_0)$ be any element, and $\beta : D^q \longrightarrow Y$, such that $\beta(s_0) = y_0$, where $x_0 \in S^{q-1} = \partial(D^q)$. Then there exists a map $\beta' : D^q \longrightarrow Y$ such that*

(a) $\beta'|_{S^{q-1}} = \beta|_{S^{q-1}}$;

(b) *the map $\beta \cup \beta' : S^q \longrightarrow Y$ represents the element $\xi \in \pi_q(Y, y_0)$.*

Proof. Let $\varphi : S^q \longrightarrow Y$ be any map representing the element $\xi \in \pi_q(Y, y_0)$. We consider the sphere $S^q_1 = D^q_N \cup_{S^{q-1}} D^q_S$. Let $p : S^q_1 \longrightarrow S^q$ be a map which takes the southern hemisphere D^q_S to the base point $s_0 \in S^q$. Clearly the composition

$$S^q_1 \xrightarrow{p} S^q \xrightarrow{\varphi} Y$$

represents the same element $\xi \in \pi_q(Y, y_0)$. Fig. 11.3 below is supposed to hint how to construct new map $\varphi' : S^q_1 \longrightarrow Y$ so that $\varphi' = \beta \cup \beta'$ represents the element $\xi \in \pi_q(Y, y_0)$. The details are left to you. $\square$

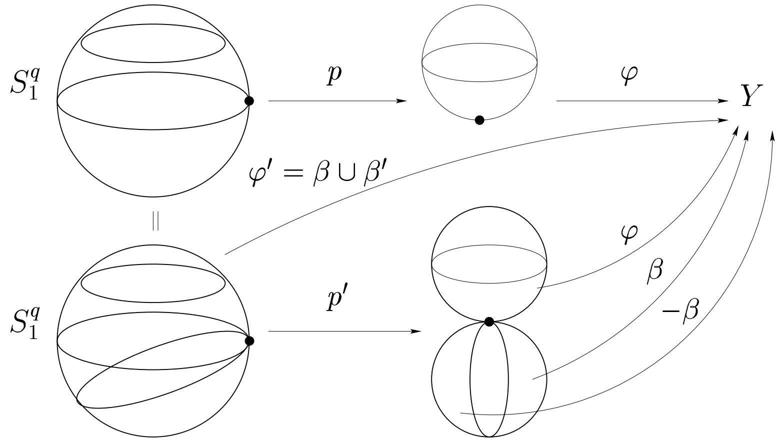

Fig. 11.3.

Now we complete a proof of the implication **(2)** $\Longrightarrow$ **(3)**. The above map $\gamma : S^{n+1} \longrightarrow Y$ gives an element $\gamma \in \pi_{n+1}(Y)$. Then we consider the element $-\gamma \in \pi_{n+1}(Y)$ and use Lemma 11.8 to find a map $\beta' : D^{n+1} \longrightarrow Y$, such that $\beta'|_{S^n} = \beta|_{S^n}$ and the map $\beta' \cup \beta$ represents the element $-\gamma$. We put together the maps we constructed to get new map $\gamma' : S^{n+1} \longrightarrow Y$ which homotopic to zero, see Fig. 11.4. Since $\gamma' \sim 0$ we are done in the case when $W = A \cup_\alpha D^{n+1}$. The general case follows then by induction: the n-th step is to do the above construction for all $(n+1)$-cells of the difference $W \setminus A$.

