10 Suspension Theorem and Whitehead product

10/1. The Freudenthal Theorem. Let X be a space with a base point x_0. We construct the homomorphism

$$\Sigma : \pi_q(X) \longrightarrow \pi_{q+1}(\Sigma X) \tag{28}$$

as follows. Let $\alpha \in \pi_q(X)$, and a map $f : S^q \longrightarrow X$ be a representative of α. The map

$$\Sigma f : \Sigma S^q = S^{q+1} \longrightarrow \Sigma X$$

defined by the formula $\Sigma f(y,t) = (f(y),t) \in \Sigma X$ gives a representative for $\Sigma(a) \in \pi_{q+1}(\Sigma X)$. It is not hard to check that

1. $f \sim g$ implies that $\Sigma f \sim \Sigma g$;
2. $\Sigma f + \Sigma g \sim \Sigma(f+g)$.

The homomorphism Σ is the *suspension homomorphism.*

Theorem 10.1. (Freudenthal Theorem) *The suspension homomorphism*

$$\Sigma : \pi_q(S^n) \longrightarrow \pi_{q+1}(S^{n+1})$$

is isomorphism for $q \leq 2n-1$ and epimorphism for $q = 2n-1$.

Remark. This is the "easy part" of the suspension Theorem. The "hard part" will be discussed later, see Theorem 10.7. The generalization of the Freudenthal Theorem goes as follows (we do not have enough tools to prove this statement at the moment):

Let X be an $(n-1)$-connected CW-complex (it implies that $\pi_i(X) = 0$ for $i < n$). Then the suspension homomorphism $\Sigma : \pi_q(X) \longrightarrow \pi_{q+1}(\Sigma X)$ is isomorphism for $q \leq 2n-1$ and epimorphism for $q = 2n-1$.

Proof that Σ is surjective. Let $f : S^{q+1} \longrightarrow S^{n+1}$ be an arbitrary map. We have to prove that we can perform a homotopy of this map f to a map Σh, where $h : S^q \longrightarrow S^n$. We will assume that $n > 0$, and $q \geq n$. In particular, the group $\pi_{q+1}(S^{n+1})$ is abelian, and $\pi_1(S^{n+1}) = 0$, so we can forget about particular choice of the base point.

Let a, b be the north and south poles of the sphere S^{n+1}. We identify the sphere S^{q+1} with the space $\mathbf{R}^n \cup \infty$, moreover, we choose this identification in such way that $f^{-1}(a), f^{-1}(b)$ do not contain the infinity.

First we should take care about the sets $f^{-1}(a)$ and $f^{-1}(b)$. We do not have any control over the map f, the only property we can use is that f continuous. However clearly $f^{-1}(a)$ and $f^{-1}(b)$ are compact sets in $\mathbf{R}^{q+1}$. Recall that if K is a finite simplicial complex in $\mathbf{R}^{q+1}$, then $\dim K$ is a maximal dimension of the simplices of K.

Lemma 10.2. *There exists a map $f_1 : S^{q+1} \longrightarrow S^{n+1}$ homotopic to f (and, actually as close to f as one may wish), such that $f_1^{-1}(a)$, $f_1^{-1}(b)$ are finite simplicial complexes in $\mathbf{R}^{q+1}$ of dimension less or equal to $q-n$.*

Proof. Here we apply the same constuction as we used in "free point Lemma".

Let me describe only main steps:

1. Find five small disks $a \in D_1^{(a)} \subset \ldots \subset D_5^{(a)}$ centered at a, and five small disks $b \in D_1^(b) \subset \ldots \subset D_5^{(b)}$ centered at b.[‡‡]

2. Find a huge simplex Δ in R^{q+1} containing $f^{-1}(D_5^{(a)}) \cup f^{-1}(D_5^{(b)})$.

3. Find fine enough barycentric triangulation $\{\Delta_\alpha\}$ of

$$\Delta = \bigcup_\alpha \Delta_\alpha,$$

such that for any simplex Δ_α satisfies the following condition: if $f(\Delta_\alpha) \cap D_i^{(j)} \neq \emptyset$, then $f(\Delta_\alpha) \subset D_{i+1}^{(j)}$ (here $i = 1, 2, 3, 4$, $j = a, b$).

4. Consider the simplicial complex

$$K = \bigcup_{f(\Delta_\alpha) \cap (D_4^{(a)} \cup D_4^{(b)}) \neq \emptyset} \Delta_\alpha.$$

5. Construct a map $f' : K \longrightarrow S^{n+1}$ which coincides with f on each vertex of K and extended linearly to all simplices.

6. "Glue" the maps f' and f to get a map f_1 which coincides with f' on $f^{-1}(D_2^{(a)} \cup D_2^{(b)})$ and with f outside of $f^{-1}(D_3^{(a)} \cup D_3^{(b)})$.

