**Homework # 2. Due to Monday October 15, 9:00 am**

**(1)** Define the Stiefel manifolds $V(n,k)$. Prove the homeomorphisms:

$$V(n,n) \cong O(n), \quad V(n,n-1) \cong SO(n), \quad V(n,1) \cong S^{n-1}.$$

**(2)** Prove the homeomorphisms:

$$\mathbf{C}V(n,n) \cong U(n), \quad \mathbf{C}V(n,n-1) \cong SU(n).$$

**(3)** Construct a homeomorphism $SO(4) \cong SO(3) \times S^3$.

**(4)** Prove the following homeomorphisms:
$S^{n-1} \cong O(n)/O(n-1) \cong SO(n)/SO(n-1)$,
$S^{2n-1} \cong U(n)/U(n-1) \cong SU(n)/SU(n-1)$,
$G(n,k) \cong O(n)/O(k) \times O(n-k)$,
$\mathbf{C}G(n,k) \cong U(n)/U(k) \times U(n-k)$.

**(5)** Define the compact-open topology on $\mathcal{C}(X,Y)$. Prove the homeomorphism:

$$\mathcal{C}(X, \mathcal{C}(Y,Z)) \cong \mathcal{C}(X \times Y, Z)$$

provided $X$, $Y$ and $Z$ are Hausdorff and locally compact topological spaces. Prove that this homeomorphism is natural.

**(6)** Define smash-product $X \wedge Y$. Prove that $S^n \wedge S^k \cong S^{n+k}$ (as pointed spaces).

**(7)** Prove that the maps $\phi^* : [X', Y] \to [X, Y]$, $\psi_* : [X, Y] \to [X, Y']$ induced by maps $\phi : X \to X'$, $\psi : Y \to Y'$ are well-defined.

**(8)** Define a cylinder and a cone of a map $f : X \to Y$. Prove that the cone of the Hopf map $H : S^{4n+3} \to \mathbf{HP}^n$ is homeomorphic to $\mathbf{HP}^{n+1}$.

**(9)** Let $X$ be a projective plane $\mathbf{RP}^2$ with $k$ small (open) disjoint disks $D_i^2$ $(i = 1, \dots, k)$ deleted. Let $Y$ be the space obtained by gluing $X$ with $k$ Mëbious bands along the boundary circles. Which surface is that?

**(10*)**[1] Consider a subspace $X$ of $S^5$, given by the equation

$$x_1x_6 - x_2x_5 + x_3x_4 = 0,$$

where $S^5 \subset \mathbf{R}^6$ is given by $x_1^2 + \cdots + x_6^2 = 1$. Prove that $X \cong S^2 \times S^2$.

[1] Usually "*" means that the excercise is optional.