$\square$

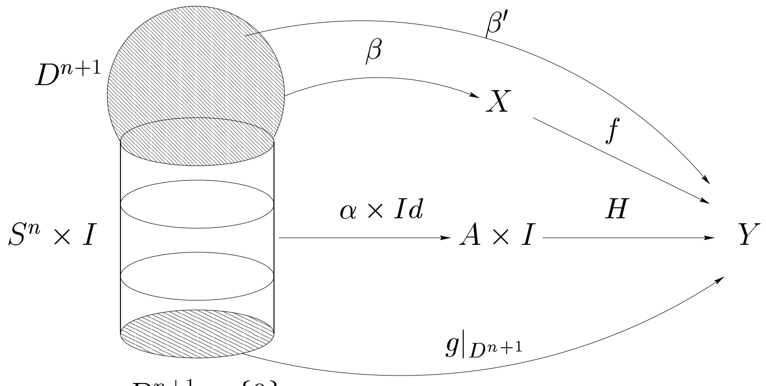

Fig. 11.4. The map $\gamma' : S^{n+1} \longrightarrow Y$.

Corollary 11.9. (Whitehead Theorem) *Let X, Y be CW-complexes. Then if a map $f_* : X \longrightarrow Y$ induces isomorphism*

$$f_* : \pi_n(X, x_0) \longrightarrow \pi_n(Y, f(x_0))$$

for all $n \geq 0$ and $x_0 \in X$, then f is a homotopy equivalence.

Exercise 11.4. Prove Corollary 11.9.

Exercise 11.5. Prove that the homotopy groups of the spaces $S^2 \times \mathbf{CP}^\infty$ and S^3 are isomorphic, and that they are not homotopy equivalent.

Exercise 11.6. Let $k \neq n$. Prove that the homotopy groups of the spaces $\mathbf{RP}^n \times S^k$ and $S^k \times \mathbf{RP}^k$ are isomorphic, and that they are not homotopy equivalent.

11/5. Cellular approximation of topological spaces. Let X be an arbitrary Hausdorff space. There is a natural question: is there a natural cellular approximation of the space X? This is the answer:

Theorem 11.10. *Let X be a Hausdorff topological space. There exists a CW-complex K and a weak homotopy equivalence $f : K \longrightarrow X$. The CW-complex K is unique up to homotopy equivalence.*

Proof. We assume that X is a path-connected space. We construct a chain of CW-complexes

$$K_0 \subset K_1 \subset K_2 \subset \cdots \subset K_{n-1} \subset K_n \subset \cdots$$

and maps $f_j : K_j \longrightarrow X$ so that

(1) $f_j|_{K_{j-1}} = f_{j-1} : K_{j-1} \longrightarrow X$,

(2) $(f_j)_* : \pi_q(K_j) \longrightarrow \pi_q(X)$ is an isomorphism for all $q \leq j$.

Let $K_0 = \{x_0\}$, and $f_0 : K_0 \longrightarrow X$ be a choice of a base point. Assume that we have constructed the maps $f_j : K_j \longrightarrow X$ for all $j \leq n-1$ satisfying the above conditions. Consider the group $\pi_n(X, x_0)$ and choose generators g_α of $\pi_n(X, x_0)$ as a $\pi_1(X, x_0)$-module and representing maps $g_\alpha : S^n_\alpha \longrightarrow X$. Let

$$K'_n = K_{n-1} \vee \left(\bigvee_\alpha S^n_\alpha \right)$$

We define $f'_n : K'_n \longrightarrow X$ to be f_{n-1} on K_{n-1} and to be $\bigvee_\alpha g_\alpha$ on $\bigvee_\alpha S^n_\alpha$. The induction hypothesis and Theorem 11.1 implies that f'_n induces isomorphism

$$\pi_q(K'_n) \xrightarrow{(f'_n)_*} \pi_q(X)$$

for $q \leq n-1$. The homomorphism $(f'_n)_* : \pi_n(K'_n) \longrightarrow \pi_n(X)$ is epimorphism since all generators g_α are in the image. However it may not monomorphic. We choose generators h_β of the kernel Ker $(f'_n)_* \subset \pi_n(K'_n)$ (as $\pi_1(K'_n)$-module) and representatives $h_\beta : S^n_\beta \longrightarrow K'_n$. Now we attach the cells e^{n+1}_β using the maps h_β as attaching maps.

