Homework # 25. Due to Wednesday, June 3, 11:00 am

(1) Let (B,A) be a CW-pair and $X^n = B^{(n)} \cup A$, where $B^{(n)}$ is the n-th skeleton of B, and Y be a homotopy sipmle space. Prove that for any map $f : X^n \to Y$ and a cochain $d \in \mathcal{E}^n(B,A;\pi_n(Y))$ there exists a map $g : X^n \to Y$ such that $f|_{X^{n-1}} = g|_{X^{n-1}}$ and $d(f,g) = d$.

(2) Consider a k-torus T^k. We identify T^k with the quotient space $\mathbf{R}^k/\sim$, where two vectors $\vec{x} \sim \vec{y}$ if and only if all coordinates of the vector $\vec{x} - \vec{y}$ are integers. It is easy to see that a linear map $\bar{f} : \mathbf{R}^k \to \mathbf{R}^\ell$ given by an $k \times \ell$-matrix A with integral entries descends to a map $f : T^k \to T^\ell$. In that case a map $f : T^k \to T^\ell$ is called *linear*. Prove that any map $f : T^k \to T^\ell$ is homotopic to a linear map.

(3) Let X be an n-dimensional CW-complex. Prove that there is a bijection:

$$H^n(X;\mathbf{Z}) \cong [X, S^n].$$

(4) Assume a CW-complex X contains S^1 such that the inclusion $i : S^1 \hookrightarrow X$ induces an injection $i_* : H_1(S^1;\mathbf{Z}) \to H_1(X;\mathbf{Z})$ with image a direct summand of $H_1(X;\mathbf{Z})$. Prove that S^1 is a retract of X.

(5) Two questions:

(a) Show that there is no map from $\mathbf{CP}^2$ to itself of degree -1.

(b) Show that there is no map from $\mathbf{CP}^2 \times \mathbf{CP}^2$ to itself of degree -1.