NOTES ON THE COURSE "ALGEBRAIC TOPOLOGY", 2016-2017

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Contents

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1. Important examples of topological spaces

1.1. **Euclidian space, spheres, disks.** The notations $\mathbf{R}^n$, $\mathbf{C}^n$ have usual meaning throughout the course. The space $\mathbf{C}^n$ is identified with $\mathbf{R}^{2n}$ by the correspondence

$$(x_1 + iy_1, \dots, y_n + ix_n) \longleftrightarrow (x_1, y_1, \dots, x_n, y_n).$$

The unit sphere in $\mathbf{R}^{n+1}$ centered in the origin is denoted by S^n, the unit disk in $\mathbf{R}^n$ by D^n, and the unit cube in $\mathbf{R}^n$ by I^n. Thus S^{n-1} is the *boundary* of the disk D^n. Just in case we give these spaces in coordinates:

$$\begin{array}{rcl} S^{n-1} & = & \left\{(x_1,\dots,x_n)\in\mathbf{R}^n \;\mid\; x_1^2+\cdots+x_n^2=1\right\}, \\[2ex] D^n & = & \left\{(x_1,\dots,x_n)\in\mathbf{R}^n \;\mid\; x_1^2+\cdots+x_n^2\le 1\right\}, \\[2ex] I^n & = & \{(x_1,\dots,x_n)\in\mathbf{R}^n \;\mid\; 0\le x_j\le 1,\ j=1,\dots,n\}. \end{array} \tag{1}$$

The symbol $\mathbf{R}^\infty$ is a union (direct limit) of the embeddings

$$\mathbf{R}^1 \subset \mathbf{R}^2 \subset \cdots \subset \mathbf{R}^n \subset \cdots .$$

Thus a point $x \in \mathbf{R}^\infty$ is a sequence of points $x = (x_1, \dots, x_n, \dots)$, where $x_n \in \mathbf{R}$ and $x_j = 0$ for j greater then some k. Topology on $\mathbf{R}^\infty$ is determined as follows. A set $F \subset \mathbf{R}^\infty$ is closed, if each intersection $F \cap \mathbf{R}^n$ is closed in $\mathbf{R}^n$. Equivalently, a set $U \subset \mathbf{R}^\infty$ is open if each intersection $U \cap \mathbf{R}^n$ is open in $\mathbf{R}^n$. In a similar way we define the spaces $\mathbf{C}^\infty$ and S^∞.

Exercise 1.1. *Let* $x^{(1)} = (a_1, 0, \dots, 0, \dots)$, $\dots$, $x^{(n)} = (0, 0, \dots, a_n, \dots)$, $\dots$ *be a sequence of elements in* $\mathbf{R}^\infty$. *Prove that the sequence* $\left\{x^{(n)}\right\}$ *converges in* $\mathbf{R}^\infty$ *if and only if the* $a_j = 0$ *if* $j \geq k$ *for some* k. [1]

Probably you already know the another version of infinite-dimensional real space, namely the Hilbert space ℓ_2 (which is the set of sequences $\{x_n\}$ so that the series $\sum_n x_n$ converges). The space ℓ_2 is a metric space, where the distance $\rho(\{x_n\}, \{y_n\})$ is defined as

$$\rho(\{x_n\}, \{y_n\}) = \sqrt{\textstyle\sum_n (y_n - x_n)^2}.$$

Clearly there is a natural map $\mathbf{R}^\infty \longrightarrow \ell_2$.

Exercise 1.2. *Is the above map* $\mathbf{R}^\infty \longrightarrow \ell_2$ *homeomorphism or not?*

Consider the unit cube I^∞ in the spaces $\mathbf{R}^\infty$, ℓ_2, i.e. $I^\infty = \{\{x_n\} \mid 0 \le x_n \le 1\}$.

Exercise 1.3. *Prove or disprove that the cube* I^∞ *is compact space (in* $\mathbf{R}^\infty$ *or* ℓ_2*).*

[1] Assume $a_n \neq 0$ for infinite number of indices and $\lim_{j\to\infty} a_j = 0$. Assume that $\lim_{j\to\infty} x^{(j)} = \mathbf{0} = (0, 0, \dots) \in \mathbf{R}^\infty$. Define the set $U = \{(x_1, \dots, x_k, \dots) \mid \ |x_j| < |a_j| \ \text{ if } a_j \neq 0\ \}$. Then by definition, U is open in $\mathbf{R}^\infty$ since $U \cap \mathbf{R}^n$ is open in $\mathbf{R}^n$. Notice that U is an open neighborhood of $\mathbf{0}$, however, U does not contain any element $x^{(j)}$ if $a_j \neq 0$.

We are going to play a little bit with the sphere S^n.

Claim 1.1. *A punctured sphere $S^n \setminus \{x_0\}$ is homeomorphic to $\mathbf{R}^n$.*

Proof. We construct a map $f : S^n \setminus \{x_0\} \longrightarrow \mathbf{R}^n$ which is known as *stereographic projection*. Let S^n be given as above (1). Let the point x_0 be the North Pole, so it has the coordinates $(0,\dots,0,1) \in \mathbf{R}^{n+1}$. Consider a point $x = (x_1,\dots,x_{n+1}) \in S^n$, $x \neq x_0$, and the line going through the points x and x_0. A directional vector of this line may be given as $\vec{v} = (-x_1,\dots,-x_n, 1-x_{n+1})$, so any point of this line could be written as

$$(0,\dots,0,1) + t(-x_1,\dots,-x_n, 1-x_{n+1}) = (-tx_1,\dots,-tx_n, 1+t(1-x_{n+1})).$$

The intersection point of this line and $\mathbf{R}^n = \{(x_1,\dots,x_n,0)\} \subset \mathbf{R}^{n+1}$ is determined by vanishing the last coordinate. Clearly the last coordinate vanishes if $t = -\frac{1}{1-x_{n+1}}$. The map $f : S^n \setminus \{pt\} \longrightarrow \mathbf{R}^n$ is given by

$$f : (x_1,\dots,x_{n+1}) \mapsto \left(\frac{x_1}{1-x_{n+1}}, \dots, \frac{x_n}{1-x_{n+1}}, 0 \right).$$

The rest of the proof is left to you. $\square$

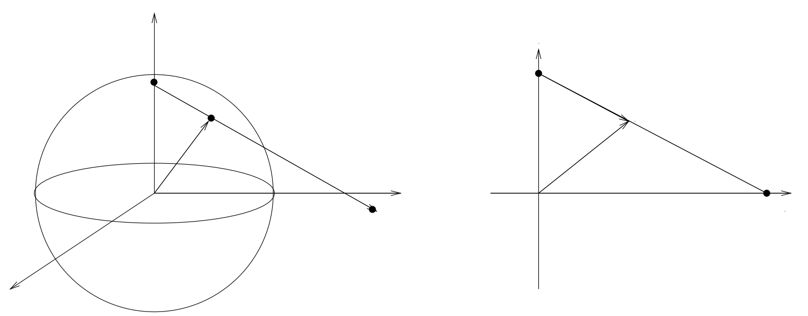


FIGURE 1. Stereographic projection

We define a hemisphere $S^n_+ = \left\{ x_1^2 + \cdots + x_{n+1}^2 = 1 \;\&\; x_{n+1} \geq 0 \right\}$.

Exercise 1.4. *Prove the that S^n_+ and D^n are homeomorphic.*

1.2. **Real projective spaces.** A real projective space $\mathbf{RP}^n$ is a set of all lines in $\mathbf{R}^{n+1}$ going through $0 \in \mathbf{R}^{n+1}$. Let $\ell \in \mathbf{RP}^n$ be a line, then we define a basis for topology on $\mathbf{RP}^n$ as follows:

$$U_\epsilon(\ell) = \left\{ \ell' \mid \text{the angle between } \ell \text{ and } \ell' \text{ less then } \epsilon \right\}.$$

Exercise 1.5. *The projective space $\mathbf{RP}^1$ is homeomorphic to the circle S^1.*

Let $(x_1,\dots,x_{n+1})$ be coordinates of a vector parallel to ℓ, then the vector $(\lambda x_1,\dots,\lambda x_{n+1})$ defines the same line ℓ (for $\lambda \neq 0$). We identify all these coordinates, the equivalence class is called *homogeneous coordinates* $(x_1 : \cdots : x_{n+1})$. Note that there is at least one x_i which is not zero. Let

$$U_j = \{ \ell = (x_1 : \cdots : x_{n+1}) \mid x_j \neq 0 \} \subset \mathbf{RP}^n$$

Then we define the map $f_j^{\mathbf{R}} : U_j \longrightarrow \mathbf{R}^n$ by the formula

$$(x_1 : \dots : x_{n+1}) \longrightarrow \left(\frac{x_1}{x_j}, \frac{x_2}{x_j}, \dots, \frac{x_{j-1}}{x_j}, 1, \frac{x_{j+1}}{x_j}, \dots, \frac{x_{n+1}}{x_i} \right).$$

Remark. The map $f_j^{\mathbf{R}}$ is a homeomorphism, it determines a *local coordinate system* in $\mathbf{RP}^n$ giving this space a structure of *smooth manifold* of dimension n.

There is natural map $c : S^n \longrightarrow \mathbf{RP}^n$ which sends each point $s = (s_1, \dots, s_{n+1}) \in S^n$ to the line going through zero and s. Note that there are exactly two points s and $-s$ which map to the same line $\ell \in \mathbf{RP}^n$. We have a chain of embeddings

$$\mathbf{RP}^1 \subset \mathbf{RP}^2 \subset \cdots \subset \mathbf{RP}^n \subset \mathbf{RP}^{n+1} \subset \cdots,$$

we define $\mathbf{RP}^\infty = \bigcup_{n \geq 1} \mathbf{RP}^n$ with the *limit topology* (similarly to the above case of $\mathbf{R}^\infty$).

1.3. **Complex projective spaces.** Let $\mathbf{CP}^n$ be the space of all complex lines in the complex space $\mathbf{C}^{n+1}$. In the same way as above we define *homogeneous coordinates* $(z_1 : \dots : z_{n+1})$ for each complex line $\ell \in \mathbf{CP}^n$, and the "local coordinate system":

$$U_i = \{ \ell = (z_1 : \dots : z_n) \mid z_i \neq 0 \} \subset \mathbf{CP}^n.$$

Clearly there is a homemorphism $f_i^{\mathbf{C}} : U_i \longrightarrow \mathbf{C}^{n+1}$.

