Homework # 22. Due to Wednesday, May 6, 11:00 am

(1) Compute the ring structure of $H^*(\mathbf{RP}^n; \mathbf{Z}_{2k})$ for any $k \geq 1$.

(2) Prove that $\mathbf{RP}^3$ and $\mathbf{RP}^2 \vee S^3$ are not homotopy equivalent.

(3) Recall that $\mathbf{RP}^{2n+1} = S^{2n+1}/\mathbf{Z}_2$, where $\mathbf{Z}_2 = \{-1, 1\}$ which acts on S^{2n+1} by taking $z \in S^{2n+1}$ to $-z$. Similarly, we identify $\mathbf{CP}^n$ with the space of orbits S^{2n+1}/S^1, where an element $e^{i\theta} \in S^1$ sends $z = (z_1, \ldots, z_{n+1}) \in S^{2n+1} \subset \mathbf{C}^{2n+2}$ to $e^{i\theta} z = (e^{i\theta} z_1, \ldots, e^{i\theta} z_{n+1})$. We identify the group $\mathbf{Z}_2 = \{-1, 1\}$ with $\{e^{i\pi}, e^{i0}\} \subset S^1$, then we have a natural map $f_n : \mathbf{RP}^{2n+1} \to \mathbf{CP}^n$ which makes the following diagram commute:

$$\begin{tikzcd}
S^{2n+1} \arrow[r, "Id"] \arrow[d, "p"] & S^n \arrow[d, "h"] \\
\mathbf{RP}^{2n+1} \arrow[r, "f_n"] & \mathbf{CP}^n
\end{tikzcd}$$

where $h : S^{2n+1} \to \mathbf{CP}^n$ is the Hopf bundle. Notice that $f_n : \mathbf{RP}^{2n+1} \to \mathbf{CP}^n$ is a fiber bundle with a fiber $S^1 = S^1/\{e^{i\pi}, e^{i0}\}$. Taking $n \to \infty$, we get a map $f_\infty : \mathbf{RP}^\infty \to \mathbf{CP}^\infty$. Finally, **the assignment: compute the ring homomorphism**

$$f_\infty^* : H^*(\mathbf{CP}^\infty; \mathbf{Z}_2) \to H^*(\mathbf{RP}^\infty; \mathbf{Z}_2).$$

(4) Let M be an oriented compact connected manifold, $\dim M = n$, it is equivalent to the fact that the homology group $H_n(M; \mathbf{Z}) \cong \mathbf{Z}$. Then a generator of $H_n(M; \mathbf{Z})$ is called a *fundamental class* and is denoted by $[M]$. Then we say that a map $f : M \to M$ has degree λ if $f_*([M]) = \lambda[M]$. Now **the assignment: let $f : \mathbf{CP}^n \to \mathbf{CP}^n$ be a map of degree 64. Find the dimension of $\mathbf{CP}^n$.**

(5) Prove that any map $f : \mathbf{CP}^{2k} \to \mathbf{CP}^{2k}$ has a fixed point.