Let K_n be the resulting CW-complex. The map $f'_n : K'_n \longrightarrow X$ may be extended to $f_n : K_n \longrightarrow X$ since each composition

$$S^n_\beta \xrightarrow{h_\beta} K'_n \xrightarrow{f'_n} X$$

is homotopic to zero. Thus $f'_n : K'_n \longrightarrow X$ may be extended to all cells e^{n+1}_β we attached. Theorem 11.1 implies that $(f_n)_* : \pi_q(K_n) \longrightarrow \pi_q(X)$ is an isomorphism for $q \leq n-1$ and also that $\pi_n(K_n) \cong \pi_n(K'_n)/\text{Ker } (f'_n)_*$. Thus $(f_n)_* : \pi_n(K_n) \longrightarrow \pi_n(X)$ is an isomorphism as well.

Exercise 11.7. Prove that the CW-complex K we constructed is unique up to homotopy.

This concludes the proof of Theorem 11.10. $\square$

Exercise 11.8. Let X, Y be two weak homotopy equivalent spaces. Prove that there exist a CW-complex K and maps $f : K \longrightarrow X$, $g : K \longrightarrow Y$ which weak homotopy equivalences.

11/6. Eilenberg-McLane spaces. Let n be a positive integer and π be a group (abelian) if $n \geq 2$. A space X is called an *Eilenberg-McLane space of the type $K(\pi, n)$* if

$$\pi_q(X) = \begin{cases} \pi & \text{if } q = n \\ 0 & \text{else.} \end{cases}$$

Theorem 11.11. *Let n be a positive integer and π be a group (abelian) if $n \geq 2$. Then the Eilenberg-McLane space of the type $K(\pi, n)$ exists and unique up to weak homotopy equivalence.*

Remark. If a space X is an Eilenberg-McLane space of the type $K(\pi, n)$, we will say that X is $K(\pi, n)$.

Proof. Let $\{g_\alpha\}$ be generators of the group π, and $\{r_\beta\}$ be relations (if $n > 1$ we mean relations in the abelian group). Let $X_n = \bigvee_\alpha S^n_\alpha$. Then $\pi_q(X_n) = 0$ if $q \leq n-1$ and $\pi_n(X_n) = \bigoplus_\alpha \mathbf{Z}$ (or free group with generators $\{g_\alpha\}$ if $n = 1$). Each relation r_β defines a unique element $r_\beta \in \pi_n(X_n)$. We choose maps $r_\beta : S^n_\beta \longrightarrow X_n$ representing the above relations and attach cells e^{n+1}_β using r_β as the attaching maps. Let X_{n+1} be the resulting space. Theorem 11.1 implies that $\pi_q(X_{n+1}) = 0$ if $q \leq n-1$ and $\pi_n(X_{n+1}) = \pi$. Then we choose generators of $\pi_{n+1}(X_{n+1})$ and attach $(n+2)$-cells using maps representing these generators as the attaching maps. Let X_{n+2} be the resulting CW-complex. Again Theorem 11.1 implies that $\pi_q(X_{n+2}) = 0$ if $q \leq n-1$ or $q = n+1$ and $\pi_n(X_{n+2}) = \pi$. Now we proceed by induction killing the homotopy group $\pi_{n+2}(X_{n+2})$ and so on.

Exercise 11.9. Prove that an Eilenberg-McLane space of the type $K(\pi, n)$ is unique up to weak homotopy equivalence, i.e. if K_1, K_2 are two Eilenberg-McLane spaces of the type $K(\pi, n)$ then there exist weak homotopy equivalences $f_1 : X \longrightarrow K_1$ and $f_2 : X \longrightarrow K_2$, where X is the space we just constructed.

This concludes the proof of Theorem 11.11 $\square$

Remark. The above construction is not algorithmic at all: we have no idea what groups $\pi_{n+k}(X_{n+k})$ we are going to get in this process.

Examples. (1) $K(\mathbf{Z}, 1) = S^1$.

(2) $K(\mathbf{Z}/2, 1) = \mathbf{RP}^\infty$.