Exercise 1.6. *Prove that the projective space $\mathbf{CP}^1$ is homeomorphic to the sphere S^2.*

Consider the sphere $S^{2n+1} \subset \mathbf{C}^{n+1}$. Each point

$$z = (z_1, \dots, z_{n+1}) \in S^{2n-1}, \quad |z_1|^2 + \cdots + |z_{n+1}|^2 = 1$$

of the sphere S^{2n+1} determines a line $\ell = (z_1 : \dots : z_{n+1}) \in \mathbf{CP}^n$. Observe that the point $e^{i\varphi} z = (e^{i\varphi} z_1, \dots, e^{i\varphi} z_{n+1}) \in S^{2n+1}$ determines the same complex line $\ell \in \mathbf{CP}^n$. We have defined the *Hopf map* $h_n : S^{2n+1} \longrightarrow \mathbf{CP}^n$.

Exercise 1.7. *Prove that the map $h_n : S^{2n+1} \longrightarrow \mathbf{CP}^n$ has a property that $h_n(\ell) = S^1$ for any $\ell \in \mathbf{CP}^n$.*

Remark. The case $n = 1$ is very interesting since $\mathbf{CP}^1 = S^2$, here we have the map $h_1 : S^3 \longrightarrow S^2$ where $h_1^{-1}(x) = S^1$ for any $x \in S^2$. This map is dicovered by Heinz Hopf in 1928-29, and h_1 gives very important example of nontrivial map $S^3 \longrightarrow S^2$. Before this discovery, mathematicians thought that there are no nontrivial maps $S^k \longrightarrow S^n$ for $k > n$ ("trivial map" means a map homotopic to the constant map).

Exercise 1.8. *Prove that $\mathbf{RP}^n$, $\mathbf{CP}^n$ are compact and connected spaces.*

Besides the real nubers $\mathbf{R}$ and complex numbers $\mathbf{C}$ there are *quaternion numbers* $\mathbf{H}$. Recall that $q \in \mathbf{H}$ may be thought as a sum $q = a + ib + jc + kd$, where $a, b, c, d \in R$, and the symbols i, j, k satisfy the identities:

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.$$

Then two quaternions $q_1 = a_1 + ib_1 + jc_1 + kd_1$ and $q_2 = a_2 + ib_2 + jc_2 + kd_2$ may be multiplied using these identies. Clearly we can identify $\mathbf{H}^n$ with $\mathbf{R}^{4n}$. We identify $\mathbf{H}$ with $\mathbf{R}^4$, then the set of *unit quaternions* $Sp(1) = \left\{ \ q = a + ib + jc + kd \mid a^2 + b^2 + c^2 + d^2 = 1 \ \right\}$ coinsides with the sphere $S^3 \subset \mathbf{R}^4$.

The product of quaternions is associative, but not commutative. However one can choose left or right multiplication to define a line in $\mathbf{H}^{n+1}$. A set of all quaternionic lines in $\mathbf{H}^{n+1}$ is the quaternion projective space $\mathbf{HP}^n$. We identify $\mathbf{H}^{n+1} \cong \mathbf{R}^{4(n+1)}$, then every line $\ell \in \mathbf{HP}^n$ is given by a non-zero vector of quaternions $(q_1, \dots, q_{n+1}) \in \mathbf{H}^{n+1} \cong \mathbf{R}^{4(n+1)}$, and, by scaling, we can assume that $|q_1|^2 + \ldots + |q_{n+1}|^2 = 1$. Thus every point $(q_1, \dots, q_{n+1}) \in S^{4n+3}$ of unit sphere determines a quaternionic line $\ell \in \mathbf{HP}^n$. This defines another *Hopf map* $H_n : S^{4n+3} \to \mathbf{HP}^n$.

Exercise 1.9. *Prove that* $\mathbf{HP}^1$ *is homeomorphic to* S^4. *Then prove that map* $H_1 : S^7 \to \mathbf{HP}^1 \cong S^4$ *has the property that* $H_1^{-1}(\ell) \cong S^3$ *for each* $\ell \in \mathbf{HP}^1$.

1.4. **Grassmannian manifolds.** These spaces generalize the projective spaces. Indeed, the space $G_k(\mathbf{R}^n)$ is a space of all k-dimensional vector subpaces of $\mathbf{R}^n$ with natural topology. Clearly $G_1(\mathbf{R}^n) = \mathbf{RP}^{n-1}$. It is not difficult to introduce local coordinates in $G_k(\mathbf{R}^n)$. Let $\pi \in G_k(\mathbf{R}^n)$ be a k-plane. Choose k linearly independent vectors $v_1, \dots, v_k$ generating π and write their coordinates in the standard basis $e_1, \dots, e_n$ of $\mathbf{R}^n$:

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{k1} & \cdots & a_{kn} \end{pmatrix}$$

Since the vectors $v_1, \dots, v_k$ are linearly independent there exist k columns of the matrix A which are linearly independent as well. In other words, there are indices $i_1, \dots, i_k$ so that a projection of the plane π on the k-plane $\langle e_{i_1}, \dots, e_{i_k} \rangle$ generated by the coordinate vectors $e_{i_1}, \dots, e_{i_k}$ is a linear isomorphism. Now it is easy to introduce local coordinates on the Grassmanian manifold $G_k(\mathbf{R}^n)$. Indeed, choose the indices $i_1, \dots, i_k$, $1 \leq i_1 < \cdots < i_k \leq n$, and consider all k-planes $\pi \in G(n, k)$ so that the projection of π on the plane $\langle e_{i_1}, \dots, e_{i_k} \rangle$ is a linear isomorphism. We denote this set of k-planes by $U_{i_1,\dots,i_k}$.

Exercise 1.10. *Construct a homeomorphism* $f_{i_1,\dots,i_k} : U_{i_1,\dots,i_k} \longrightarrow \mathbf{R}^{k(n-k)}$.

The result of this exercise shows that the Grassmannian manifold $G_k(\mathbf{R}^n)$ is a smooth manifold of dimension $k(n-k)$. The projective spaces and Grassmannian manifolds are very important examples of spaces which we will see many times in our course.

Exercise 1.11. *Define a complex Grassmannian manifold $G_k(\mathbf{C}^n)$ and construct a local coordinate system for $G_k(\mathbf{C}^n)$. In particular, find its dimension.*

We have a chain of spaces:

$$G_k(\mathbf{R}^k) \subset G_k(\mathbf{R}^{k+1}) \subset \cdots \subset G_k(\mathbf{R}^n) \subset G_k(\mathbf{R}^{n+1}) \subset \cdots .$$

Let $G(\infty, k)$ be the union (inductive limit) of these spaces. The topology of $G_k(R^\infty)$ is given in the same way as to $\mathbf{R}^\infty$: a set $F \subset G_k(\mathbf{R}^\infty)$ is closed if and only if the intersection $F \cap G_k(\mathbf{R}^n)$ is closed for each n. This topology is known as a *topology of an inductive limit.*

Exercise 1.12. *Prove that the Grassmannian manifolds $G_k(\mathbf{R}^n)$ and $G_k(\mathbf{C}^n)$ are compact and connected.*

1.5. **Flag manifolds.** Here we just mention these examples without further considerations (we are not ready for this yet). Let $1 \leq k_1 < \cdots < k_s \leq n-1$. A *flag of the type* $(k_1, \ldots, k_s)$ is a chain of vector subspaces $V_1 \subset \cdots \subset V_s$ of $\mathbf{R}^n$ such that $\dim V_i = k_i$. A set of flags of the given type is the flag manifold $F(n; k_1, \ldots, k_s)$. Hopefully we shall return to these spaces: they are very interesting and popular creatures.

1.6. **Classic Lie groups.** The first example here is the group $GL(\mathbf{R}^n)$ of nondegenerated linear transformations of $\mathbf{R}^n$. Once we choose a basis $e_1, \ldots, e_n$ of $\mathbf{R}^n$, each element $A \in GL(\mathbf{R}^n)$ may be identified with an $n \times n$ matrix A with $\det A \neq 0$. Clearly we may identify the space of all $n \times n$ matrices with the space $\mathbf{R}^{n^2}$. The determinant gives a continuous function $\det : \mathbf{R}^{n^2} \longrightarrow \mathbf{R}$, and the space $GL(\mathbf{R}^n)$ is an open subset of $\mathbf{R}^{n^2}$:

$$GL(\mathbf{R}^n) = \mathbf{R}^{n^2} \setminus \det{}^{-1}(0).$$

In particular, this identification defines a topology on $GL(\mathbf{R}^k)$. In the same way one may construct an embedding $GL(\mathbf{C}^n) \subset \mathbf{C}^{n^2}$. The orthogonal and special orthogonal groups $O(k)$, $SO(k)$ are subgroups of $GL(\mathbf{R}^k)$, and the groups $U(k)$, $SU(k)$ are subgroups of $GL(\mathbf{C}^k)$. (Recall that $O(n)$ (or $U(n)$) is a group of those linear transformations of $\mathbf{R}^n$ (or $\mathbf{C}^n$) which preserve a Euclidian (or Hermitian) metric on $\mathbf{R}^n$ (or $\mathbf{C}^n$), and the groups $SO(k)$ and $SU(k)$ are subgroups of $O(k)$ and $U(k)$ of matrices with the determinant 1.)

Exercise 1.13. *Prove that $SO(2)$ and $U(1)$ are homeomorphic to S^1, and that $SO(3)$ is homeomorphic to $\mathbf{RP}^3$.*

Hint: To prove that $SO(3)$ is homeomorphic to $\mathbf{RP}^3$ you have to analyze $SO(3)$: the key fact is the geometric description of an orthogonal transformation $\alpha \in SO(3)$, it is given by rotating a plane (by an angle φ) about a line ℓ perpendicular to that plane. You should use the line ℓ and the angle φ as major parameters to construct a homeomorphism $SO(3) \to \mathbf{RP}^3$, where it is important to use a particular model of $\mathbf{RP}^3$, namely a disk D^3 where one identifies the opposite points on $S^2 = \partial D^3 \subset D^3$.

Exercise 1.14. *Prove that the spaces $O(n)$, $SO(n)$, $U(n)$, $SU(n)$ are compact.*

Exercise 1.15. *Prove that the space $O(n)$ has two path-connected components, and that the spaces $SO(n)$, $U(n)$, $SU(n)$ are path-connected.*

Exercise 1.16. *Prove that each matrix $A \in SU(2)$ may be presented as:*

$$A = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}, \textit{where } \alpha, \beta \in \mathbf{C},\ |\alpha|^2 + |\beta|^2 = 1$$

Use this presentation to prove that $SU(2)$ is homeomorphic to S^3.

