(1) Define the Stiefel manifolds $V(n, k)$. Prove the homeomorphisms:

$$V(n, n) \cong O(n), \quad V(n, n - 1) \cong SO(n), \quad V(n, 1) \cong S^{n-1}.$$ 

(2) Prove the homeomorphisms:

$$CV(n, n) \cong U(n), \quad CV(n, n - 1) \cong SU(n).$$ 

(3) Construct a homeomorphism $SO(4) \cong SO(3) \times S^3$.

(4) Prove the following homeomorphisms:

$$S^{n-1} \cong O(n)/O(n - 1) \cong SO(n)/SO(n - 1),$$
$$S^{2n-1} \cong U(n)/U(n - 1) \cong SU(n)/SU(n - 1),$$
$$G(n, k) \cong O(n)/O(k) \times O(n - k),$$
$$CG(n, k) \cong U(n)/U(k) \times U(n - k).$$

(5) Define the compact-open topology on $\mathcal{C}(X, Y)$. Prove the homeomorphism:

$$\mathcal{C}(X, \mathcal{C}(Y, Z)) \cong \mathcal{C}(X \times Y, Z)$$

provided $X$, $Y$ and $Z$ are Hausdorff and locally compact topological spaces. Prove that this homeomorphism is natural.

(6) Define smash-product $X \wedge Y$. Prove that $S^n \wedge S^k \cong S^{n+k}$ (as pointed spaces).

(7) Prove that the maps $\phi^* : [X', Y] \to [X, Y]$, $\psi_* : [X, Y] \to [X, Y']$ induced by maps $\phi : X \to X'$, $\psi : Y \to Y'$ are well-defined.

(8) Define a cylinder and a cone of a map $f : X \to Y$. Prove that the cone of the Hopf map $H : S^{4n+3} \to \mathbb{HP}^n$ is homeomorphic to $\mathbb{HP}^{n+1}$.

(9) Let $X$ be a projective plane $\mathbb{RP}^2$ with $k$ small (open) disjoint disks $D^2_i$ ($i = 1, \ldots, k$) deleted. Let $Y$ be the space obtained by gluing $X$ with $k$ Möbius bands along the boundary circles. Which surface is that?

(10$^*$) Consider a subspace $X$ of $S^5$, given by the equation

$$x_1x_6 - x_2x_5 + x_3x_4 = 0,$$

where $S^5 \subset \mathbb{R}^6$ is given by $x_1^2 + \cdots + x_6^2 = 1$. Prove that $X \cong S^2 \times S^2$.

$^1$Usually "$^*$" means that the exercise is optional.