Homework # 16. Due to Wednesday, March 4, 11:00 am

(1) Let $A : S^n \longrightarrow S^n$ be the antipodal map, $A : x \mapsto -x$, and $\iota_n \in \pi_n(S^n)$ be the generator represented by the identity map $S^n \longrightarrow S^n$. Prove that the homotopy class $[A] \in \pi_n(S^n)$ is equal to

$$[A] = \begin{cases} \iota_n, & \text{if } n \text{ is odd,} \\ -\iota_n, & \text{if } n \text{ is even.} \end{cases}$$

(2) Let $e^0, \ldots, e^n$ be the cells in the standard cell decomposition of $\mathbf{RP}^n$. Prove that

$$[e^q : e^{q-1}] = \begin{cases} 2 & \text{if } q \text{ is odd,} \\ 0, & \text{if } q \text{ is even.} \end{cases}$$

(3) Compute the homology groups $H_q(\mathbf{RP}^{2n} \# \mathbf{CP}^n)$.

(4) Let M_g^2 be an oriented surface of genus g. Compute $H_q(M_g^2)$.

(5) Let N_g^2 be an non-oriented surface of genus g. Compute $H_q(N_g^2)$.

(6) Construct a map $f : S^{2n-1} \to S^{2n-1}$ without fixed points.

(7) Let $f, g : S^n \to S^n$ be two maps. Assume that $f(x) \neq -g(x)$ for all $x \in S^n$. Prove that $f \sim g$.

(8) Let $f : S^n \to S^n$ be a map with $\deg f \neq (-1)^{n+1}$. Then there exists $x \in S^n$ with $f(x) = x$.

(9) Let $f : S^{2n} \to S^{2n}$ be a map. Prove that there exists a point $x \in S^{2n}$ such that either $f(x) = x$ or $f(x) = -x$.

(10) Let $f : S^n \to S^n$ be a map of degree zero. Prove that there exist two points $x, y \in S^n$ with $f(x) = x$ and $f(y) = -y$.

(11) Construct a surjective map $f : S^n \to S^n$ of degree zero.

(12) Let $n \neq k$. Prove that the spaces $S^n \times \mathbf{RP}^k$ and $\mathbf{RP}^n \times S^k$ are not homotopy equivalent.