It is important to emphasize that the classic groups $O(n)$, $SO(n)$, $U(n)$, $SU(n)$ are all *manifolds*, i.e. for each point α there there exists an open neighborhood homeomorphic to a Euclidian space.

Exercise 1.17. *Prove that the space for any point $\alpha \in SO(n)$ there exists an open neighborhood homeomorphic to the Euclidian space of the dimension $\frac{n(n-1)}{2}$.*

Exercise 1.18. *Prove that the spaces $U(n)$, $SU(n)$ are manifolds and find their dimension.*

The next set of examples is also very important.

1.7. **Stiefel manifolds.** Again, we consider the vector space $\mathbf{R}^n$. We call vectors $v_1, \ldots, v_k$ a k-*frame* if they are linearly independent. A k-frame $v_1, \ldots, v_k$ is called an *orthonormal k-frame* if the vectors $v_1, \ldots, v_k$ are of unit length and orthogonal to each other. The space of all orthonormal k-frames in $\mathbf{R}^n$ is denoted by $V_k(\mathbf{R}^n)$. There are analogous complex and quaternionic versions of these spaces, they are denoted as $V_k(\mathbf{C}^n)$ and $V_k(\mathbf{H}^n)$ respectively. Here is an exercise where your knowledge of basic linear algebra may be crucial:

Exercise 1.19. *Prove the following homeomorphisms: $V_n(\mathbf{R}^n) \cong O(n)$, $V_{n-1}(\mathbf{R}^n) \cong SO(n)$, $V_n(\mathbf{C}^n) \cong U(n)$, $V_{n-1}(\mathbf{C}^n) \cong SU(n)$, $V_1(\mathbf{R}^n) \cong S^{n-1}$, $V_1(\mathbf{C}^n) \cong S^{2n-1}$, $V_1(\mathbf{H}^n) \cong S^{4n-1}$.*

We note that the group $O(n)$ acts on the spaces $V_k(\mathbf{R}^n)$ and $G_k(\mathbf{R}^n)$: indeed, if $\alpha \in O(n)$ and $v_1, \ldots, v_k$ is an orthonormal k-frame, then $\alpha(v_1), \ldots, \alpha(v_k)$ is also an orthonormal k-frame. As for the Grassmannian manifold, one can easily see that $\alpha(\Pi) \subset \mathbf{R}^n$ is a k-dimensional subspace if $\Pi \in G_k(\mathbf{R}^n)$.

The group $O(n)$ contains a subgroup $O(j)$ which acts on $\mathbf{R}^j \subset \mathbf{R}^n$, where $\mathbf{R}^j = \langle e_1, \ldots, e_j \rangle$ is generated by the first j vectors $e_1, \ldots, e_j$ of the standard basis $e_1, \ldots, e_n$ of $\mathbf{R}^n$. Similarly $U(n)$ acts on the spaces $G_k(\mathbf{C}^n)$ and $V_k(\mathbf{C}^n)$, and $U(j)$ is a subgroup of $U(n)$.

Exercise 1.20. *Prove the following homeomorphisms:*

(a) $S^{n-1} \cong O(n)/O(n-1) \cong SO(n)/SO(n-1)$,

(b) $S^{2n-1} \cong U(n)/U(n-1) \cong SU(n)/SU(n-1)$,

(c) $G_k(\mathbf{R}^n) \cong O(n)/O(k) \times O(n-k)$,

(c) $G_k(\mathbf{C}^n) \cong U(n)/U(k) \times U(n-k)$.

We note here that $O(k) \times O(n-k)$ is a subgroup of $O(n)$ of orthogonal matrices with two diagonal blocks of the sizes $k \times k$ and $(n-k) \times (n-k)$ and zeros otherwise.

There is also the following natural action of the orthogonal group $O(k)$ on the Stieffel manifold $V_k(\mathbf{R}^n)$. Let $v_1, \ldots, v_k$ be an orthonormal k-frame then $O(k)$ acts on the space $V = \langle v_1, \ldots, v_k \rangle$, in particular, if $\alpha \in O(k)$, then $\alpha(v_1), \ldots, \alpha(v_k)$ is also an orthonormal k-frame. Similarly there is a natural action of $U(k)$ on $V_k(\mathbf{C}^n)$.

Exercise 1.21. *Prove that the above actions of $O(k)$ on $V_k(\mathbf{R}^n)$ and of $U(k)$ on $V_k(\mathbf{C}^n)$ are free.*

Exercise 1.22. *Prove the following homeomorphisms:*

(a) $V_k(\mathbf{R}^n)/O(k) \cong G_k(\mathbf{R}^n)$,

(b) $V_k(\mathbf{C}^n)/U(k) \cong G_k(\mathbf{C}^n)$.

There are obvious maps $V_k(\mathbf{R}^n) \xrightarrow{p} G_k(\mathbf{R}^n)$, $V_k(\mathbf{C}^n) \xrightarrow{p} \mathbf{C}G_k(\mathbf{C}^n)$ (where each orthonormal k-frame $v_1, \ldots, v_k$ maps to the k-plane $\pi = \langle v_1, \ldots, v_k \rangle$ generated by this frame). It is easy to see that the inverse image $p^{-1}(\pi)$ may be identified with $O(k)$ (in the real case) and $U(k)$ (in the complex case). We shall return to these spaces later on. In particular, we shall describe a cell-structure of these spaces and compute their homology and cohomology groups.

1.8. **Surfaces.** Here I refer to Chapter 1 of Massey, Algebraic topology, for details. I would like for you to read this Chapter carefully even though most of you have seen this material before. Here I briefly remind some constructions and give exercises. The section 4 of the reffered Massey book gives the examples of surfaces. In particular, the torus T^2 is described in three different ways:

(a) A product $S^1 \times S^1$.

(b) A subspace of $\mathbf{R}^3$ given by: $\left\{ (x,y,z) \in \mathbf{R}^3 \mid (\sqrt{x^2+y^2}-2)^2 + z^2 = 1 \right\}$.

(c) A unit square $I^2 = \left\{ (x,y) \in \mathbf{R}^2 \mid 0 \le x \le 1, 0 \le y \le 1 \right\}$ with the identification:

$$(x,0) \equiv (x,1) \quad (0,y) \equiv (1,y) \quad \text{for all } 0 \le x \le 1, \quad 0 \le y \le 1.$$

Exercise 1.23. *Prove that the spaces described in* **(a)**, **(b)**, **(c)** *are indeed homeomorphic.*

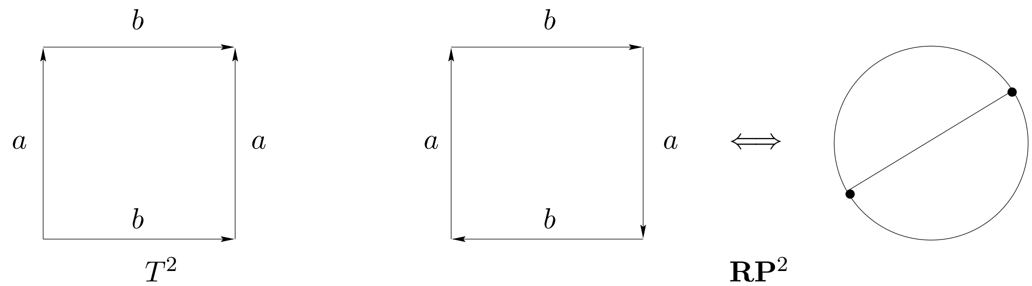


FIGURE 2. Torus and projective plane

The next surface we want to become our best friend is the projective space $\mathbf{RP}^2$. Earlier we defined $\mathbf{RP}^2$ as a space of lines in $\mathbf{R}^3$ going through the origin.

Exercise 1.24. *Prove that the projective plane* $\mathbf{RP}^2$ *is homeomorphic to the following spaces:*

(a) *The unit disk* $D^2 = \{(x,y) \in \mathbf{R}^2 \mid x^2+y^2 \leq 1\}$ *with the opposite points* $(x,y) \equiv (-x,-y)$ *of the circle* $S^1 = \{(x,y) \in \mathbf{R}^2 \mid x^2+y^2 = 1\} \subset D^2$ *have been identified.*
(b) *The unit square, see Fig. 3, with the arrows* a *and* b *identified as it is shown.*
(c) *The Mëbius band which boundary (the circle) is identified with the boundary of the disk* D^2*, see Fig. 3.*

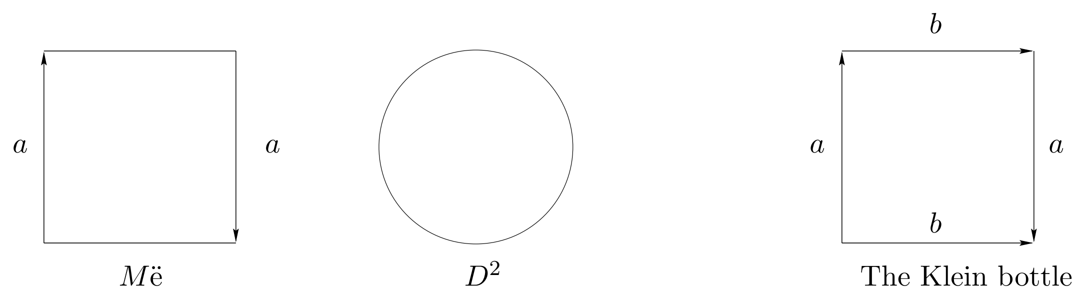


FIGURE 3

Here the Mëbius band is constructed from a square by identifying the arrows a. The Klein bottle Kl^2 may be described as a square with arrows identified as it is shown in Fig. 3.

Exercise 1.25. *Prove that the Klein bottle* Kl^2 *is homeomorphic to the union of two Mëbius bands along the circle.*

Massey carefully defines connected sum $S_1 \# S_2$ of two surfaces S_1 and S_2.

Exercise 1.26. *Prove that* $Kl^2 \# \mathbf{RP}^2$ *is homeomorphic to* $\mathbf{RP}^2 \# T^2$.

Exercise 1.27. *Prove that* $Kl^2 \# Kl^2$ *is homeomorphic to* $Kl^2 \# T^2$.

Exercise 1.28. *Prove that* $\mathbf{RP}^2 \# \mathbf{RP}^2$ *is homeomorphic to* Kl^2.