This gives us a map f_1 (which is homotopic to f) with the following property:

The inverse image $f_1(D_1^{(a)})$ (and $f_1(D_1^{(b)})$) is covered by finite number of $q+1$-simplices Δ_α, such that $f_1|_{\Delta_i}$ is a linear map.

Assume for a moment that there is such a simplex Δ_α that the simplex $f_1(\Delta_\alpha) \subset S^{n+1}$ has dimesion less than $n+1$, and $a \in f_1(\Delta_\alpha)$. Then we can change a little bit the map f_1 (it is enough to change a value of f_1 at one vertex!) to get a map f_2 such that $a \notin f_1(\Delta_\alpha)$.

This observation allows us to assume that if $a \in f_2(\Delta_\alpha)$, then the simplex Δ_α has dimension $q+1$. Since the restriction $f_2|_{\Delta_\alpha}$ is a linear map of maximal rank, than

[‡‡] Once again. we identify a neighborhood of a (respectively of b) with an open subset of $\mathbf{R}^{n+1}$ via, say, the stereographic projection.

$f_2^{-1}(a) = K$ consists of simplices of dimension at most $q+1-(n+1) = q-n$. This proves that $K = f_2^{-1}(a)$, $L = f_2^{-1}(b)$ are simlicial complexes of dimension less or equal to $n-q$. $\square$

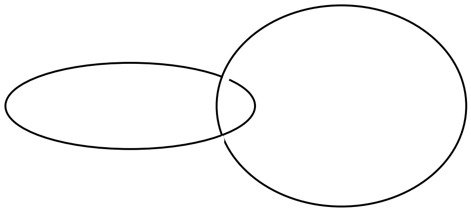

Fig. 10.1.

Now we have to introduce a couple of definititions.

A homotopy $F : \mathbf{R}^p \times I \longrightarrow \mathbf{R}^p$ is an **isotopy** if $F_t : \mathbf{R}^p \longrightarrow \mathbf{R}^p$ is an homeomorphism for eact $t \in I$. A hyperplane $\Pi \subset \mathbf{R}^p$ divides $\mathbf{R}^p \setminus \Pi$ into two half-spaces: $\mathbf{R}^p_-$ and $\mathbf{R}^p_+$.

We say that two simlicial complexes $K, L \subset \mathbf{R}^p$ **are not linked** if there exist a hyperplane $\Pi \subset \mathbf{R}^p$, and an isotopy $F_t : \mathbf{R}^p \longrightarrow \mathbf{R}^p$, so that $F_0 = Id$, and the sets $F_1(K)$ and $F_1(L)$ are separated by the hyperplane Π. Fig. 10.1 shows an example of two linked circles.

Lemma 10.3. *Let $K, L \subset \mathbf{R}^p$ be two finite simplicial complexes fo dimensions k, l respectively. Let $k+l+1 < p$. Then the simplicial complexes K and L are not linked.*

Proof. First let $\Pi \subset \mathbf{R}^p$ be a hyperplane such that $K \cap \Pi = \emptyset$, and $K \cap \Pi = \emptyset$. If K and L are in the different half-spaces, then we are done. Let K and L be in $\mathbf{R}^p_+$. We want to produce an isotopy $F_t : \mathbf{R}^p \longrightarrow \mathbf{R}^p$ such that $F_0 = Id_{\mathbf{R}^p}$ and $F_1(K)$ and $F_1(L)$ are separated by the hyperplane Π. We need the following statement.

Claim 10.1. *There exists a point $x_0 \in \mathbf{R}^p_-$ such that any line going through x_0 does not intersect both K and L.*

Proof of Claim 10.1. Let $W_1, \ldots, W_\nu \subset \mathbf{R}^p$ be planes (of minimal dimensions) containing the simplices $\Delta_1, \ldots, \Delta_\mu$ of the simplicial complex K, and let $U_1, \ldots, U_\mu \subset \mathbf{R}^p$ be the corresponding planes containing the simplices of L. Notice that $\dim W_i \leq k$ and $\dim U_j \leq l$, $i = 1, \ldots, \nu$, $j = 1, \ldots, \mu$. Let Π_{ij} be a minimal plane containing W_i and U_j. Notice that the maximal dimension of Π_{ij} is $k+l+1$. Indeed, let $w \in W_i$, $u \in U_j$ be any points. Then a basis of W_i, a basis of U_j, and the vector $w-u$ generate Π_{ij}, see Fig. 10.2. Since $k+l+1 < p$, there exists a point x_0 of $\mathbf{R}^p_-$, such that $x_0 \notin \bigcup P_{ij}$. $\square$

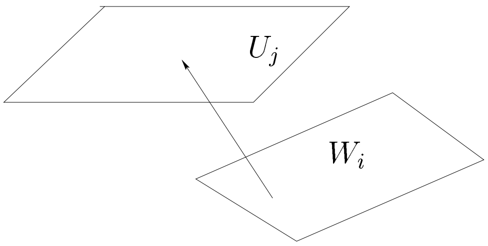

Fig. 10.2: The plane Π_{ij}.

