Notes on the course Algebraic Topology

Boris Botvinnik

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1 First set of the most important topological spaces

Here I assume that you are familliar with the standard concepts of general topology: topological spaces, continuity, compactness, connectedness and so on. We are not going to discuss these concepts and constructions again starting with definitions. Instead we make our first steps with the most important examples of topological spaces. One of the major goals we have in this course is to develop a "geometric" intuition. This is why we have to study carefully particular examples of topological spaces.

1/1. Euclidian space, spheres, disks. The notations $\mathbf{R}^n$, $\mathbf{C}^n$ have usual meaning throughout the course. The space $\mathbf{C}^n$ is identified with $\mathbf{R}^{2n}$ by the correspondence

$$(x_1 + iy_1, \ldots, y_n + ix_n) \longleftrightarrow (x_1, y_1, \ldots, x_n, y_n).$$

The unit sphere in $\mathbf{R}^{n+1}$ centered in the origin is denoted by S^n, the unit disk in $\mathbf{R}^n$ by D^n, and the unit cube in $\mathbf{R}^n$ by I^n. Thus S^{n-1} is the *boundary* of the disk D^n. Just in case we give these spaces in coordinates:

$$\begin{array}{lcl} S^{n-1} & = & \{(x_1, \ldots, x_n) \in \mathbf{R}^n \ \mid \ x_1^2 + \cdots + x_n^2 = 1\}\,, \\ D^n & = & \{(x_1, \ldots, x_n) \in \mathbf{R}^n \ \mid \ x_1^2 + \cdots + x_n^2 \leq 1\}\,, \\ I^n & = & \{(x_1, \ldots, x_n) \in \mathbf{R}^n \ \mid \ 0 \leq x_j \leq 1, \ j = 1, \ldots, n\}\,. \end{array} \tag{1}$$

The symbol $\mathbf{R}^\infty$ is a union (direct limit) of the embeddings $\mathbf{R}^1 \subset \mathbf{R}^2 \subset \cdots \subset \mathbf{R}^n \subset \cdots$. Thus a point $x \in \mathbf{R}^\infty$ is a sequence of points $x = (x_1, \ldots, x_n, \ldots)$, where $x_n \in \mathbf{R}$ and $x_j = 0$ for j greater then some k. Topology on $\mathbf{R}^\infty$ is determined as follows. A set $F \subset \mathbf{R}^\infty$ is closed, if each intersection $F \cap \mathbf{R}^n$ is closed in $\mathbf{R}^n$. In a similar way we define the spaces $\mathbf{C}^\infty$ and S^∞.

Exercise 1.1. *Let $x^{(1)} = (a_1, 0, \ldots, 0, \ldots)$, $\ldots$, $x^{(n)} = (0, 0, \ldots, a_n, \ldots)$, $\ldots$ be a sequence of elements in $\mathbf{R}^\infty$. Prove that the sequence $\{x^{(n)}\}$ converges in $\mathbf{R}^\infty$ if and only if the sequence of numbers $\{a_n\}$ is finite.*

Probably you already know the another "version" of $\mathbf{R}^\infty$, namely the Hilbert space ℓ_2 (which is the set of sequences $\{x_n\}$ so that the series $\sum_n x_n$ converges). The space ℓ_2 is a metric space, where the distance $\rho(\{x_n\},\{y_n\})$ is defined as $\rho(\{x_n\},\{y_n\}) = \sqrt{\sum_n (y_n - x_n)^2}$. Clearly there is a natural map $\mathbf{R}^\infty \longrightarrow \ell_2$.

Remark. The optional exercises are labeled by *.

Exercise 1.2. *Is the above map $\mathbf{R}^\infty \longrightarrow \ell_2$ homeomorphism or not?*

Consider the unit cube I^∞ in the spaces $\mathbf{R}^\infty$, ℓ_2, i.e. $I^\infty = \{\{x_n\} \mid 0 \leq x_n \leq 1\}$.

Exercise 1.3. *Prove or disprove that the cube I^∞ is compact space (in $\mathbf{R}^\infty$ or ℓ_2).*

We are going to play a little bit with the sphere S^n.