2. Constructions

2.1. **Product.** Recall that a product $X \times Y$ of X, Y is a set of pairs (x, y), $x \in X, y \in Y$. If X, Y are topological spaces then a *basis* for *product topology* on $X \times Y$ is given by the products $U \times V$, where $U \subset X$, $V \subset Y$ are open. Here are the first examples:

Example. The torus $T^n = S^1 \times \cdots \times S^1$ which is homeomorphic to $U(1) \times \cdots \times U(1) \subset U(n)$ (diagonal orthogonal complex matrices).

Exercise 2.1. (Challenging) *Consider the surface X in S^5, given by the equation*

$$x_1x_6 - x_2x_5 + x_3x_4 = 0$$

(where $S^5 \subset \mathbf{R}^6$ is given by $x_1^2 + \cdots + x_6^2 = 1$). Prove that $X \cong S^2 \times S^2$.

Exercise 2.2. *Prove that the space $SO(4)$ is homeomorphic to $S^3 \times \mathbf{RP}^3$.*

Hint: Consider carefully the map $SO(4) \longrightarrow S^3 = SO(4)/SO(3)$ and use the fact that S^3 has a natural group structure: it is a group of unit quaternions. It should be emphasized that $SO(n)$ is not homeomorphic to the product $S^{n-1} \times SO(n-1)$ if $n > 4$.

We note also that there are standard projections $X \times Y \xrightarrow{pr_X} X$ and $X \times Y \xrightarrow{pr_Y} Y$, and to give a map $f : Z \longrightarrow X \times Y$ is the same as to give two maps $f_X : Z \longrightarrow X$ and $f_Y : Z \longrightarrow Y$.

2.2. **Cylinder, suspension.** Let $I = [0, 1] \subset \mathbf{R}$. The space $X \times I$ is called a *cylinder over* X, and the subspaces $X \times \{0\}$, $X \times \{1\}$ are the bottom and top "bases". Now we will construct new spaces out of the cylinder $X \times I$.

Remark: quotient topology. Let "$\sim$" be an equivalence relation on the topological space X. We denote by $X/\sim$ the set of equivalence classes. There is a natural map (not continuos so far) $p : X \longrightarrow X/\sim$. We define the following topology on $X/\sim$: the set $U \subset X/\sim$ is open if and only if $p^{-1}(U)$ is open. This topology is called a *quotient topology.*

The first example: let $A \subset X$ be a closed set. Then we define the relation "$\sim$" on X as follows ([ ] denote an equivalence class):

$$[x] = \begin{cases} \{x\} & \text{if } x \notin A, \\ A & \text{if } x \in A. \end{cases}$$

The space $X/\sim$ is denoted by X/A.

The space $C(X) = X \times I/X \times \{1\}$ is a *cone over* X. A *suspention* ΣX over X is the space $C(X)/X \times \{0\}$.

Exercise 2.3. *Prove that the spaces $C(S^n)$ and ΣS^n are homeomorphic to D^{n+1} and S^{n+1} respectively.*

Here is a picture of these spaces:

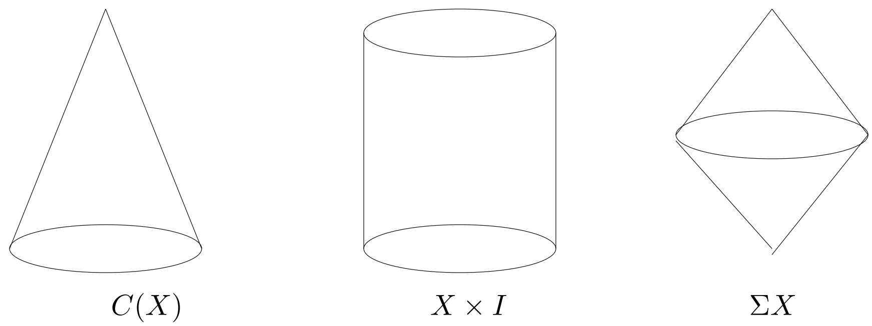


FIGURE 4

2.3. **Glueing.** Let X and Y be topological spaces, $A \subset Y$ and $\varphi : A \longrightarrow X$ be a map. We consider a disjoint union $X \cup Y$, and then we identify a point $a \in A$ with the point $\varphi(a) \in X$. The quotient space $X \cup Y/\sim$ under this identification will be denoted as $X \cup_\varphi Y$, and this procedure will be called *glueing* X *and* Y *by means of* φ. There are two special cases of this construction.

Let $f : X \longrightarrow Y$ be a map. We identify X with the bottom base $X \times \{0\}$ of the cylinder $X \times I$. The space $X \times I \cup_f Y = Cyl(f)$ is called a *cylinder of the map* f. The space $C(X) \cup_f Y$ is called a *cone of the map* f. Note that the space $Cyl(f)$ contains X and Y as subspaces, and the space $C(f)$ contains X.

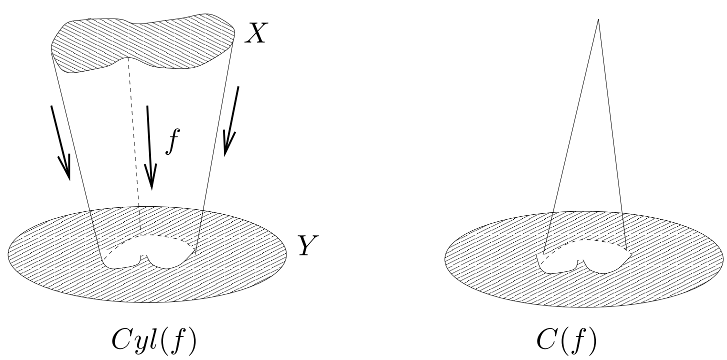


FIGURE 5

Let $f : S^n \longrightarrow \mathbf{RP}^n$ be the (we have studied before) map which takes a vector $\vec{v} \in S^n$ to the line $\ell = \langle \vec{v} \rangle$ spanned by $\vec{v}$.

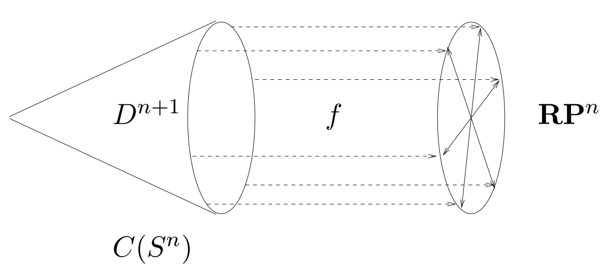


FIGURE 6

Claim 2.1. *The cone $C(f)$ is homeomorphic to the projective space $\mathbf{RP}^{n+1}$.*

Proof (outline). Consider the cone over S^n, clearly $C(S^n) \cong D^{n+1}$ (Exercise 2.3). Now the cone $C(f)$ is a disk D^{n+1} with the opposite points of S^n identified, see Fig. 6. $\square$

In particular, a cone of the map $f : S^1 \longrightarrow S^1 = \mathbf{RP}^1$ (given by the formula $e^{i\varphi} \mapsto e^{2i\varphi}$) coincides with the projective plane $\mathbf{RP}^2$.

Exercise 2.4. *Prove that a cone $C(h)$ of the Hopf map $h : S^{2n+1} \longrightarrow \mathbf{CP}^n$ is homeomorphic to the projective space $\mathbf{CP}^{n+1}$.*

Here is the construction which should help you with Exercise 2.4. Let us take one more look at the Hopf map $h : S^{2k+1} \longrightarrow \mathbf{CP}^k$: we take a point $(z_1, \dots, z_{k+1}) \in S^{2k+1}$, where $|z_1|^2+\cdots+|z_{k+1}|^2 = 1$, then h takes it to the line $(z_1 : \cdots : z_{k+1}) \in \mathbf{CP}^k$. Moreover $h(z_1, \dots, z_{k+1}) = (z'_1, \dots, z'_{k+1})$ if and only if $z'_j = e^{i\varphi} z_j$. Thus we can identify $\mathbf{CP}^k$ with the following quotient space:

$$\mathbf{CP}^k = S^{2k+1}/\sim, \quad \text{where } (z_1, \dots, z_{k+1}) \sim (e^{i\varphi} z_1, \dots, e^{i\varphi} z_{k+1}). \tag{2}$$

Now consider a subset of lines in $\mathbf{CP}^k$ where the last homogeneous coordinate is nonzero:

$$U_{k+1} = \{(z_1 : \cdots : z_{k+1}) \mid z_{k+1} \neq 0\}.$$

We already know that U_{k+1} is homeomorphic to $\mathbf{C}^k$ by means of the map

$$(z_1 : \cdots : z_{k+1}) \mapsto \left(\tfrac{z_1}{z_{k+1}}, \dots, \tfrac{z_k}{z_{k+1}}\right)$$

Now we use (2) to identify U_{k+1} with an open disk $D^{2k} \subset \mathbf{C}^k$ as follows. Let us think about $U_{k+1} \subset S^{2k+1}/\sim$ as above. Let $\ell \in U_{k+1}$. Choose a point $(z_1, \dots, z_{k+1}) \in S^{2k+1}$ representing ℓ. Then we have that

$$|z_1|^2 + \cdots + |z_{k+1}|^2 = 1, \quad \text{and} \quad z_{k+1} \neq 0.$$

A complex number z_{k+1} has a unique representation $z_{k+1} = re^{i\alpha}$, where $r = |z_{k+1}|$. Notice that $0 < r \leq 1$. Then the point

$$(e^{-i\alpha} z_1, \dots, e^{-i\alpha} z_k, e^{-i\alpha} z_{k+1}) = (e^{-i\alpha} z_1, \dots, e^{-i\alpha} z_k, r) \in S^{2k+1}$$

represents the same line $\ell \in U_{k+1}$. Moreover, this representation is unique. We have:

$$|z_1|^2 + \cdots + |z_k|^2 = 1 - r^2$$

which describes the sphere $S^{2k-1}_{\sqrt{1-r^2}} \subset \mathbf{C}^k$ of radius $\sqrt{1-r^2}$. The union of the spheres $S^{2k-1}_{\sqrt{1-r^2}}$ over $0 < r \leq 1$ is nothing but an open unit disk in $\mathbf{C}^k$. Then we notice that we can let z_{k+1} to be equal to zero: $z_{k+1} = 0$ corresponds to the points

$$(z_1, \dots, z_k, 0) \in S^{2k+1} \quad \text{with} \quad |z_1|^2 + \cdots + |z_k|^2 = 1,$$

i.e. the sphere $S^{2k-1} \subset \mathbf{C}^k$ modulo the equivalence relation $(z_1, \dots, z_k, 0) \sim (e^{i\varphi} z_1, \dots, e^{i\varphi} z_k, 0)$. This is nothing but the projective space $\mathbf{CP}^{k-1}$. We summarize our construction:

Lemma 2.1. *There is a homeomorphism*

$$\mathbf{CP}^k \equiv D^{2k}/\sim,$$

where $(z_1,\dots,z_k)\sim(z'_1,\dots,z'_k)$ *if and only if*

$$\begin{cases} |z_1|^2+\cdots+|z_k|^2=1,\ |z'_1|^2+\cdots+|z'_k|^2=1, & \textit{and} \\ z'_j=e^{i\varphi}z_j & \textit{for all } j=1,\dots,k. \end{cases}$$

2.4. **Join.** A *join* $X*Y$ of spaces X Y is a union of all linear paths $I_{x,y}$ starting at $x\in X$ and ending at $y\in Y$; the union is taken over all points $x\in X$ and $y\in Y$. For example, a joint of two intervals I_1 and I_2 lying on two non-parallel and non-intersecting lines is a tetrahedron: A formal

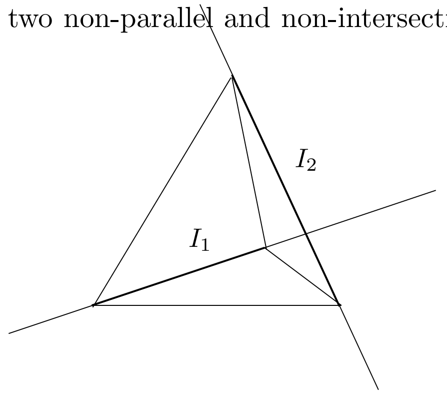


Figure 7

definition of $X*Y$ is the following. We start with the product $X\times Y\times I$: here there is a linear path (x,y,t), $t\in I$ for given points $x\in X$, $y\in Y$. Then we identify the following points:

$$\begin{array}{ll}(x,y',1)\sim(x,y'',1) & \text{for any } x\in X,\ y',y''\in Y,\\ (x',y,0)\sim(x'',y,0) & \text{for any } x',x''\in X,\ y\in Y.\end{array}$$

Exercise 2.5. *Prove the homeomorphisms*

- **(a)** $X*\{\textit{one point}\}\cong C(X)$*;*
- **(b)** $X*\{\textit{two points}\}\cong \Sigma(X)$*;*
- **(c)** $S^n*S^k\cong S^{n+k+1}$*. Hint: prove first that* $S^1*S^1\cong S^3$*.*

2.5. **Spaces of maps, loop spaces, path spaces.** Let X, Y are topological spaces. We consider the space $\mathcal{C}(X,Y)$ of all continuous maps from X to Y. To define a topology of the functional space $\mathcal{C}(X,Y)$ it is enough to describe a basis. The basis of the *compact-open topology* is given as follows. Let $K\subset X$ be a compact set, and $O\subset Y$ be an open set. We denote by $\mathcal{U}(K,O)$ the set of all continuous maps $f:X\longrightarrow Y$ such that $f(K)\subset O$, this is (by definition) a basis for the compact-open topology on $\mathcal{C}(X,Y)$.

Examples. Let X be a point. Then the space $\mathcal{C}(X,Y)$ is homeomorphic to Y. If X be a space consisting of n points, then $\mathcal{C}(X,Y)\cong Y\times\cdots\times Y$ (n times).

Let X, Y, and Z be Hausdorff and locally compact[2] topological spaces. There is a natural map

$$T : \mathcal{C}(X, \mathcal{C}(Y,Z)) \longrightarrow \mathcal{C}(X \times Y, Z),$$

given by the formula: $\{f : X \longrightarrow \mathcal{C}(Y,Z)\} \longrightarrow \{(x,y) \longrightarrow (f(x))(y)\}$.

Exercise 2.6. *Prove that the map $T : \mathcal{C}(X, \mathcal{C}(Y,Z)) \longrightarrow \mathcal{C}(X \times Y, Z)$ is a homeomorphism.*

Recall we call a map $f : I \longrightarrow X$ a path, and the points $f(0) = x_0$ $f(1) = x_1$ are the beginning and the end points of the path f. The space of all paths $\mathcal{C}(I,X)$ contains two important subspaces:

1. $\mathcal{E}(X, x_0, x_1)$ is the subspace of paths $f : I \longrightarrow X$ such that $f(0) = x_0$ and $f(1) = x_1$;
2. $\mathcal{E}(X, x_0)$ is the space of all paths with x_0 the begining point.
3. $\Omega(X, x_0) = \mathcal{E}(X, x_0, x_0)$ is the loop space with the begining point x_0.

Exercise 2.7. *Prove that the spaces $\Omega(S^n, x)$ and $\Omega(S^n, x')$ are homeomorphic for any points $x,\ x' \in S^n$.*

Exercise 2.8. *Give examples of a space X other than S^n for which $\Omega(X,x)$ and $\Omega(X,x')$ are homeomorphic for any points $x,\ x' \in X$. Why does it fail for an arbitrary space X? Give an example when this is not true.*

The loop spaces $\Omega(X,x)$ are rather difficult to describe even in the case of $X = S^n$, however, the spaces X and $\Omega(X,x)$ are intimately related. To see that, consider the following map

$$p : \mathcal{E}(X, x_0) \longrightarrow X \qquad (3)$$

which sends a path $f : I \longrightarrow X$, $f(0) = x_0$, to the point $x = f(1)$. Notice that $p^{-1}(x_0) \cong \Omega(X, x_0)$. The map (3) may be considered as a map of pointed spaces (see the definitions below):

$$p : (\mathcal{E}(X, x_0), *) \longrightarrow (X, *),$$

where the path $* : I \longrightarrow X$ sends the interval to the point $*(t) = x_0$ for all $t \in I$. Clearly $p(*) = x_0$.

2.6. **Pointed spaces.** A *pointed space* (X, x_0) is a topological space X together with a base point $x_0 \in X$. A map $f : (X, x_0) \longrightarrow (Y, y_0)$ is a continuous map $f : X \longrightarrow Y$ such that $f(x_0) = y_0$. Many operations preserve base points, for example the product $X \times Y$ of pointed spaces (X, x_0), (Y, y_0) have the base point $(x_0, y_0) \in X \times Y$. Some other operations have to be modified.

The cone $C(X, x_0) = C(X)/\{x_0\} \times I$: here we identify with the point all interval over the base point x_0, and the image of $\{x_0\} \times I$ in $C(X, x_0)$ is the base point of this space.

The suspension:

$$\Sigma(X, x_0) = \Sigma(X)/\{x_0\} \times I = C(X)/(X \times \{0\} \cup x_0 \times I) = C(X, x_0)/(X \times \{0\}).$$

[2] A topological space X is called *locally compact* if for each point $x \in X$ and an open neighborhood U of X there exists an open neighbourhood $V \subset U$ such that the closure $\overline{V}$ of V is compact.

The space of maps $\mathcal{C}(X, x_0, Y, y_0)$ for pointed spaces[3] (X, x_0) and (Y, y_0) is the space of continuous maps $f : X \longrightarrow Y$ such that $f(x_0) = y_0$ (with the same compact-open topology). The base point in the space $\mathcal{C}(X, Y)$ is the map $c : X \longrightarrow Y$ which sends all space X to the point $y_0 \in Y$.

If X is a pointed space, then $\Omega(X, x_0)$ is the space of loops begining and ending at the base point $x_0 \in X$, and the space $\mathcal{E}(X, x_0)$ is the space of paths starting at the base point x_0.

Exercise 2.9. *Let X and Y are pointed space. Prove that the space $C(\Sigma(X), Y)$ and the space $C(X, \Omega(Y))$ are homeomorphic.*

Exercise 2.10. *Let $S^1 = \{e^{i\varphi}\}$ be a circle and $s_0 = 1$ ($\varphi = 0$) be a base point. How many path-connected components does the space $\Omega(S^1)$ (a space of loops with s_0 the begining point) have? Try the same question for $\Omega(\mathbf{RP}^2)$.*

There are two more operations which are specific for pointed spaces.

1. A one-point-union (or a bouquet) $X \vee Y$ of pointed spaces (X, x_0) and (Y, y_0) is a disjoint union $X \cup Y$ with the points x_0 and y_0 identified, see Fig. 8.

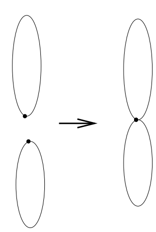


FIGURE 8

2. A smash-product $X \wedge Y$ is the factor-space: $X \wedge Y = X \times Y/((x_0 \times Y) \cup (X \times y_0))$, see Fig. 9:

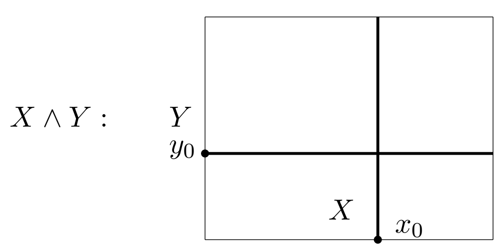


FIGURE 9

Exercise 2.11. *Prove that the space $S^n \wedge S^n$ is homeomorphic to S^{n+m} as pointed spaces.*

Exercise 2.12. *Prove that $X \wedge S^1$ is homeomorphic to $\Sigma(X)$ as pointed spaces.*

[3] We will denote this space by $\mathcal{C}(X, Y)$ when it is clear what the base points are.

Remark. We have mentioned several *natural* homeomorphisms, for instance, the homeomorphisms

(a) $\mathcal{C}(\Sigma(X), Y) \xrightarrow{F_{X,Y}} \mathcal{C}(X, \Omega(Y))$,

(b) $X \wedge S^1 \xrightarrow{G} \Sigma(X)$

are *natural*. We would like to give more details.

