Homework # 1. Due to Friday, October 5, 9:00 a.m.

(1) Do Exercises # 5.1 (p. 15), # 7.6 (p. 28) and # 8.1, 8.2 (p. 34) from Massey.

(2) Give a detailed proof of Lemma 7.1 (p. 26) from Massey.

(3) Define the quaternionic projective space $\mathbf{HP}^n$. Prove that $\mathbf{HP}^1$ is homeomorphic to S^4.

(4) Prove that the projective spaces $\mathbf{RP}^n$, $\mathbf{CP}^n$, $\mathbf{HP}^n$ are connected and compact.

(5) Define the Hopf maps $h : S^{2n+1} \to \mathbf{CP}^n$ and $H : S^{4n+3} \to \mathbf{HP}^n$. Prove that $h^{-1}(x) \cong S^1$ for each $x \in \mathbf{CP}^n$ and $H^{-1}(x) \cong S^3$ for each $x \in \mathbf{HP}^n$.

(6) Prove that the Grassmannian manifolds $G(n,k)$ and $\mathbf{C}G(n,k)$ are compact and connected.

(7) Prove that the spaces $O(n)$, $SO(n)$, $U(n)$, $SU(n)$, $Sp(n)$ are compact. How many connected components does each of these spaces have?

(8) Find dimensions of the manifolds $G(n,k)$ $\mathbf{C}G(n,k)$, $O(n)$, $SO(n)$, $U(n)$, $SU(n)$, $Sp(n)$.

(9) Give a detailed proof that $SO(3)$ is homeomorphic to $\mathbf{RP}^3$.

(10) Prove that each matrix $A \in SU(2)$ may be presented as:

$$A = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}, \text{where } \alpha, \beta \in \mathbf{C},\ |\alpha|^2 + |\beta|^2 = 1$$

Use this presentation to prove that $SU(2)$ is homeomorphic to S^3.

Solutions.

(9) We identify $\mathbf{RP}^3$ with the disk $D^3/\sim$, where D^3 is given by the inequality

$$x_1^2 + x_2^2 + x_3^2 \leq \sqrt{\pi},$$

and the equivalence relation $\sim$ is given as

$$(x_1, x_2, x_3) \sim (y_1, y_2, y_3) \quad \text{if and only if} \quad \begin{cases} x_1^2 + x_2^2 + x_3^2 = y_1^2 + y_2^2 + y_3^2 = \sqrt{\pi} \\ \\ (x_1, x_2, x_3) = (-y_1, -y_2, -y_3). \end{cases}$$

Now we choose a standard orthonormal basis (e_1, e_2, e_3) of $\mathbf{R}^3$. Let $\alpha \in SO(3)$. Then there exists a unique orthonormal basis (v_1, v_2, v_3) of $\mathbf{R}^3$ such that a matrix A of α (with respect to the basis (v_1, v_2, v_3)) is given as

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix}.$$

Here v_1 spans a line $L = \langle v_1 \rangle$ which is fixed by α, and ϕ is the angle of rotation in the plane $\langle v_2, v_3 \rangle = L^{\perp}$.