(3) $K(\mathbf{Z}, 2) = \mathbf{CP}^\infty$.

(4) Let $L^{2n-1}(\mathbf{Z}/m)$ be the lens space we defined at the end of Section 7, and let

$$L^\infty(\mathbf{Z}/m) = \lim_{n \longrightarrow \infty} L^{2n-1}(\mathbf{Z}/m).$$

Then $L^\infty(\mathbf{Z}/m) = K(\mathbf{Z}/m, 1)$.

Exercise 11.10. Construct the space $K(\pi, 1)$, where π is a finitely generated abelian group.

Exercise 11.11. Let $\mathcal{B}_n^{col} = \{(z_1, \ldots, z_n) \in \mathbf{C}^n \mid z_j \neq z_k \text{ if } j \neq k\}$. Prove that $\mathcal{B}_n^{col} = K(B_n^{col}, 1)$, where B_n^{col} is the *pure braid* group on n strings. The symmetric group Σ_n acts freely on the space $\mathcal{B}_n^{col}$. Let $\mathcal{B}_n = \mathcal{B}_n^{col}/\Sigma_n$ be the orbit space. Prove that $\mathcal{B}_n = K(B_n, 1)$, where B_n is the braid group on n strings.

Exercise 11.12. Let $X = K(\pi, n)$. Prove that $\Omega X = K(\pi, n-1)$.

11/7. Killing the homotopy groups. There are two constructions we discuss here. The first construction was we used implicitly several times. Let X be a space, then for each n there is a space X_n and a map $f_n : X \longrightarrow X_n$, such that

(1) $\pi_q(X_n) = \begin{cases} \pi_q(X) & \text{if } q \leq n \\ 0 & \text{else} \end{cases}$

(2) $(f_n)_* : \pi_q(X) \longrightarrow \pi_q(X_n)$ is isomorphism if $q \leq n$.

We know how to construct X_n: start with generators $\{g_\alpha\}$ of the group $\pi_{n+1}(X)$, then attach the cells e_α^{n+2} using the maps $g_\alpha : S_\alpha^{n+1} \longrightarrow X$. Then the resulting space Y_{n+1} has the homotopy groups $\pi_{n+1}(Y_{n+1}) = 0$ and $\pi_q(Y_{n+1}) = \pi_q(X)$ if $q \leq n$. Then one kills in the same way the homotopy group $\pi_{n+2}(Y_{n+1})$ to construct the space Y_{n+2} with $\pi_{n+1}(Y_{n+2}) = 0$, $\pi_{n+2}(Y_{n+2}) = 0$, and $\pi_q(Y_{n+2}) = \pi_q(X)$ if $q \leq n$, and so on. The limiting space is X_n with the above properties. The map $f_n : X \longrightarrow X_n$ is the embedding.

Let X be $(n-1)$-connected. Then $X_n = K(\pi_n(X), n)$. This construction may be organized so that there is a commutative diagram

$$
\begin{tikzcd}
 & \vdots \arrow[d, "i_{n+k+1}"'] \\
 & X_{n+k} \arrow[d, "i_{n+k}"'] \\
 & \vdots \arrow[d, "i_{n+2}"'] \\
X \arrow[uur, "f_{n+k}"] \arrow[r, "f_{n+1}"] \arrow[dr, "f_n"] & X_{n+1} \arrow[d, "i_{n+1}"'] \\
 & K(\pi_n, n)
\end{tikzcd}
\tag{38}
$$

where $\pi_q = \pi_q(X)$. The maps $i_q : X_q \longrightarrow X_{q-1}$ in the diagram (38) are homotopy equivalent to Serre fiber bundles, so that the diagram (38) becomes commutative up to homotopy. Let F_q be the fiber of the Serre bundle $X_q \xrightarrow{i_q} X_{q-1}$. The exact sequence in homotopy