First, let $f : X \longrightarrow X'$, and $g : Y \longrightarrow Y'$ be maps of pointed spaces, then there the maps

$$f^* : \mathcal{C}(X', Y) \longrightarrow \mathcal{C}(X, Y),$$

$$g_* : \mathcal{C}(X, Y) \longrightarrow \mathcal{C}(X, Y'),$$

given by the formula:

$$f^* : (\varphi : X' \longrightarrow Y) \mapsto (X \xrightarrow{f} X' \xrightarrow{\varphi} Y),$$

$$g_* : (\psi : X \longrightarrow Y) \mapsto (X \xrightarrow{\psi} Y \xrightarrow{g} Y').$$

We have the following diagram of maps:

$$\begin{tikzcd} \mathcal{C}(X,Y) \arrow[d, "g_*"] & \mathcal{C}(X',Y) \arrow[l, "f^*"'] \arrow[d, "g_*"] \\ \mathcal{C}(X,Y') & \mathcal{C}(X',Y') \arrow[l, "f^*"] \end{tikzcd} \tag{4}$$

We claim that the diagram (4) is commutative. Let $\varphi : X' \to Y$ be an element in the right top corner of (4). By definition, we obtain the following diagram:

$$\begin{tikzcd} \left\{X \xrightarrow{f} X' \xrightarrow{\varphi} Y\right\} \arrow[d, "g_*"] & \left\{X' \xrightarrow{\varphi} Y\right\} \arrow[l, "f^*"'] \arrow[d, "g_*"] \\ \left\{X \xrightarrow{f} X' \xrightarrow{\varphi} Y \xrightarrow{g} Y'\right\} & \left\{X' \xrightarrow{\varphi} Y \xrightarrow{g} Y'\right\} \arrow[l, "f^*"] \end{tikzcd}$$

Clearly both ways from the right top corner to the bottom left one give the same result.

Next, we notice that the maps $f : X \longrightarrow X'$, and $g : Y \longrightarrow Y'$ induce the maps

$$\Sigma f : \Sigma X \longrightarrow \Sigma X', \quad \Omega g : \Omega Y \longrightarrow \Omega Y'$$

given by the formula

$$\Sigma f(x, t) = (f(x), t), \quad \Omega(g) : (\gamma : I \longrightarrow Y) \mapsto (g \circ \gamma : I \longrightarrow Y').$$

We call the homeomorphism $F_{X,Y} : \mathcal{C}(\Sigma(X), Y) \longrightarrow \mathcal{C}(X, \Omega(Y))$ *natural* since for any maps

$$f : X \longrightarrow X', \quad g : Y \longrightarrow Y'$$

the following diagram of pointed spaces and maps commutes:

$$
\begin{tikzcd}
 & \mathcal{C}(\Sigma(X), Y') \arrow[rr, "F_{X,Y'}"] & & \mathcal{C}(X, \Omega(Y')) \\
\mathcal{C}(\Sigma(X), Y) \arrow[ur, "g_*"] \arrow[rr, "F_{X,Y}"] & & \mathcal{C}(X, \Omega(Y)) \arrow[ur, "\Omega g_*"] & \\
 & \mathcal{C}(\Sigma(X'), Y') \arrow[uu, "\Sigma f^*"] \arrow[rr, "F_{X',Y'}"] & & \mathcal{C}(X', \Omega(Y')) \arrow[uu, "f^*"'] \\
\mathcal{C}(\Sigma(X'), Y) \arrow[uu, "\Sigma f^*"] \arrow[ur, "g_*"] \arrow[rr, "F_{X',Y}"] & & \mathcal{C}(X', \Omega(Y)) \arrow[uu, "f^*"] \arrow[ur, "\Omega g_*"] &
\end{tikzcd}
\tag{5}
$$

Exercise 2.13. *Check commutativity of the diagram (5).*

Exercise 2.14. *Show that the homeomorphism* $X \wedge S^1 \xrightarrow{G} \Sigma(X)$ *is natural.*

3. Homotopy and homotopy equivalence

3.1. **Definition of a homotopy.** Let X and Y be topological spaces. Two maps

$$f_0 : X \longrightarrow Y \text{ and } f_1 : X \longrightarrow Y$$

are *homotopic* (notation: $f_0 \sim f_1$) if there exists a map $F : X \times I \longrightarrow Y$ such that the restriction $F|_{X\times\{0\}}$ coincides with f_0, and the restriction $F|_{X\times\{1\}}$ coincides with f_1.

The map $F : X \times I \longrightarrow Y$ is called a *homotopy.* We can think also that a homotopy between maps f_0 and f_1 is a continuous family of maps $\varphi_t : X \longrightarrow Y$, $0 \leq t \leq 1$, such that $\varphi_0 = f_0$, $\varphi_1 = f_1$, and the map $F : X \times I \longrightarrow Y$, $F(x,t) = \varphi_t(x)$ is a continuous map for every $t \in I$.

If the spaces X and Y are “good spaces” (like our examples S^n, $\mathbf{RP}^n$, $\mathbf{CP}^n$, $\mathbf{HP}^n$, $G_k(\mathbf{R}^n)$ $V_k(\mathbf{R}^n)$ and so on), then we can think about homotopy between f_0 and f_1 as a path in the space of continuous maps $\mathcal{C}(X,Y)$ joining f_0 and f_1. Furthemore, in such case, the set of homotopy classes $[X,Y]$ (see below) may be identified with the set of path-components of the space $\mathcal{C}(X,Y)$.

If a map $f : X \longrightarrow Y$ is homotopic to a constant map $X \longrightarrow pt \in Y$, we call the map f *null-homotopic.*

Example. Let $Y \subset \mathbf{R}^n$ (or $\mathbf{R}^\infty$) be a convex subset. Then for any space X any two maps $f_0 : X \longrightarrow Y$ and $f_1 : X \longrightarrow Y$ are homotopic. Indeed, the map

$$F : x \longrightarrow (1-t)f_0(x) + tf_1(x)$$

defines a corresponding homotopy.

3.2. **Homotopy classes of maps.** Clearly a homotopy determines an equivalence relation on the space of maps $\mathcal{C}(X,Y)$. The set of equivalence classes is denoted by $[X,Y]$ and it is called a *set of homotopy classes.*

Examples. 1. The set $[X,*]$ consists of one point for any space X.

2. The space $[*,Y]$ is the set of path-connected components of Y.

Let $\varphi : X \longrightarrow X'$ be a map (continuous), then we define the map (not continuous since we do not have a topology on the set $[X,Y]$) $\varphi^* : [X',Y] \longrightarrow [X,Y]$ as follows. Let $a \in [X',Y]$ be a homotopy class. Choose any representative $f : X' \longrightarrow Y$ of the class a, then $\varphi^*(a)$ is a homotopy class contaning the map $f \circ \varphi : X \longrightarrow Y$.

Now let $\psi : Y \longrightarrow Y'$ be a map. Then the map $\psi_* : [X,Y] \longrightarrow [X,Y']$ is defined as follows. For any $b \in [X,Y]$ and a representative $g : X \longrightarrow Y$ the map $\psi \circ g : X \longrightarrow Y'$ determines a homotopy class $\psi(b) = [\psi \circ g]$.

Exercise 3.1. *Prove that the maps φ^* and ψ_* are well-defined.*

3.3. **Homotopy equivalence.** We will give three different definitions of homotopy equivalence.

Definition 3.1. (HE-I) *Two spaces X_1 and X_2 are homotopy equivalent ($X_1 \sim X_2$) if there exist maps $f : X_1 \longrightarrow X_2$ and $g : X_2 \longrightarrow X_1$ such that the compositions $g \circ f : X_1 \longrightarrow X_1$ and $f \circ g : X_2 \longrightarrow X_2$ are homotopy equivalent to the identity maps I_{X_1} and I_{X_2} respectively.*

In this case we call maps f and g *mutually inverse homotopy equivalences*, and both maps f and g are *homotopy equivalences*.

Remark. If the maps $g \circ f$ and $f \circ g$ are the identity maps, then f and g are *mutually inverse homeomorphisms*.

Definition 3.2. (HE-II) *Two spaces X_1 and X_2 are homotopy equivalent ($X_1 \sim X_2$) if there is a rule assigning for any space Y a one-to-one map $\varphi_Y : [X_1, Y] \longrightarrow [X_2, Y]$ such that for any map $h : Y \longrightarrow Y'$ the diagram*

$$\begin{tikzcd}
{[X_1, Y]} \arrow[r, "\varphi_Y"] \arrow[d, "h_*"'] & {[X_2, Y]} \arrow[d, "h_*"'] \\
{[X_1, Y']} \arrow[r, "\varphi_{Y'}"] & {[X_2, Y']}
\end{tikzcd} \tag{6}$$

commutes, i.e. $\varphi_{Y'} \circ h_* = h_* \circ \varphi_Y$.

Definition 3.3. (HE-III) *Two spaces X_1 and X_2 are homotopy equivalent ($X_1 \sim X_2$) if there is a rule assigning for any space Y a one-to-one map $\varphi^Y : [Y, X_1] \longrightarrow [Y, X_2]$ such that for any map $h : Y \longrightarrow Y'$ the diagram*

$$\begin{tikzcd}
{[Y, X_1]} \arrow[r, "\varphi^Y"] & {[Y, X_2]} \\
{[Y', X_1]} \arrow[r, "\varphi_{Y'}"] \arrow[u, "h^*"] & {[Y', X_2]} \arrow[u, "h^*"]
\end{tikzcd} \tag{7}$$

commutes, i.e. $\varphi^Y \circ h^* = h^* \circ \varphi^{Y'}$.

Theorem 3.4. *Definitions 3.1, 3.2 and 3.3 are equivalent.*

Proof. Here we prove only that Definitions 3.1 and 3.2 are equivalent. Let $X_1 \sim X_2$ in the sence of Definition 3.2. Then there is one-to-one map $\varphi_{X_2} : [X_1, X_2] \longrightarrow [X_2, X_2]$. Let I_{X_2} be the identity map, $I_{X_2} \in [X_2, X_2]$. Let $\alpha = \varphi^{-1}([I_{X_2}]) \in [X_1, X_2]$ and $f \in \alpha$, $f : X_1 \longrightarrow X_2$ be a representative.

There is also a one-to-one map $\varphi_{X_1} : [X_1, X_1] \longrightarrow [X_2, X_1]$. We let $\beta = \varphi_{X_1}([I_{X_1}])$ and we choose a map $g : X_2 \longrightarrow X_1$, $g \in \beta$. We shall show that $f \circ g \sim I_{X_2}$. The diagram

(8)
$$\begin{tikzcd}
{[X_1, X_1]} \arrow[r, "\varphi_{X_1}"] \arrow[d, "f_*"'] & {[X_2, X_1]} \arrow[d, "f_*"'] \\
{[X_1, X_2]} \arrow[r, "\varphi_{X_2}"] & {[X_2, X_2]}
\end{tikzcd}$$

commutes by Definition 3.2. It implies that $\varphi_{X_2} \circ f_* = f_* \circ \varphi_{X_1}$. Let us consider the image of the element $[I_{X_1}]$ in the diagram (8). We have:

$$f_*([I_{X_1}] = [f \circ I_{X_1}] = [f], \qquad \varphi_{X_2}([f]) = [I_{X_2}]$$

by definition and by the choice of f. It implies that

$$\varphi_{X_2} \circ f_*([I_{X_1}]) = [I_{X_2}].$$

On the other hand, we have:

$$f_* \circ \varphi_{X_1}([I_{X_1}]) = f_*([g]) = [f \circ g].$$

Commutativity of (8) implies that $[f \circ g] = [I_{X_2}]$, i.e. $f \circ g \sim I_{X_2}$.