Now we continue the proof of Lemma 10.3. The isotopy F_t may be costructed as follows. Consider the space of all lines going through the point $x_0 \in \mathbf{R}^p$. This is the projective space $\mathbf{RP}^{p-1}$. Choose a continuous nonnegative function $\varphi : \mathbf{RP}^{p-1} \longrightarrow \mathbf{R}$ such that $\varphi(\lambda) = 0$ if $\lambda \cap L \neq \emptyset$, and $\varphi(\lambda) = v_0 > 0$ if $\lambda \cap L \neq \emptyset$.

Now the isotopy $F_t : \mathbf{R}^p \longrightarrow \mathbf{R}^p$ moves a point $x \in \mathbf{R}^p$ along the line λ (connecting x and x_0) toward x_0 with the velocity $\varphi(\lambda)$, where φ is as above.

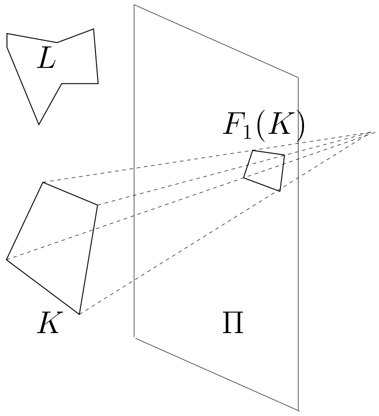

Fig. 10.3: The isotopy F_t.

Clearly at some moment the image of K will be inside of $\mathbf{R}^p_-$, see Fig. 10.3. □

We complete the proof that Σ is surjective. We use Lemma 10.2 and Lemma 10.3 to construct a map $f_2 : S^{q+1} \longrightarrow S^{n+1}$ homotopic to f such that the inverse images $f_2^{-1}(a)$ and $f_2^{-1}(b)$ are located in the "northern" and respectively "southern" parts of the sphere S^{q+1} (we use here the following obvious estimation: $(q-n)+(q-n)+1 = 2q-(2n-1) \leq 2q-q < q+1$ provided $q \leq 2n-1$). Furthermore, there are two "ice caps", disks A and B centered at the poles a and b respectively, which do not touch the equator of S^{n+1}, and such that $f_2^{-1}(A)$ and $f_2^{-1}(B)$ do not touch the equator of S^{q+1} as well, see the picture below:

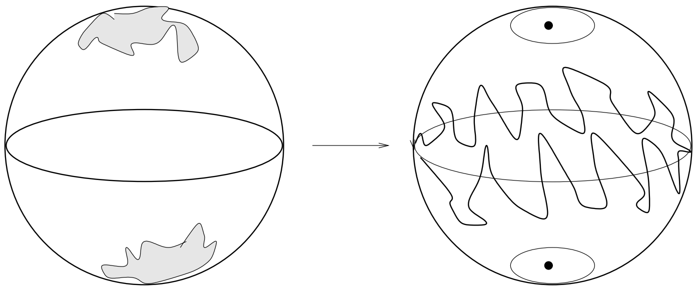

Fig. 10.4.

Now we make a homotopy $S^{n+1} \longrightarrow S^{n+1}$ which sretches A and B to the north and the south hemispheres respectively, and squeezes the remainder onto the equator sphere $S^n \subset S^{n+1}$. By composing this map with f_2, we obtain a map f_3 which sends the equator of S^{q+1} to the equator of S^{n+1}, and the north and south poles of S^{q+1} sends to the north and south poles of S^{n+1}. Now we look at the spheres S^{q+1} and S^{n+1} from the North:

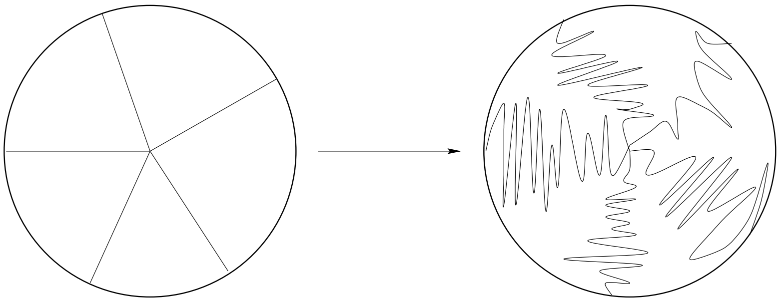

Fig. 10.5.