Claim 1.1. (Stereographic projection) *A punctured sphere $S^n \setminus \{x_0\}$ is homeomorphic to $\mathbf{R}^n$.*

Proof. We construct a map $f : S^n \setminus \{x_0\} \longrightarrow \mathbf{R}^n$. Let S^n be given as above (1). Let the point x_0 be the North Pole, so it has the coordinates $(0,\dots,0,1) \in \mathbf{R}^{n+1}$. Consider a point $x = (x_1,\dots,x_{n+1}) \in S^n$, $x \neq x_0$, and the line going through the points x and x_0. A directional vector of this line may be given as $\vec{v} = (-x_1,\dots,-x_n, 1-x_{n+1})$, so any point of this line could be written as

$$(0,\dots,0,1) + t(-x_1,\dots,-x_n,1-x_{n+1}) = (-tx_1,\dots,-tx_n, 1+t(1-x_{n+1})).$$

The intersection point of this line and $\mathbf{R}^n = \{(x_1,\dots,x_n,0)\} \subset \mathbf{R}^{n+1}$ is determined by vanishing the last coordinate. Clearly the last coordinate vanishes if $t = -\frac{1}{1-x_{n+1}}$. The map $f : S^n \setminus \{pt\} \longrightarrow \mathbf{R}^n$ is given by

$$f : (x_1,\dots,x_{n+1}) \mapsto \left(\frac{x_1}{1-x_{n+1}},\dots,\frac{x_n}{1-x_{n+1}},0\right).$$

The rest of the proof is left to you. $\square$

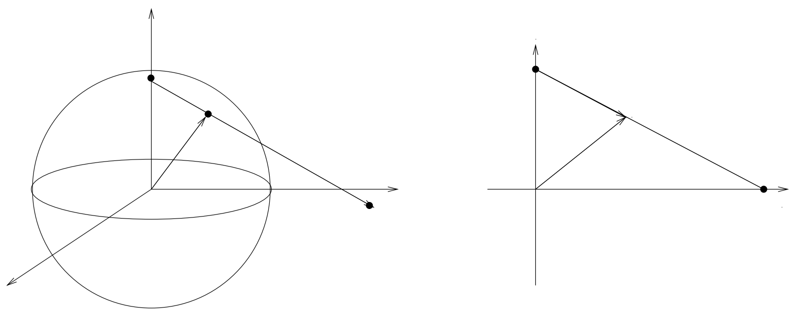

Fig. 1.1. Stereographic projection.

We define a hemisphere $S^n_+ = \left\{x_1^2 + \cdots + x_{n+1}^2 = 1 \;\&\; x_{n+1} \geq 0\right\}$.

Exercise 1.4. *Prove the that S^n_+ and D^n are homeomorphic.*

1/2. Real projective spaces. A real projective space $\mathbf{RP}^n$ is a set of all lines in $\mathbf{R}^{n+1}$ going through $0 \in \mathbf{R}^{n+1}$. Let $\ell \in \mathbf{RP}^n$ be a line, then we define a basis for topology on $\mathbf{RP}^n$ as follows:

$$U_\epsilon(\ell) = \{\ell' \mid \text{the angle between } \ell \text{ and } \ell' \text{ less then } \epsilon\}.$$

Exercise 1.5. *A projective space $\mathbf{RP}^1$ is homeomorphic to the circle S^1.*

Let $(x_1, \ldots, x_{n+1})$ be coordinates of a vector parallel to ℓ, then the vector $(\lambda x_1, \ldots, \lambda x_{n+1})$ defines the same line ℓ (for $\lambda \neq 0$). We identify all these coordinates, the equivalence class is called *homogeneous coordinates* $(x_1 : \ldots : x_{n+1})$. Note that there is at least one x_i which is not zero. Let

$$U_j = \{\ell = (x_1 : \ldots : x_{n+1}) \mid x_j \neq 0 \} \subset \mathbf{RP}^n$$

Then we define the map $f_j^{\mathbf{R}} : U_j \longrightarrow R^n$ by the formula

$$(x_1 : \ldots : x_{n+1}) \longrightarrow \left(\frac{x_1}{x_j}, \frac{x_2}{x_j}, \ldots, \frac{x_{j-1}}{x_j}, 1, \frac{x_{j-1}}{x_j}, \ldots, \frac{x_{n+1}}{x_i}\right).$$

Remark. The map $f_j^{\mathbf{R}}$ is a homeomorphism, it determines a *local coordinate system* in $\mathbf{RP}^n$ giving this space a structure of *smooth manifold* of dimension n.