A similar argument proves that $g \circ f \sim I_{X_1}$. It means that $X_1 \sim X_2$ in the sence of Definition 3.1.

Now assume that $X_1 \sim X_2$ in the sence of Definition 3.1, i.e. there are maps $f : X_1 \longrightarrow X_2$ and $g : X_1 \longrightarrow X_1$ such that $f \circ g \sim I_{X_2}$ and $g \circ f \sim I_{X_1}$. Let Y be any space and define

$$\varphi_Y = g^* : [X_1, Y] \longrightarrow [X_2, Y].$$

We shall show that this map is inverse to the map

$$f^* : [X_2, Y] \longrightarrow [X_1, Y].$$

Indeed, let $h \in \mathcal{C}(X_1, Y)$, then

$$f^* \circ g^*([h]) = f^*([h \circ g]) = (\text{by definition of } f^*) = [h \circ (g \circ f)] = [h] \quad (\text{since } g \circ f \sim I_{X_1}).$$

This shows that f^* is inverse to g^*. With a similar argument we prove that g^* is inverse to f^*. Thus $\varphi_Y = g^*$ is a bijection. Now we have to check naturality.

Let Y' be a space and $k : Y \longrightarrow Y'$ be a map. We show that the diagram

(9)
$$\begin{tikzcd}
{[X_1, Y]} \arrow[r, "\varphi_Y = g^*"] \arrow[d, "k_*"'] & {[X_2, Y]} \arrow[d, "k_*"'] \\
{[X_1, Y']} \arrow[r, "\varphi_{Y'} = g^*"] & {[X_2, Y']}
\end{tikzcd}$$

commutes. Let $h \in \mathcal{C}(X_1, Y)$ be a map. Then we have

$$k_*([h]) = [k \circ h], \quad g^*([k \circ h]) = [(k \circ h) \circ g],$$

and also

$$g^*([h]) = [h \circ g], \quad k_*([h \circ g]) = [k \circ (h \circ g)].$$

It means that (9) commutes. Thus Definitions 3.1 and 3.2 are equivalent. □

Exercise 3.2. *Prove the equivalence of Definitions 3.1 and 3.3.*

We call a class of homotopy equivalent spaces a *homotopy type*. Obviously any homeomorphic spaces are homotopy equivalent. The simpest example of spaces which are homotopy equivalent, but not homeomorphic is the following: X_1 is a circle, and X_2 is an annulus, see Fig. 24.

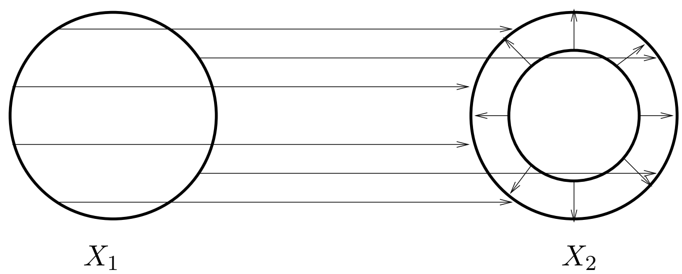


FIGURE 10

Exercise 3.3. *Give 3 examples of spaces homotopy equivalent and not homeomorphic spaces.*

We call a space X a *contractible space* if the identity map $I : X \longrightarrow X$ null-homotopic, i.e. it is homotopic to the "constant map" $* : X \longrightarrow X$, mapping all X to a single point.

Exercise 3.4. *Prove that a space X is contractible if and only if it is homotopy equivalent to a point.*

Exercise 3.5. *Prove that a space X is contractible if and only if every map $f : Y \longrightarrow X$ is null-homotopic.*

Exercise 3.6. *Prove that the space of paths $\mathcal{E}(X, x_0)$ is contractible for any X.*

Exercise 3.7. *Let X_1, X_2 be pointed spaces. Prove that if $X_1 \sim X_2$ then $\Sigma(X_1) \sim \Sigma(X_2)$ and $\Omega(X_1) \sim \Omega(X_2)$.*

3.4. **Retracts.** We call a subspace A of a topological space X its *retract* if there exists a map $r : X \longrightarrow X$ (a *retraction*) such that $r(X) = A$ and $r(a) = a$ for any $a \in A$.

Examples. 1. A single point $x \in X$ is a retact of the space X since a constant map $r : X \longrightarrow x$ is a retraction.

2. The subspace $A = \{0\} \cup \{1\}$ of the interval $I = [0,1]$ *is not a retract of* I, otherwise we would map I to the disconnected space A.

3. In general, the sphere S^n *is not a retract of the disk* D^{n+1} for any n, however we do not have enough tools in our hands to prove it now.

4. The "base" $X \times \{0\}$ is a retract of the cylinder $X \times I$.

Exercise 3.8. *Prove that the "base" $X \times \{0\}$ of the cone $C(X)$ is a retract of $C(X)$ if and only if the space X is contractible.*

Sometimes a retraction $r : X \longrightarrow X$ (where $r(X) = A$) is homotopic to the identity map $Id : X \longrightarrow X$, in that case we call A a *deformation retract of* X; moreover if this homotopy may be chosen to be the identity map on A,[4] then we call A a *strict deformation retract of* X.

Lemma 3.5. *A subspace A is a deformation retract of X if and only if the inclusion $A \longrightarrow X$ is a homotopy equivalence.*

Exercise 3.9. *Prove Lemma 3.5.*

Lemma 3.5 shows that a concept of deformation retract is not really new for us; a concept of strict deformation retract is more restrictive, however these two concepts are different only in some pathological cases.

Exercise 3.10. *Let $A \subset X$, and $r^{(0)} : X \longrightarrow A$, $r^{(1)} : X \longrightarrow A$ be two deformation retractions. Prove that the retractions $r^{(0)}$, $r^{(1)}$ may be joined by a continuous family of deformation retractions $r^{(s)} : X \longrightarrow A$, $0 \leq s \leq 1$. Note: It is important here that $r^{(0)}$, $r^{(1)}$ are both homotopic to the identity map I_X.*

3.5. **The case of "pointed" spaces.** The definitions of homotopy, homotopy equivalence have to be changed (in an obvious way) for spaces with base points. The set of homotopy clases of "pointed" maps $f : X \longrightarrow Y$ will be also denoted as $[X, Y]$. We need one more generalization.

Definition 3.6. *A pair (X, A) is just a space X with a labeled subspace $A \subset X$; a map of pairs $f : (X, A) \longrightarrow (Y, B)$ is a continuous map $f : X \longrightarrow Y$ such that $f(A) \subset B$. Two maps $(X, A) \longrightarrow (Y, B)$, $f_1 : (X, A) \longrightarrow (Y, B)$ are homotopic if there exist a map $F : (X \times I, A \times I) \longrightarrow (Y, B)$ such that*

$$F|_{(X \times \{0\}, A \times \{0\}} = f_0, \quad F|_{(X \times \{1\}, A \times \{1\}} = f_1.$$

[4] i.e. a homotopy $h : X \times I \longrightarrow X$ between $r : X \longrightarrow X$ and the identity map $Id : X \longrightarrow X$ has the following property: $h(a, t) = a$ for any $a \in A$.

We have seen already the example of pairs and their maps. Let me recall that the cones of the maps $c : S^n \longrightarrow \mathbf{RP}^n$ and $h : S^{2n+1} \longrightarrow \mathbf{CP}^n$ give us the commutative diagrams:

$$
\begin{tikzcd}
D^{n+1} \arrow[r, "f"] & \mathbf{RP}^{n+1} & D^{2n+2} \arrow[r, "g"] & \mathbf{CP}^{n+1} \\
S^n \arrow[u, "i"] \arrow[r, "c"] & \mathbf{RP}^n \arrow[u, "i"] & S^{2n+1} \arrow[u, "i"] \arrow[r, "h"] & \mathbf{CP}^n \arrow[u, "i"]
\end{tikzcd}
\tag{10}
$$

which are the maps of pairs:

$$f : (D^{n+1}, S^n) \longrightarrow (\mathbf{RP}^{n+1}, \mathbf{RP}^n), \quad g : (D^{2n+2}, S^{2n+1}) \longrightarrow (\mathbf{CP}^{n+1}, \mathbf{CP}^n).$$

4. CW-complexes

Algebraic topologists rarely study arbitrary topological spaces: there is not much one can prove about an abstract topological space. However, there is very well-developed area known as general topology which studies *simple properties* (such as conectivity, the Hausdorff property, compactness and so on) *of complicated spaces*. There is a giant Zoo out there of very complicated spaces endowed with all possible degrees of pathology, i.e. when one or another simple property fails or holds. Some of these spaces are extremely useful, such as the *Cantor set* or *fractals*, they help us to understand very delicate phenomenas observed in mathematics and physics. In algebraic topology we mostly study *complicated properties of simple spaces*.

It turns out that the most important spaces which are important for mathematics have some additional structures. The first algebraic topologist, Poincarè, studied mostly the spaces endowed with "analytic" structures, i.e. when a space X has natural differential structure or Riemannian metric and so on. The major advantage of these structures is that they all are natural, so we should not really care about their existence: they are given! There is the other type of natural structures on topological spaces: so called *combinatorial structures*, i.e. when a space X comes equipped with a decomposition into more or less "standard pieces", so that one could study the whole space X by examination the mutual geometric and algebraic relations between those "standard pieces". Below we formalize this concept: these spaces are known as CW-complexes. For instance, all examples we studied so far are like that.