Here we see only the northern hemispheres. We have here all possible meridians of S^{q+1} and their images under the map f_3. The further homotopy which finally turns the map f_4 into the suspension map may be constructed as follows:

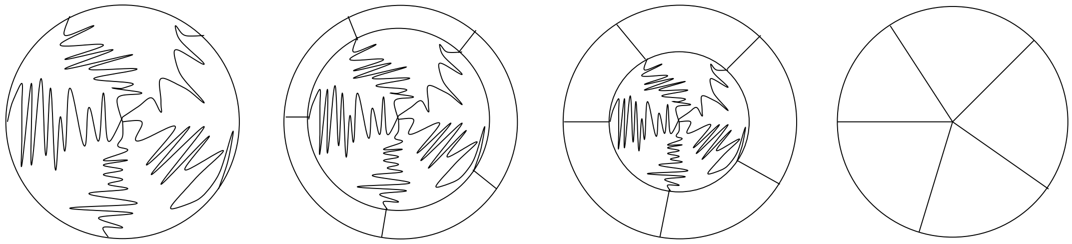

Fig. 10.6.

This construction due to J. Alexander. □

Exercise 10.1. Describe the last homotopy in more detail.

Proof that Σ is injective for $q < 2n-1$. Let $f_0 = \Sigma h_0 : S^{q+1} \longrightarrow S^{n+1}$, and $f_1 = \Sigma h_1 : S^{q+1} \longrightarrow S^{n+1}$, and $f_0 \sim f_1$. We should show that $h_0 \sim h_1$.

We consider the homotopy $F : S^{q+1} \times I \longrightarrow S^{n+1}$. Again, we examine $F^{-1}(a)$ and $F^{-1}(b)$, and by Lemma 10.2 (to be precise, its generalization) we conclude that F is homotopic to F_1 such that $F_1^{-1}(a) = K$ and $F_1^{-1}(b) = L$ are finite simplicial complexes of dimension at most $q+1-n$. The condition $q < 2n-1$ and Lemma 10.3 imply that the simplicial complexes K and L may be separated. The rest of the arguments are very similar to those applied in the above proof. □

Exercise 10.2. Prove the injectivity of Σ in detail.

10/2. First applications.

Theorem 10.4. (Hopf) *$\pi_n(S^n) \cong \mathbf{Z}$ for each $n \geq 1$.*

Exercise 10.3. Prove Theorem 10.4.

Exercise 10.4. Prove that $\pi_3(S^2) \cong \mathbf{Z}$, and the Hopf map $S^3 \longrightarrow S^2$ is a representative of the generator of $\pi_3(S^2)$.

Corollary 10.5. *The sphere S^n is not contractible.*

10/3. A degree of a map $S^n \longrightarrow S^n$. A map $f : S^n \longrightarrow S^n$ gives a representative of some element $\alpha \in \pi_n(S^n) \cong \mathbf{Z}$. We choose the generator ι_n of $\pi_n(S^n)$ as a homotopy class of the identity map. Thus $[f] = \alpha = \lambda\iota_n$. The integer $\lambda \in \mathbf{Z}$ is called a degree of the map f. The notation is $\deg f$.

Exercise 10.5. Prove the following properties of the degree:

(a) Two maps $f, g : S^n \longrightarrow S^n$ are homotopic if and only if $\deg f = \deg g$.

(b) A map $f : S^n \longrightarrow S^n$, $\deg f = \lambda$ induces the homomorphism $f_* : \pi_n(S^n) \longrightarrow \pi_n(S^n)$ which is a multiplication by λ.

(c) The suspension $\Sigma f : \Sigma S^n \longrightarrow \Sigma S^n$ has degree λ if and only if the map $f : S^n \longrightarrow S^n$ has degree λ.

10/4. Stable homotopy groups of spheres. Consider the following chain of the suspension homomorphisms:

$$\pi_{k+1}S^1 \xrightarrow{\Sigma} \pi_{k+2}S^2 \xrightarrow{\Sigma} \cdots \xrightarrow{\Sigma} \pi_{k+n}S^n \xrightarrow{\Sigma} \pi_{k+n+1}S^{n+1} \xrightarrow{\Sigma} \cdots$$

By the Suspension Theorem the homomorphism $\Sigma : \pi_{k+n}S^n \longrightarrow \pi_{k+n+1}S^{n+1}$ is isomorphism provided that $n \geq k+2$. The group $\pi_{k+n}S^n$ with $n \geq k+2$ is called the stable homotopy group of sphere. The notation:

$$\pi_k^s(S^0) = \pi_{k+n}S^n \quad \text{where } n \geq k+2.$$

So far we proved that $\pi_0(S^0) = \pi_n S^n \cong \mathbf{Z}$. The problem to compute the stable homotopy groups of spheres is highly nontrivial. We shall return to this problem later.

10/5. Whitehead product. Consider the product $S^n \times S^k$ as a CW-complex. Clearly we can choose a cell decomposition of $S^n \times S^k$ into four cells of dimensions $0, n, k, n+k$. The first three cells give us the wedge $S^n \vee S^k \subset S^n \times S^k$. The last cell $e^{n+k} \subset S^n \times S^k$ has the attaching map $w : S^{n+k-1} \longrightarrow S^n \vee S^k$. This attaching map is called the *Whitehead map.* It is convenient to have a particular construction of the map w.