There is natural map $c : S^n \longrightarrow \mathbf{RP}^n$ which sends each point $s = (s_1, \ldots, s_{n+1}) \in S^n$ to the line going through zero and s. Note that there are exactly two points s and $-s$ which map to the same line $\ell \in \mathbf{RP}^n$.

We have a chain of embeddings $\mathbf{RP}^1 \subset \mathbf{RP}^2 \subset \cdots \subset \mathbf{RP}^n \subset \mathbf{RP}^{n+1} \subset \cdots$, we define $\mathbf{RP}^\infty = \bigcup_{n \geq 1} \mathbf{RP}^n$ with the *limit topology* (similarly to the above case of $\mathbf{R}^\infty$).

1/3. Complex projective spaces. Let $\mathbf{CP}^n$ be the space of all complex lines in the complex space $\mathbf{C}^{n+1}$. In the same way as above we define *homogeneous coordinates* $(z_1 : \ldots : z_{n+1})$ for each complex line $\ell \in \mathbf{CP}^n$, and the "local coordinate system":

$$U_i = \{\ell = (z_1 : \ldots : z_n) \mid z_i \neq 0 \} \subset \mathbf{CP}^n.$$

Clearly there is a homemorphism $f_i^{\mathbf{C}} : U_i \longrightarrow \mathbf{C}^{n+1}$.

Exercise 1.6. *Prove that the projective space $\mathbf{CP}^1$ is homeomorphic to the sphere S^2.*

Consider the sphere $S^{2n-1} \subset \mathbf{C}^n$. Each point $z = (z_1, \ldots, z_{n+1}) \in S^{2n-1}$, $|z_1|^2 + \cdots + |z_{n+1}|^2 = 1$ determines a line $\ell = (z_1 : \ldots : z_{n+1}) \in \mathbf{CP}^n$. Observe that the point $e^{i\varphi}z = (e^{i\varphi}z_1, \ldots, e^{i\varphi}z_{n+1}) \in S^{2n+1}$ and it determines the same complex line ℓ. We have defined the map $g(n) : S^{2n+1} \longrightarrow \mathbf{CP}^n$.

Exercise 1.7. *Prove that the map $g(n) : S^{2n+1} \longrightarrow \mathbf{CP}^n$ has a property that $g(n)^{-1}(\ell) = S^1$ for any $\ell \in \mathbf{CP}^n$.*

The case $n = 1$ is very interesting since $\mathbf{CP}^1 = S^2$, here we have the map $g(1) : S^3 \longrightarrow S^2$ where $g(1)^{-1}(x) = S^1$ for any $x \in S^2$. This map is the *Hopf map*, it gives very important example of nontrivial map $S^3 \longrightarrow S^2$. Before this map was discovered by Hopf, people thought that there are no nontrivial maps $S^k \longrightarrow S^n$ for $k > n$ ("trivial map" means a map homotopic to the constant map).

Exercise 1.8. *Prove that $\mathbf{RP}^n$, $\mathbf{CP}^n$ are compact and connected spaces.*

Besides the reals $\mathbf{R}$ and complex numbers $\mathbf{C}$ there are quaternion numbers $\mathbf{H}$. Recall that $q \in \mathbf{H}$ may be thought as a sum $q = a + ib + jc + kd$, where $a, b, c, d \in R$, and the symbols i, j, k satisfy the identities:

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.$$

Then two quaternions $q_1 = a_1 + ib_1 + jc_1 + kd_1$ and $q_2 = a_2 + ib_2 + jc_2 + kd_2$ may be multiplied using these identies. The product here is not commutative, however one can choose left or right multiplication to define a line in $\mathbf{H}^{n+1}$. A set of all quaternionic lines in $\mathbf{H}^{n+1}$ is the quaternion projective space $\mathbf{HP}^n$.

