Homework # 2. Due to Wednesday, October 23, 11:00 am

(1) Define the Stiefel manifolds $V_k(\mathbf{R}^n)$. Prove the homeomorphisms:

$$V_n(\mathbf{R}^n) \cong O(n), \quad V_{n-1}(\mathbf{R}^n) \cong SO(n), \quad V_1(\mathbf{R}^n) \cong S^{n-1}.$$

(2) Prove the homeomorphisms:

$$V_n(\mathbf{C}^n) \cong U(n), \quad V_{n-1}(\mathbf{C}^n) \cong SU(n).$$

(3) Construct a homeomorphism $SO(4) \cong SO(3) \times S^3$.

(4) Define a natural actions of the groups $O(k)$ and $U(k)$ on the spaces the $V_k(\mathbf{R}^n)$ and $V_k(\mathbf{C}^n)$ respectively.

    (a) Prove that those actions are free.

    (b) Prove the homeomorphisms:

$$V_k(\mathbf{R}^n)/O(k) \cong G_k(\mathbf{R}^n), \quad V_k(\mathbf{C}^n)/U(k) \cong G_k(\mathbf{C}^n)$$

(5) Let X, Y, Z be Hausdorff and locally-compact topological spaces. Define the compact-open topology on $\mathcal{C}(X, Y)$. Prove the homeomorphism:

$$\mathcal{C}(X, \mathcal{C}(Y, Z)) \cong \mathcal{C}(X \times Y, Z).$$

Prove that this homeomorphism is natural.

(6) Define smash-product $X \wedge Y$. Prove that $S^n \wedge S^k \cong S^{n+k}$ (as pointed spaces).

(7) Prove that the maps $\phi^* : [X', Y] \longrightarrow [X, Y]$, $\psi_* : [X, Y] \longrightarrow [X, Y']$ induced by maps $\phi : X \longrightarrow X'$, $\psi : Y \longrightarrow Y'$ are well-defined.

(8) Define a cylinder and a cone of a map $f : X \longrightarrow Y$. Prove that the cone of the Hopf map $H : S^{4n+3} \longrightarrow \mathbf{HP}^n$ is homeomorphic to $\mathbf{HP}^{n+1}$.

(9) Let X be a projective plane $\mathbf{RP}^2$ with k small (open) disjoint disks D_i^2 ($i = 1, \ldots, k$) deleted. Let Y be the space obtained by gluing X with k Mëbious bands along the boundary circles. Which surface is that?