We can think about the sphere S^{n+k-1} as a boundary of the unit disk $D^{n+k} \subset \mathbf{R}^{n+k}$. Thus a point $x \in S^{n+k-1}$ has coordinates $(x_1, \ldots, x_{n+k})$, where $x_1^2 + \cdots + x_{n+k}^2 = 1$. We define

$$U = \left\{(x_1, \ldots, x_{n+k}) \in S^{n+k-1} \mid x_1^2 + \cdots + x_n^2 \leq 1/2\right\}$$

$$V = \left\{(x_1, \ldots, x_{n+k}) \in S^{n+k-1} \mid x_{n+1}^2 + \cdots + x_{n+k}^2 \leq 1/2\right\}$$

Exercise 10.6. Prove that U is homeomorphic to $D^n \vee S^{k-1}$, V is homeomorphic to $S^{n-1} \vee D^k$, and that

$$S^{n+k-1} \cong D^n \vee S^{k-1} \cup_{S^{n-1} \vee S^{k-1}} S^{n-1} \vee D^k$$

Remark. The same decomposition may be constructed by using the homeomorphisms:

$$S^{n+k-1} = \partial(D^{n+k}) = \partial(D^n \times D^k) = \partial(D^n) \times D^k \cup D^n \times \partial(D^k) = S^{n-1} \vee D^k \cup D^n \vee S^{k-1}.$$

The map $w : S^{n+k-1} \longrightarrow S^n \vee S^k$ is defined as follows. First we construct the maps $\varphi_U : U \longrightarrow S^n \vee S^k$ and $\varphi_V : V \longrightarrow S^n \vee S^k$ as the compositions:

$$\varphi_U : U \xrightarrow{\cong} D^n \vee S^{k-1} \xrightarrow{pr} D^n \longrightarrow> D^n/S^{n-1} \xrightarrow{\cong} S^n \longrightarrow S^n \vee S^k,$$

$$\varphi_V : V \xrightarrow{\cong} S^{n-1} \vee D^k \xrightarrow{pr} D^k \longrightarrow> D^k/S^{k-1} \xrightarrow{\cong} S^k \longrightarrow S^n \vee S^k.$$

Clearly we have that

$$\varphi_U|_{S^{n-1} \vee S^{k-1}} = * = \varphi_V|_{S^{n-1} \vee S^{k-1}}$$

and hence the maps φ_U, φ_V define the map $w : S^{n+k-1} \longrightarrow S^n \vee S^k$.

Remark. It is easy to see that the map $w : S^{n+k-1} \longrightarrow S^n \vee S^k$ we constructed is the attaching map for the cell e^{n+1} in the product $S^n \times S^k$.

Now let $\alpha \in \pi_n(X, x_0)$ and $\beta \in \pi_k(X, x_0)$ be represented by maps

$$f : S^n \longrightarrow X, \quad g : S^k \longrightarrow X.$$

We define a map $h : S^{n+k-1} \longrightarrow X$ as the composition:

$$h : S^{n+k-1} \xrightarrow{w} S^n \vee S^k \xrightarrow{f \vee g} X.$$

A homotopy class of the map h defines an element $[\alpha, \beta] \in \pi_{n+k-1}(X, x_0)$ which is called the *Whitehead product.*

Claim 10.2. *The Whitehead product satisfies the following properties:*

(1) *Naturality: Let $f : (X, x_0) \longrightarrow (Y, y_0)$ be a map, $\alpha \in \pi_n(X, x_0))$ and $\beta \in \pi_k(X, x_0)$. Then*
$$f_*([\alpha, \beta]) = [f_*(\alpha), f_*(\beta)],$$
where $f_* : \pi_*(X, x_0) \longrightarrow \pi_*(Y, y_0)$ is the homomorphism induced by the map f.

(2) $[\alpha + \beta, \gamma] = [\alpha, \gamma] + [\beta, \gamma]$.

(3) *If $\alpha \in \pi_n(X)$, $\beta \in \pi_k(X)$ then $[\alpha, \beta] = (-1)^{nk}[\beta, \alpha]$.*

(4) *If $\alpha \in \pi_n(X)$, $\beta \in \pi_k(X)$, $\gamma \in \pi_l(X)$, then (the Jacobi identity)*
$$(-1)^{nl}[[\alpha, \beta], \gamma] + (-1)^{nk}[[\beta, \gamma], \alpha] + (-1)^{kl}[[\gamma, \alpha], \beta] = 0.$$

(5) *If $\alpha \in \pi_1(X)$, $\beta \in \pi_1(X)$ then $[\alpha, \beta] = \alpha\beta\alpha^{-1}\beta^{-1}$.*

Exercise 10.7. Prove the above property (3).

Exercise 10.8. Prove the above property (5).

To prove more about the Whitehead product we have to figure out several facts about the Whitehead map $w : S^{n+k-1} \longrightarrow S^n \vee S^k$. The map w defines an element $w \in \pi_{n+k-1}(S^n \vee S^k)$.