Exercise 1.9. *Give details of the above definition. In particular, check that the space $\mathbf{HP}^n$ is well-defined. Identify the quaternionic line $\mathbf{HP}^1$ with some well-known topological space.*

1/4. Grassmannian manifolds. These spaces generalize the projective spaces. Indeed, the space $G(n, k)$ is a space of all k-dimensional vector subpaces of $\mathbf{R}^n$ with natural topology. Clearly $G(n, 1) = \mathbf{RP}^1$. It is not difficult to introduce local coordinates in $G(n, k)$. Let $\pi \in G(n, k)$ be a k-plane. Choose k linearly independent vectors $v_1, \ldots, v_k$ generating π and write their coordinates in the standard basis $e_1, \ldots, e_n$ of $\mathbf{R}^n$:

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{k1} & \cdots & a_{kn} \end{pmatrix}$$

Since the vectors $v_1, \ldots, v_k$ are linearly independent there exist k columns of the matrix A which are linearly independent as well. In other words, there are indices $i_1, \ldots, i_k$ so that a projection of the plane π on the k-plane $\langle e_{i_1}, \ldots, e_{i_k} \rangle$ generated by the coordinate vectors $e_{i_1}, \ldots, e_{i_k}$ is a linear isomorphism. Now it is easy to introduce local coordinates on the Grassmanian manifold $G(n, k)$. Indeed, choose the indices $i_1, \ldots, i_k$, $1 \leq i_1 < \cdots < i_k \leq n$, and consider all k-planes $\pi \in G(n, k)$ so that the projection of π on the plane $\langle e_{i_1}, \ldots, e_{i_k} \rangle$ is a linear isomorphism. We denote this set of k-planes by $U_{i_1,\ldots,i_k}$.

Exercise 1.10. *Construct a homeomorphism* $f_{i_1,\dots,i_k} : U_{i_1,\dots,i_k} \longrightarrow \mathbf{R}^{k(n-k)}$.

The result of this exercise shows that the Grassmannian manifold $G(n,k)$ is a smooth manifold of dimension $k(n-k)$. The projective spaces and Grassmannian manifolds are very important examples of spaces which we will see many times in our course.

Exercise 1.11. *Define a complex Grassmannian manifold* $\mathbf{C}G(n,k)$ *and construct a local coordinate system for* $\mathbf{C}G(n,k)$*. In particular, find its dimension.*

We have a chain of spaces: $G(k,k) \subset G(k+1,k) \subset \cdots \subset G(n,k) \subset G(n+1,k) \subset \cdots$. Let $G(\infty,k)$ be the union (inductive limit) of these spaces. The topology of $G(\infty,k)$ is given in the same way as to $\mathbf{R}^\infty$: a set $F \subset G(\infty,k)$ is closed if and only if the intersection $F \cap G(n,k)$ is closed for each n. This topology is known as a *topology of an inductive limit*.

Exercise 1.12. *Prove that the Grassmannian manifolds* $G(n,k)$ *and* $\mathbf{C}G(n,k)$ *are compact and connected.*

1/6. Flag manifolds. Here we just mention these examples without further considerations (we are not ready for this yet). Let $1 \leq k_1 < \cdots < k_s \leq n-1$. A *flag of the type* $(k_1,\dots,k_s)$ is a chain of vector subspaces $V_1 \subset \cdots \subset V_s$ of $\mathbf{R}^n$ such that $\dim V_i = k_i$. A set of flags of the given type is the flag manifold $F(n;k_1,\dots,k_s)$. Hopefully we shall return to these spaces: they are very interesting and popular creatures.

1/7. Classic Lie groups. The first example here is the group $GL(\mathbf{R}^n)$ of nondegenerated linear transformations of $\mathbf{R}^n$. Once we choose a basis $e_1,\dots,e_n$ of $\mathbf{R}^n$, each element $A \in GL(\mathbf{R}^n)$ may be identified with an $n \times n$ matrix A with $\det A \neq 0$. Clearly we may identify the space of all $n \times n$ matrices with the space $\mathbf{R}^{n^2}$. The determinant gives a continuous function $\det : \mathbf{R}^{n^2} \longrightarrow \mathbf{R}$, and the space $GL(\mathbf{R}^n)$ is an open subset of $\mathbf{R}^{n^2}$:

$$GL(\mathbf{R}^n) = \mathbf{R}^{n^2} \setminus \det^{-1}(0).$$

In particular, this identification defines a topology on $GL(\mathbf{R}^k)$. In the same way one may construct an embedding $GL(\mathbf{C}^n) \subset \mathbf{C}^{n^2}$. The orthogonal and special orthogonal groups $O(k)$, $SO(k)$ are subgroups of $GL(\mathbf{R}^k)$, and the groups $U(k)$, $SU(k)$ are subgroups of $GL(\mathbf{C}^k)$. (Recall that $O(n)$ (or $U(n)$) is a group of those linear transformations of $\mathbf{R}^n$ (or $\mathbf{C}^n$) which preserve a Euclidian (or Hermitian) metric on $\mathbf{R}^n$ (or $\mathbf{C}^n$), and the groups $SO(k)$ and $SU(k)$ are subgroups of $O(k)$ and $U(k)$ of matrices with the determinant 1.)

Exercise 1.13. *Prove that* $SO(2)$ *and* $U(1)$ *are homeomorphic to* S^1*, and that* $SO(3)$ *is homeomorphic to* $\mathbf{RP}^3$.

Hint: *To prove that* $SO(3)$ *is homeomorphic to* $\mathbf{RP}^3$ *you have to analyze* $SO(3)$*: the key fact is the geometric description of an orthogonal transformation* $\alpha \in SO(3)$*, it is* *given by rotating a plane (by an angle φ) about a line ℓ perpendicular to that plane. You should use the line ℓ and the angle φ as major parameters to construct a homeomorphism $SO(3) \to \mathbf{RP}^3$, where it is important to use a particular model of $\mathbf{RP}^3$, namely a disk D^3 where one identifies the opposite points on $S^2 = \partial D^3 \subset D^3$.*

Exercise 1.14. *Prove that the spaces $O(n)$, $SO(n)$, $U(n)$, $SU(n)$ are compact.*

Exercise 1.15. *Prove that the space $O(n)$ has two path-connected components, and that the spaces $SO(n)$, $U(n)$, $SU(n)$ are path-connected.*

Exercise 1.16. *Prove that each matrix $A \in SU(2)$ may be presented as:*

$$A = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}, \text{ where } \alpha, \beta \in \mathbf{C}, \ |\alpha|^2 + |\beta|^2 = 1$$

Use this presentation to prove that $SU(2)$ is homeomorphic to S^3.

It is important to emphasize that the classic groups $O(n)$, $SO(n)$, $U(n)$, $SU(n)$ are all *manifolds*, i.e. for each point α there there exists an open neighborhood homeomorphic to a Euclidian space.

Exercise 1.17. *Prove that the space for any point $\alpha \in SO(n)$ there exists an open neighborhood homeomorphic to the Euclidian space of the dimension $\frac{n(n-1)}{2}$.*

Exercise 1.18. *Prove that the spaces $U(n)$, $SU(n)$ are manifolds and find their dimension.*

The next set of examples is also very important.

1/8. Stiefel manifolds. Again, we consider the vector space $\mathbf{R}^n$. We call vectors $v_1, \ldots, v_k$ a *k-frame* if they are linearly independent. A k-frame $v_1, \ldots, v_k$ is called an *orthonormal k-frame* if the vectors $v_1, \ldots, v_k$ are of unit length and orthogonal to each other. The space of all orthonormal k-frames in $\mathbf{R}^n$ is denoted by $V(n,k)$. There are analogous complex and quaternionic versions of these spaces, they are denoted as $\mathbf{C}V(n,k)$ and $\mathbf{H}V(n,k)$ respectively. Here is an exercise where your knowledge of basic linear algebra may be crucial:

Exercise 1.19. *Prove the following homeomorphisms: $V(n,n) \cong O(n)$, $V(n,n-1) \cong SO(n)$, $\mathbf{C}V(n,n) \cong U(n)$, $\mathbf{C}V(n,n-1) \cong SU(n)$, $V(n,1) \cong S^{n-1}$, $\mathbf{C}V(n,1) \cong S^{2n-1}$, $\mathbf{H}V(n,1) \cong S^{4n-1}$.*

We note that the group $O(n)$ acts on the spaces $V(n,k)$ and $G(n,k)$: indeed, if $\alpha \in O(n)$ and $v_1, \dots, v_k$ is an orthonormal k-frame, then $\alpha(v_1), \dots, \alpha(v_k)$ is also an orthonormal k-frame. As for the Grassmannian manifold, one can easily see that $\alpha(\Pi)$ is a k-dimensional subspace in $\mathbf{R}^n$ if Π is.

