Homework # 1. Due to Friday, October 5, 9:00 a.m.

(1) Do Exercises # 5.1 (p. 15), # 7.6 (p. 28) and # 8.1, 8.2 (p. 34) from Massey.

(2) Give a detailed proof of Lemma 7.1 (p. 26) from Massey.

(3) Define the quaternionic projective space $\mathbb{HP}^n$. Prove that $\mathbb{HP}^1$ is homeomorphic to $S^4$.

(4) Prove that the projective spaces $\mathbb{RP}^n$, $\mathbb{CP}^n$, $\mathbb{HP}^n$ are connected and compact.

(5) Define the Hopf maps $h : S^{2n+1} \to \mathbb{CP}^n$ and $H : S^{4n+3} \to \mathbb{HP}^n$. Prove that $h^{-1}(x) \cong S^1$ for each $x \in \mathbb{CP}^n$ and $H^{-1}(x) \cong S^3$ for each $x \in \mathbb{HP}^n$.

(6) Prove that the Grassmannian manifolds $G(n,k)$ and $CG(n,k)$ are compact and connected.

(7) Prove that the spaces $O(n)$, $SO(n)$, $U(n)$, $SU(n)$, $Sp(n)$ are compact. How many connected components does each of these spaces have?

(8) Find dimensions of the manifolds $G(n,k)$, $CG(n,k)$, $O(n)$, $SO(n)$, $U(n)$, $SU(n)$, $Sp(n)$.

(9) Give a detailed proof that $SO(3)$ is homeomorphic to $\mathbb{RP}^3$.

(10) Prove that each matrix $A \in SU(2)$ may be presented as:

$$A = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}, \text{ where } \alpha, \beta \in \mathbb{C}, \ |\alpha|^2 + |\beta|^2 = 1$$

Use this presentation to prove that $SU(2)$ is homeomorphic to $S^3$. 

(9) We identify $\mathbb{RP}^3$ with the disk $D^3/\sim$, where $D^3$ is given by the inequality
\[ x_1^2 + x_2^2 + x_3^2 \leq \sqrt{\pi}, \]
and the equivalence relation $\sim$ is given as
\[(x_1, x_2, x_3) \sim (y_1, y_2, y_3) \text{ if and only if } \begin{cases} x_1^2 + x_2^2 + x_3^2 = y_1^2 + y_2^2 + y_3^2 = \sqrt{\pi} \\ (x_1, x_2, x_3) = (-y_1, -y_2, -y_3). \end{cases} \]

Now we choose a standard orthonormal basis $(e_1, e_2, e_3)$ of $\mathbb{R}^3$. Let $\alpha \in SO(3)$. Then there exists a unique orthonormal basis $(v_1, v_2, v_3)$ of $\mathbb{R}^3$ such that a matrix $A$ of $\alpha$ (with respect to the basis $(v_1, v_2, v_3)$) is given as
\[ A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \phi & -\sin \phi \\ 0 & \sin \phi & \cos \phi \end{bmatrix}. \]

Here $v_1$ spans a line $L = \langle v_1 \rangle$ which is fixed by $\alpha$, and $\phi$ is the angle of rotation in the plane $\langle v_2, v_3 \rangle = L^\perp$. 