Remark. Denote $\iota_n \in \pi_n(S^n)$, $\iota_k \in \pi_k(S^k)$ the generators given by the identity maps $Id : S^n \longrightarrow S^n$, $Id : S^k \longrightarrow S^k$ respectively. We denote also by ι_n, ι_k the image of the elements ι_n, ι_k in $\pi_n(S^n \vee S^k)$, $\pi_k(S^n \vee S^k)$ respectively. Comparing the definitions of the Whitehead map $w : S^{n+k-1} \longrightarrow S^n \vee S^k$ and of the Whitehead product gives the identity:
$$w = [\iota_n, \iota_k] \in \pi_{n+k-1}(S^n \vee S^k).$$

Claim 10.3. *The element $w \in \pi_{n+k-1}(S^n \vee S^k)$ has infinite order. In particular, the group $\pi_{n+k-1}(S^n \vee S^k)$ is infinite.*

Proof. The map w is the attaching map of the cell e^{n+k} in the product $S^n \times S^k$. It gives us the commutative diagram:

$$\begin{tikzcd} S^{n+k-1} \arrow[r, "w"] \arrow[d, "p'"'] & S^n \vee S^k \arrow[d, "i"] \\ D^{n+k} \arrow[r, "\Phi"] & S^n \times S^k \end{tikzcd}$$

Clearly the map $\Phi : D^{n+k} \longrightarrow S^n \times S^k$ determines an element $\iota \in \pi_{n+k}(S^n \times S^k, S^n \vee S^k)$. Consider the map

$$j : (S^n \times S^k, S^n \vee S^k) \longrightarrow (S^{n+k}, s_0)$$

which maps $S^n \vee S^k$ to the base point $s_0 \in S^{n+k}$. The composition $j \circ \Phi : D^{n+k} \longrightarrow S^{n+k}$ is a representative of a generator of the group $\pi_{n+k}(S^{n+k}) \cong \mathbf{Z}$. Thus we conclude that the element $\iota \in \pi_{n+k}(S^n \times S^k, S^n \vee S^k)$ is nontrivial and has infinite order.

Next we consider the long exact sequence in homotopy:

$$\pi_{n+k}(S^n \vee S^k) \xrightarrow{i_*} \pi_{n+k}(S^n \times S^k) \xrightarrow{j_*} \pi_{n+k}(S^n \times S^k, S^n \vee S^k) \xrightarrow{\partial} \pi_{n+k-1}(S^n \vee S^k)$$

We claim that i_* is epimorphic since $\pi_{n+k}(S^n \times S^k) = \pi_{n+k}(S^n) \oplus \pi_{n+k}(S^k)$. Thus the homomorphism j_* is zero, and ∂ is monomorphims. Since $w = \partial(\iota)$ it follows that the group $\pi_{n+k-1}(S^n \vee S^k)$ is infinite and w has an infinite order. $\square$

Remark. Claim 10.3 implies that $w = [\iota_n, \iota_k] \in \pi_{n+k-1}(S^n \vee S^k)$ has infinite order.

Claim 10.4. *The element $w \in \pi_{n+k-1}(S^n \vee S^k)$ is in a kernel of each of the following homomorphisms:*

(1) $i_* : \pi_{n+k-1}(S^n \vee S^k) \longrightarrow \pi_{n+k-1}(S^n \times S^k)$,

(2) $pr_*^{(n)} : \pi_{n+k-1}(S^n \vee S^k) \longrightarrow \pi_{n+k-1}(S^n)$,

(3) $pr_*^{(k)} : \pi_{n+k-1}(S^n \vee S^k) \longrightarrow \pi_{n+k-1}(S^k)$.

Proof. The exact sequence

$$\longrightarrow \pi_{n+k}(S^n \times S^k, S^n \vee S^k) \xrightarrow{\partial} \pi_{n+k-1}(S^n \vee S^k) \xrightarrow{i_*} \pi_{n+k-1}(S^n \times S^k) \longrightarrow$$

implies that $w \in \operatorname{Ker} i_*$ since $w = \partial(\iota)$.

The commutative diagram

$$\begin{tikzcd}
\pi_{n+k-1}(S^n \vee S^k) \arrow[r, "i_*"] \arrow[dr, "pr_*^{(n)}"] & \pi_{n+k-1}(S^n \times S^k) \arrow[d, "pr_*"] \\
& \pi_{n+k-1}(S^n)
\end{tikzcd}$$

(where $pr : S^n \times S^k \longrightarrow S^n$ is a map collapsing S^k to the base point) implies that $w \in \text{Ker } pr_*^{(n)}$ and similarly $w \in \text{Ker } pr_*^{(n)}$. □

Now consider the suspension homomorphism

$$\Sigma : \pi_q(S^n \times S^k) \longrightarrow \pi_{q+1}(\Sigma(S^n \times S^k)).$$