The group $O(n)$ contains a subgroup $O(j)$ which acts on $\mathbf{R}^j \subset \mathbf{R}^n$, where $\mathbf{R}^j = \langle e_1, \dots, e_j \rangle$ is generated by the first j vectors $e_1, \dots, e_j$ of the standard basis $e_1, \dots, e_n$ of $\mathbf{R}^n$. Similarly $U(n)$ acts on the spaces $\mathbf{C}G(n,k)$ and $\mathbf{C}V(n,k)$, and $U(j)$ is a subgroup of $U(n)$.

Exercise 1.20. *Prove the following homeomorphisms:*

- **(a)** $S^{n-1} \cong O(n)/O(n-1) \cong SO(n)/SO(n-1)$,
- **(b)** $S^{2n-1} \cong U(n)/U(n-1) \cong SU(n)/SU(n-1)$,
- **(c)** $G(n,k) \cong O(n)/O(k) \times O(n-k)$,
- **(c)** $\mathbf{C}G(n,k) \cong U(n)/U(k) \times U(n-k)$.

We note here that $O(k) \times O(n-k)$ is a subgroup of $O(n)$ of orthogonal matrices with two diagonal blocks of the sizes $k \times k$ and $(n-k) \times (n-k)$ and zeros otherwise.

There is also the following natural action of the orthogonal group $O(k)$ on the Stieffel manifold $V(n,k)$. Let $v_1, \dots, v_k$ be an orthonormal k-frame then $O(k)$ acts on the space $V = \langle v_1, \dots, v_k \rangle$, in particular, if $\alpha \in O(k)$, then $\alpha(v_1), \dots, \alpha(v_k)$ is also an orthonormal k-frame. Similarly there is a natural action of $U(k)$ on $\mathbf{C}V(n,k)$.

Exercise 1.21. *Prove that the above action of $O(k)$ on $V(n,k)$ is free, as well as the action of $U(k)$ on $\mathbf{C}V(n,k)$ is free as well.*

Exercise 1.22. *Prove the following homeomorphisms:*

- **(a)** $V(n,k)/O(k) \cong G(n,k)$,
- **(b)** $\mathbf{C}V(n,k)/U(k) \cong \mathbf{C}G(n,k)$.

There are obvious maps $V(n,k) \xrightarrow{p} G(n,k)$, $\mathbf{C}V(n,k) \xrightarrow{p} \mathbf{C}G(n,k)$ (where each orthonormal k-frame $v_1, \dots, v_k$ maps to the k-plane $\pi = \langle v_1, \dots, v_k \rangle$ generated by this frame). It is easy to see that the inverse image $p^{-1}(\pi)$ may be identified with $O(k)$ (in the real case) and $U(k)$ (in the complex case). We shall return to these spaces later on. In particular, we shall describe a cell-structure of these spaces and compute their homology and cohomology groups.

1/9. Surfaces. Here I refer to Chapter 1 of Massey, Algebraic topology, for details. I would like for you to read this Chapter carefully even though most of you have seen this material before. Here I briefly remind some constructions and give exercises. The section 4 of the reffered Massey book gives the examples of surfaces. In particular, the torus T^2 is described in three different ways:

(a) A product $S^1 \times S^1$.

(b) A subspace of $\mathbf{R}^3$ given by: $\left\{(x, y, z) \in \mathbf{R}^3 \mid (\sqrt{x^2+y^2}-2)^2 + z^2 = 1 \right\}$.