$$\cdots \longrightarrow \pi_j(F_q) \longrightarrow \pi_j(X_q) \longrightarrow \pi_j(X_{q-1}) \longrightarrow \pi_{j-1}(F_q) \longrightarrow \cdots$$

for the Serre bundle $X_q \xrightarrow{i_q} X_{q-1}$ immediately implies that $F_q = K(\pi_q, q) = \Omega K(\pi_q, q+1)$. It turns out that any Serre bundle with a fiber $\Omega K(\pi_q, q+1)$ may be induced by some map $k_q : X_{q-1} \longrightarrow K(\pi_q, q+1)$ so that the following diagram:

$$
\begin{tikzcd}
X_q \arrow[r, "\hat{k}_q"] \arrow[d, "\Omega K(\pi_q,q+1)"', "i_q"] & E(K(\pi_q, q+1)) \arrow[d, "\Omega K(\pi_q,q+1)"', "p"] \\
X_{q-1} \arrow[r, "k_q"] & K(\pi_q, q+1)
\end{tikzcd}
\tag{39}
$$

commutes. The maps $k_q : X_{q-1} \longrightarrow K(\pi_{q+1}, q+2)$ are known as the *Postnikov invariants* of the space X. The diagram

$$
\begin{tikzcd}
 & \vdots \arrow[d, "i_{n+k+1}"'] & \\
 & X_{n+k} \arrow[r, "k_{n+k+1}"] \arrow[d, "i_{n+k}"', "{K(\pi_{n+k},n+k)}"] & K(\pi_{n+k}, n+k+1) \\
 & \vdots \arrow[d, "i_{n+2}"', "{K(\pi_{n+2},n+2)}"] & \\
X \arrow[ruu, "f_{n+k}"] \arrow[r, "f_{n+1}"] \arrow[rd, "f_n"'] & X_{n+1} \arrow[r, "k_{n+2}"] \arrow[d, "i_{n+1}"', "{K(\pi_{n+1},n+1)}"] & K(\pi_{n+2}, n+3) \\
 & K(\pi_n, n) \arrow[r, "k_{n+1}"] & K(\pi_{n+1}, n+2)
\end{tikzcd}
\tag{40}
$$

is called the *Postnikov tower* of the space X. The Postnikov tower exists and unique up to homotopy under some restrictions on X. For instance, it exists when X is a simply-connected CW-complex. The existence of diagram (40) shows that the Eilenberg-McLane spaces are the "elmentary building blocks" for any simply connected space X. The Postnikov tower also shows that there are many spaces with the same homotopy groups, while these spaces are not homotopy equivalent. Again, this construction does not provide an algorithm to compute the homotopy groups, however it leads to some computational procedure called the Adams spectral sequence. We are not ready even to discuss this, and we shall return to the above constructions later on.

There is the second way to kill homotopy groups. Let X be $(n-1)$-connected as above. The map $f_n : X \longrightarrow X_n = K(\pi_n, n)$ may be turned into Serre fiber bundle. Let $X|_n$ be its fiber, and $j_n : X|_n \longrightarrow X$ be the inclusion map. The exact sequence in homotopy for the fiber bundle $X \xrightarrow{X|_n} K(\pi_n, n)$ implies that the map $j_n : X|_n \longrightarrow X$ induces isomorphism $\pi_q(X|_n) \cong \pi_q(X)$ if $q \geq n+1$, and also that $\pi_q(X|_n) = 0$ if $q \leq n$. One can iterate this construction to build the space $X|_{n+k}$ and the map $j_{n+k} : X|_{n+k} \longrightarrow X$ so that the induced homomorphism $\pi_q(X|_{n+k}) \longrightarrow \pi_q(X)$ is isomorphism if $q \geq n+k$ and $\pi_q(X|_{n+k}) = 0$ if $q \leq n+k-1$.

Exercise 11.13. Let $X = S^2$. Prove that $X|_3 = S^3$.

Exercise 11.14. Let $X = \mathbf{CP}^n$. Prove that $X|_3 = X|_{2n+1} = S^{2n+1}$.