Claim 10.5. *The element $w \in \pi_{n+k-1}(S^n \vee S^k)$ is in the kernel of the suspension homomorphism*

$$\Sigma : \pi_{n+k-1}(S^n \times S^k) \longrightarrow \pi_{n+k}(\Sigma(S^n \times S^k)).$$

Proof. Consider the commutative diagram:

$$\begin{tikzcd}
\pi_{n+k-1}(S^n) \arrow[d, "\Sigma"] & \pi_{n+k-1}(S^n \vee S^k) \arrow[l, "pr_*^{(n)}"'] \arrow[r, "pr_*^{(k)}"] \arrow[d, "\Sigma"] & \pi_{n+k-1}(S^k) \arrow[d, "\Sigma"] \\
\pi_{n+k}(S^{n+1}) & \pi_{n+k}(\Sigma(S^n \vee S^k)) \arrow[l, "\Sigma(pr_*^{(n)})"'] \arrow[r, "\Sigma(pr_*^{(k)})"] & \pi_{n+k}(S^{k+1})
\end{tikzcd} \tag{29}$$

where pr denote the collapsing maps. By Claim 10.4 $w \in \text{Ker } pr_*^{(n)}$, $w \in \text{Ker } pr_*^{(k)}$. Notice that $\Sigma(S^n \vee S^k) \sim S^{n+1} \vee S^{k+1}$. We need the following lemma.

Lemma 10.6. *There is an isomorphism*

$$\pi_{n+k}(S^{n+1} \vee S^{k+1}) \cong \pi_{n+k}(S^{n+1}) \oplus \pi_{n+k}(S^{k+1})$$

Proof. Consider the long exact sequence for the pair $(S^{n+1} \times S^{k+1}, S^{n+1} \vee S^{k+1})$:

$$\begin{aligned}\pi_{n+k+1}(S^{n+1} \times S^{k+1}, S^{n+1} \vee S^{k+1}) \xrightarrow{\partial} \pi_{n+k}(S^{n+1} \vee S^{k+1}) \xrightarrow{i_*} \pi_{n+k}(S^{n+1} \times S^{k+1}) \\ \xrightarrow{j_*} \pi_{n+k}(S^{n+1} \times S^{k+1}, S^{n+1} \vee S^{k+1}) \longrightarrow\end{aligned} \tag{30}$$

We notice that the $(n+k+1)$-skeleton of the product $S^{n+1} \times S^{k+1}$ is the wedge $S^{n+1} \vee S^{k+1}$. Thus any map $D^{k+n+1} \longrightarrow S^{n+1} \times S^{k+1}$ may be deformed to the subcomplex $S^{n+1} \vee S^{k+1}$. Thus $\pi_{n+k+1}(S^{n+1} \times S^{k+1}, S^{n+1} \vee S^{k+1}) = 0$. The same argument gives that

$$\pi_{n+k}(S^{n+1} \times S^{k+1}, S^{n+1} \vee S^{k+1}) = 0.$$

Thus the long exact sequence (30) gives the isomorphism:

$$i_* : \pi_{n+k}(S^{n+1} \vee S^{k+1}) \xrightarrow{\cong} \pi_{n+k}(S^{n+1} \times S^{k+1}) \cong \pi_{n+k}(S^{n+1}) \oplus \pi_{n+k}(S^{k+1}). \quad \square$$

To complete the proof of Claim 10.5 we notice that Lemma 10.6 and the diagram (29) imply that $w \in \text{Ker } \Sigma$. $\square$

Claim 10.6. *Let $\alpha \in \pi_n(X)$, $\beta \in \pi_k(X)$. Then $[\alpha, \beta] \in Ker\ \Sigma$, where*

$$\Sigma : \pi_{n+k-1}(X) \longrightarrow \pi_{n+k}(\Sigma X)$$

is the suspension homomorphism.

Exercise 10.9. Prove Claim 10.6.

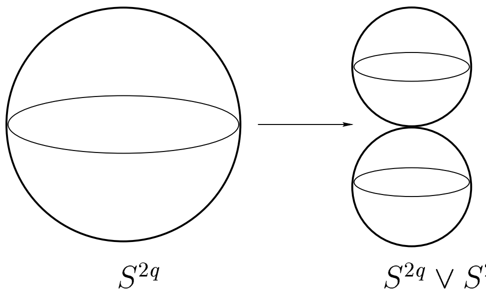

Fig. 10.7.

Now we want to study a particular case. Consider the map $\mu : S^{2q} \longrightarrow S^{2q} \vee S^{2q}$ which collapses the equator sphere, see Fig. 10.7. It induces the homomorphism

$$\mu_* : \pi_*(S^{2q}) \longrightarrow \pi_*(S^{2q} \vee S^{2q}).$$

Let $\iota_{2q} \in \pi_{2q}(S^{2q})$ be the generator represented by the identity map $Id : S^{2q} \longrightarrow S^{2q}$. Let $\iota_{2q}^{(1)} \in \pi_{2q}(S^{2q} \vee S^{2q})$ be the image of the element ι_{2q} under the inclusion map $i^{(1)} : S^{2q} \longrightarrow S^{2q} \vee S^{2q}$ of the sphere S^{2q} to the first sphere in $S^{2q} \vee S^{2q}$. Let $\iota_{2q}^{(2)}$ be the corresponding element for the second sphere in $S^{2q} \vee S^{2q}$. Clearly $\mu_*(\iota_{2q}) = \iota_{2q}^{(1)} + \iota_{2q}^{(2)}$.