(c) A unit square $I^2 = \{(x, y) \in \mathbf{R}^2 \mid 0 \le x \le 1, 0 \le y \le 1\ \}$ with the identification:

$$(x,0) \equiv (x,1) \quad (0,y) \equiv (1,y) \ \text{ for all } 0 \le x \le 1, \quad 0 \le y \le 1.$$

Exercise 1.23. *Prove that the spaces described in (a), (b), (c) are indeed homeomorphic.*

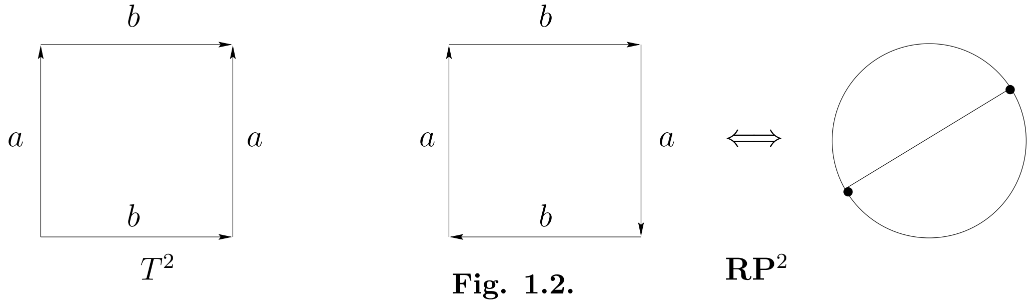

Fig. 1.2.

The next surface we want to become our best friend is the projective space $\mathbf{RP}^2$. Earlier we defined $\mathbf{RP}^2$ as a space of lines in $\mathbf{R}^3$ going through the origin.

Exercise 1.24. *Prove that the projective plane* $\mathbf{RP}^2$ *is homeomorphic to the following spaces:*

(a) *The unit disk* $D^2 = \{(x, y) \in \mathbf{R}^2 \mid x^2 + y^2 \le 1\}$ *with the opposite points* $(x, y) \equiv (-x, -y)$ *of the circle* $S^1 = \{(x, y) \in \mathbf{R}^2 \mid x^2 + y^2 = 1\} \subset D^2$ *have been identified.*

(b) *The unit square (see Fig. 1.3) with the arrows* a *and* b *identified as it is shown.*

(c) *The Mëbius band which boundary (the circle) is identified with the boundary of the disk* D^2*, see Fig. 1.3.*

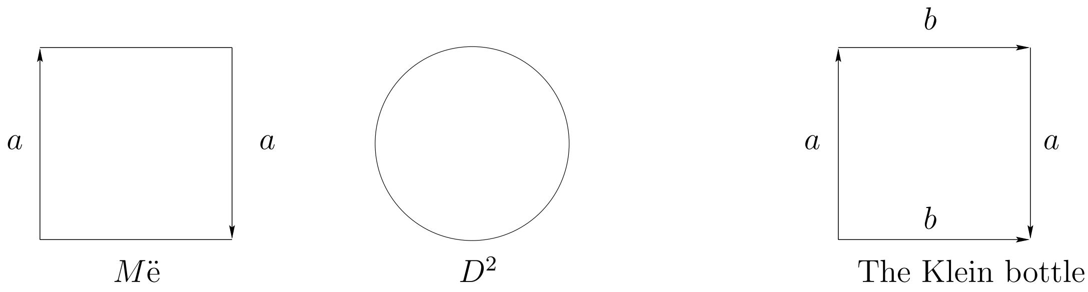

Fig. 1.3.

Here the Mëbius band is constructed from a square by identifying the arrows a. The Klein bottle Kl^2 may be described as a square with arrows identified as it is shown above on Fig. 1.2.

Exercise 1.25. *Prove that the Klein bottle Kl^2 is homeomorphic to the union of two Mëbius bands along the circle.*

Massey carefully defines connected sum $S_1 \# S_2$ of two surfaces S_1 and S_2.

Exercise 1.26. *Prove that $Kl^2 \# \mathbf{RP}^2$ is homeomorphic to $\mathbf{RP}^2 \# T^2$.*

Exercise 1.27. *Prove that $Kl^2 \# Kl^2$ is homeomorphic to Kl^2.*

Exercise 1.28. *Prove that $\mathbf{RP}^2 \# \mathbf{RP}^2$ is homeomorphic to Kl^2.*