Homework # 12. Due to Wednesday, February 5, 11:00 am

(1) Prove that Whitehead product is natural.

(2) Prove that if $\alpha \in \pi_1(X)$, $\beta \in \pi_1(X)$, then $[\alpha, \beta] = \alpha\beta\alpha^{-1}\beta^{-1}$.

(3) Prove that the embedding $i : S^n \vee S^k \subset S^n \times S^k$ induces a surjection

$$i_* : \pi_{n+k}(S^n \vee S^k) \to \pi_{n+k}(S^n \times S^k).$$

(4) Prove the isomorphism $\pi_{n+k}(S^{n+1} \vee S^{k+1}) \cong \pi_{n+k}(S^{n+1}) \oplus \pi_{n+k}(S^{k+1})$.

(5) Prove that the suspension $\Sigma(S^n \times S^k)$ is homotopy equivalent to the wedge $S^{n+1} \vee S^{k+1} \vee S^{n+k+1}$.

(6) Prove that the homotopy groups of the spaces $S^3 \times \mathbf{CP}^\infty$ and S^2 are isomorphic, and prove that they are not homotopy equivalent.

(7) Consider the map

$$g : S^{2n-2} \times S^3 \xrightarrow{\text{projection}} (S^{2n-2} \times S^3)/(S^{2n-2} \vee S^3) = S^{2n+1} \xrightarrow{\text{Hopf}} \mathbf{CP}^n.$$

Prove that g induces trivial homomorphism in homotopy groups, however g is not homotopic to a constant map.