Claim 10.7. *The Whitehead product $[\iota_{2q}, \iota_{2q}] \in \pi_{4q-1}(S^{2q})$ is a nontrivial element of infinite order.*

Proof. The map $\mu : S^{2q} \vee S^{2q} \longrightarrow S^{2q}$ induces the homomorphism

$$\mu_* : \pi_{4q-1}(S^{2q}) \longrightarrow \pi_{4q-1}(S^{2q} \vee S^{2q}).$$

By naturality we have that

$$\mu_*([\iota_{2q}, \iota_{2q}]) = [\mu_*(\iota_{2q}), \mu_*(\iota_{2q})] = [\iota_{2q}^{(1)} + \iota_{2q}^{(2)}, \iota_{2q}^{(1)} + \iota_{2q}^{(2)}].$$

By additivity and commutativity (Claim 10.2 (2), (3)) we also have:

$$
\begin{aligned}
\mu_*([\iota_{2q},\iota_{2q}]) &= [\iota_{2q}^{(1)}+\iota_{2q}^{(2)},\iota_{2q}^{(1)}+\iota_{2q}^{(2)}] \\
&= [\iota_{2q}^{(1)},\iota_{2q}^{(1)}]+[\iota_{2q}^{(1)},\iota_{2q}^{(2)}]+[\iota_{2q}^{(2)},\iota_{2q}^{(1)}]+[\iota_{2q}^{(2)},\iota_{2q}^{(2)}] \\
&= [\iota_{2q}^{(1)},\iota_{2q}^{(1)}]+[\iota_{2q}^{(1)},\iota_{2q}^{(2)}]+(-1)^{4q}[\iota_{2q}^{(1)},\iota_{2q}^{(2)}]+[\iota_{2q}^{(2)},\iota_{2q}^{(2)}] \\
&= [\iota_{2q}^{(1)},\iota_{2q}^{(1)}]+[\iota_{2q}^{(2)},\iota_{2q}^{(2)}]+2w_{2q}
\end{aligned}
$$

where $w_{2q} = [\iota_{2q}^{(1)},\iota_{2q}^{(2)}]$. Notice that we used the fact that the sphere S^{2q} is even-dimensional. Now assume that the element $[\iota_{2q},\iota_{2q}]\in\pi_{4q-1}(S^{2q})$ has finite order. Then the elements $[\iota_{2q}^{(j)},\iota_{2q}^{(j)}]\in\pi_{4q-1}(S^{2q})$, $j=1,2$ also have finite order since $\iota_{2q}^{(j)}$ is the image of the generator ι_{2q} under the homomorphism $\pi_{2q}(S^{2q})\longrightarrow\pi_{2q}(S^{2q}\vee S^{2q})$. Then it follows that for some integer λ

$$0=\mu_*(\lambda[\iota_{2q},\iota_{2q}])=\lambda[\iota_{2q}^{(1)},\iota_{2q}^{(1)}]+\lambda[\iota_{2q}^{(2)},\iota_{2q}^{(2)}]+2\lambda w_{2q}=2\lambda w_{2q}.$$

This contradicts to Claim 10.3. Thus the element $[\iota_{2q},\iota_{2q}]\in\pi_{4q-1}(S^{2q})$ has infinite order. □

We specify Claims 10.6, 10.7 to get so called "hard part" of the Suspension Theorem.

Theorem 10.7. **(1)** *The Whitehead product $[\iota_{2q},\iota_{2q}]\in\pi_{4q-1}(S^{2q})$ has infinite order.*

(2) *The Whitehead product $[\iota_{2q},\iota_{2q}]\in\pi_{4q-1}(S^{2q})$ is in the kernel of the suspension homomorphism, i.e. $\Sigma([\iota_{2q},\iota_{2q}])=0$ in $\pi_{4q}(S^{2q+1})$.*

Remark. Actually, $\pi_{4q-1}(S^{2q})\cong\mathbf{Z}\oplus\{\text{finite group}\}$ and these groups are the only homotopy groups of spheres (besides $\pi_n(S^n)$) which are infinite. We shall return to the Whitehead product to study the Hopf invariant.

Now we consider the product $S^n\times S^k$.

Corollary 10.8. *The suspension $\Sigma(S^n\times S^k)$ is homotopy equivalent to the wedge*

$$S^{n+1}\vee S^{k+1}\vee S^{n+k+1}.$$

Exercise 10.10. Prove Corollary 10.8.