Homework # 18. Due to Wednesday, April 8, 11:00 am

(1) Let $\beta_q(X) = \text{Rank } H_q(X)$ be the Betti number of X. Prove that
$$\beta_q(X \times X') = \sum_{r+s=q} \beta_r(X)\beta_s(X').$$

(2) Let X, X' be such spaces that their Euler characteristics $\chi(X)$, $\chi(X')$ are finite. Prove that $\chi(X \times X') = \chi(X) \cdot \chi(X')$.

(3) Let $\mathcal{C}$ be a chain complex, $\mathcal{C} = \{\mathcal{C}_q\}$, such that $\mathcal{C}_q = 0$ for $q \geq n$ (for some n). Let $\phi : \mathcal{C} \longrightarrow \mathcal{C}$ be a chain map, and $\phi_* : H_*\mathcal{C} \longrightarrow H_*\mathcal{C}$ be the induced homomorphism in homology groups. Prove that
$$\text{Lef}(\phi) = \text{Lef}(\phi_*).$$

(4) Let X be a finite contractible CW-complex. Prove that any map $f : X \longrightarrow X$ has a fixed point.

(5) Let $f : \mathbf{RP}^{2n} \longrightarrow \mathbf{RP}^{2n}$ be a map. Prove that f always has a fixed point. Give an example that the above statement fails for a map $f : \mathbf{RP}^{2n+1} \longrightarrow \mathbf{RP}^{2n+1}$.

(6) Let $n \neq k$. Prove that $\mathbf{R}^n$ is not homeomorphic to $\mathbf{R}^k$.

(7) Let $f : S^n \longrightarrow S^n$ be a map, and $\deg(f)$ be the degree of f. Prove that $\text{Lef}(f) = 1 + (-1)^n \deg(f)$.

(8) Prove that there is no tangent vector field $v(x)$ on the sphere S^{2n} such that $v(x) \neq 0$ for all $x \in S^{2n}$.