4.1. **Basic definitions.** We will call an open disk D^n (as well as any space homeomorphic to D^n) by n-cell. Notation: e^n. We will use the notation $\bar{e}^n$ for a "closed cell" which is homeomorphic to D^n. For $n = 0$ we let $e^0 = D^0$ (point). Let ∂e^n be a "boundary" of the cell e^n; ∂e^n is homeomorphic to the sphere S^{n-1}. Recall that if we have a map $\varphi : \partial e^n \longrightarrow K$, then we can construct the space $K \cup_\varphi e^n$, such that the diagram

$$\begin{tikzcd}
\overline{e}^n \arrow[r, "\Phi"] & K \cup_\varphi e^n \\
\partial e^n \arrow[u, "i"] \arrow[r, "\varphi"] & K \arrow[u, "i"']
\end{tikzcd}$$

commutes. We will call this procedure an *attaching of the cell* e^n *to the space* K. The map $\varphi : \partial e^n \longrightarrow K$ is the *attaching map*, and the map $\Phi : e^n \longrightarrow K \cup_\varphi e^n$ the *characteristic map of the cell* e^n. Notice that Φ is a homeomorphism of the open cell e^n on its image.

An example of this construction is the diagram (10), where the maps $c : S^n \longrightarrow \mathbf{RP}^n$ and $h : S^{2n+1} \longrightarrow \mathbf{CP}^n$ are the attaching maps of the corresponding cells e^{n+1} and e^{2n+2}. As we shall see below,

$$\mathbf{RP}^n \cup_c e^{n+1} \cong \mathbf{RP}^{n+1} \quad \text{and} \quad \mathbf{CP}^n \cup_h e^{2n+2} \cong \mathbf{CP}^{n+1}.$$

We return to this particular construction a bit later.

Definition 4.1. *A Hausdorff topological space X is a CW-complex (or cell-complex) if it is decomposed as a union of cells:*

$$X = \bigcup_{q=0}^{\infty}\left(\bigcup_{i\in I_q} e_i^q\right),$$

where the cells $e_i^q \cap e_j^p = \emptyset$ unless $q=p$, $i=j$, and for each e_i^q there exists a characteristic map $\Phi : D^q \longrightarrow X$ such that its restriction $\Phi_{\overset{\circ}{D}{}^q}$ gives a homeomorphism $\Phi|_{\overset{\circ}{D}{}^q} : \overset{\circ}{D}{}^q \longrightarrow e_i^q$. It is required that the following axioms are satisfied:

(C) (closure finite) *The boundary $\partial e_i^q = \bar{e}_i^q \setminus e_i^q$ of the cell e_i^q is a subset of the union of finite number of cells e_j^r, where $r<q$.*

(W) (weak topology) *A set $F \subset X$ is closed if and only if the intersection $F \cap \bar{e}_i^q$ is closed for every cell e_i^q.*

Example 1. The sphere S^n. There are two standard cell decompositions of S^n:

(a) Let e^0 be a point (say, the north pole $(0,0,\dots,0,1)$ and $e^n = S^n \setminus e^0$, so $S^n = e^0 \cup e^n$. A characteristic map $D^n \longrightarrow S^n$ which corresponds to the cell e^n may be defined by

$$(x_1, x_2, \dots, x_n) \longrightarrow (x_1 \sin\pi\rho, \dots, x_n \sin\pi\rho, \cos\pi\rho), \quad \text{where } \rho = \sqrt{x_1^2 + \dots + x_n^2}$$

(b) We define $S^n = \bigcup_{q=0}^n e_{\pm}^q$, where

$$e_{\pm}^q = \{(x_1, \dots x_{n+1}) \in S^n \mid x_{q+2} = \dots = x_{n+1} = 0, \text{ and } \pm x_{q+1} > 0\}, \quad \text{see Fig. 11.}$$

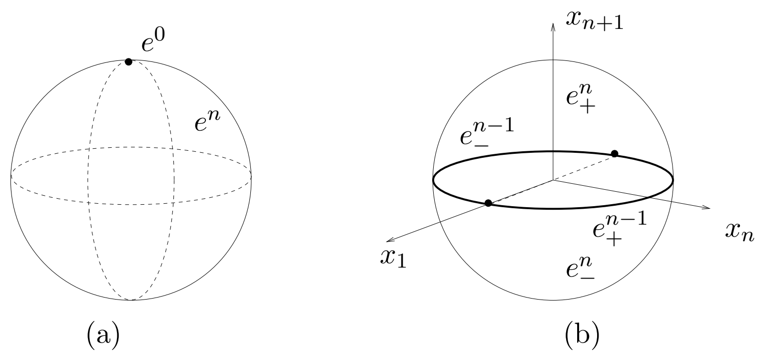


FIGURE 11

There exist a lot more cell decompositions of the sphere S^n: one can decompose S^n on $(3^{n+1}-1)$ cells as a boundary of $(n+1)$-dimension simplex[5] Δ^{n+1}, or on $(2^{n+2}-2)$ cells as a boundary of the cube I^n.

[5] A simplex Δ^k is determined as follows:
$\Delta^k = \left\{(x_1, \dots, x_{k+1}) \in \mathbf{R}^{k+1} \mid x_1 \geq 0, \dots, x_{k+1} \geq 0, \ \Sigma_{i=1}^{k+1} x_i = 1\right\}$.

Exercise 4.1. *Describe these cell decompositions of* S^n.

Example 2. Any of the above cell decompositions of the sphere S^{n-1} may be used to construct a cell decomposition of the disk D^n by adding one more cell: $Id : D^n \longrightarrow D^n$. The most simple one gives us three cells.

4.2. **Some comments on the definition of a** CW**-complex.** $\mathbf{1}^o$ Let X be a CW-complex. We denote $X^{(n)}$ the union of all cells in X of dimension $\leq n$. This is the n-th skeleton of X. The n-th skeleton $X^{(n)}$ is an example (very important one) of a subcomplex of a CW-complex. A subcomplex $A \subset X$ is a closed subset of A which is a union of some cells of X. In particular, the n-th skeleton $A^{(n)}$ is a subcomplex of $X^{(n)}$ for each $n \geq 0$. A map $f : X \longrightarrow Y$ of CW-complexes is a cellular map if $f|_{X^{(n)}}$ maps the n-th skeleton to the n-skeleton $Y^{(n)}$ for each $n \geq 0$. In particular, the inclusion $A \subset X$ of a subcomplex is a cellular map. A CW-complex is called *finite* if it has a finite number of cells. A CW-complex is called *locally finite* if X has a finite number of cells in each dimension. Finally (X, x_0) is a pointed CW-complex, if x_0 is a 0-cell.

Exercise 4.2. *Prove that a* CW*-complex compact if and only if it is finite.*

$\mathbf{2}^o$ It turns out that a closure of a cell within a CW-complex may be not a CW-complex.

Exercise 4.3. *Construct a cellular decomposion of the wedge* $X = S^1 \vee S^2$ *(with a single 2-cell* e^2*) such that a closure of the cell* e^2 *is not a* CW*-subcomplex of* X.

$\mathbf{3}^o$ **(Warning)** The axiom (W) does not imply the axiom (C). Indeed, consider a decomposition of the disk D^2 into 2-cell e^2 which is the interior of the disk D^2 and each point of the circle S^1 is considered as a zero cell.

Exercise 4.4. *Prove that the disk* D^2 *with the cellular decomposition described above satisfies* (W), *and does not satisfy* (C).

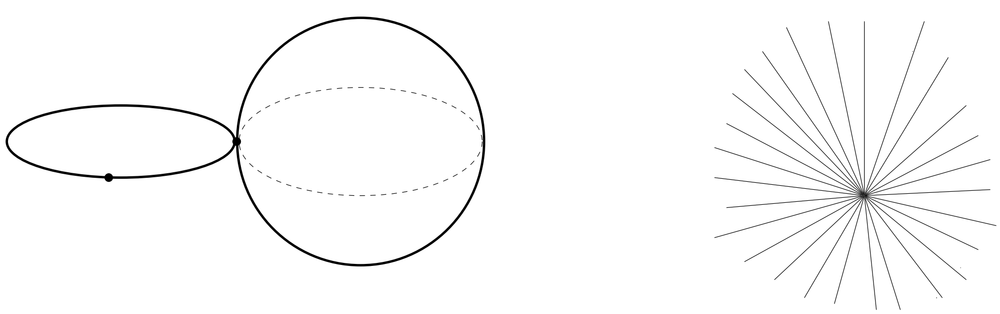


FIGURE 12

$\mathbf{4}^o$ **(Warning)** The axiom (C) does not imply the axiom (W). Indeed, consider the following space X. We start with an infinite (even countable) family I_α of unit intervals. Let $X = \bigvee_\alpha I_\alpha$, where we identify zero points of all intervals I_α. We define a topology on X by means of the following metric. Let $t' \in I_{\alpha'}$ and $t'' \in I_{\alpha''}$. Then a distance is defined by

$$\rho(t', t'') = \begin{cases} |t' - t''| & \text{if } \alpha' = \alpha'' \\ t' + t'' & \text{if } \alpha' \neq \alpha'' \end{cases}$$

Exercise 4.5. *Check that a natural cellular decomposition of X into the interiors of I_α and remaining points (zero cells) does not satisfy the axiom (W).*

4.3. **Operations on CW-complexes.** All operations we considered are well-defined on the category of CW-complexes, however we have to be a bit careful. If one of the CW-complexes X and Y is locally finite, then the product $X \times Y$ has a canonical CW-structure. The same holds for a smash-product $X \wedge Y$ of pointed CW-complexes. The cone $C(X)$, cylinder $X \times I$, and suspension $\Sigma(X)$ has canonical CW-structure determined by X. We can glue CW-complexes $X \cup_f Y$ if $f : A \longrightarrow Y$ a cellular map, and $A \subset X$ is a subcomplex. Also the quotient space X/A is a CW-complex if (A, X) is a CW-pair. The functional spaces $\mathcal{C}(X, Y)$ are two big to have natural CW-structure, however, a space $\mathcal{C}(X, Y)$ is homotopy equivalent to a CW-complex if X and Y are CW-complexes. The last statement is a nontrivial result due to J. Milnor (1958).

4.4. **More examples of CW-complexes.** Real projective space $\mathbf{RP}^n$. Here we choose in $\mathbf{RP}^n$ a sequence of projective subspaces:

$$* = \mathbf{RP}^0 \subset \mathbf{RP}^1 \subset \ldots \subset \mathbf{RP}^{n-1} \subset \mathbf{RP}^n.$$

and set $e^0 = \mathbf{RP}^0$, $e^1 = \mathbf{RP}^1 \setminus \mathbf{RP}^0, \ldots e^n = \mathbf{RP}^n \setminus \mathbf{RP}^{n-1}$. The diagram (10) shows that the map $c : S^{k-1} \longrightarrow \mathbf{RP}^k$ is an attaching map, and its extension to the cone over S^{k-1} (the disk D^k) is a characteristic map of the cell e^k. Alternatively this decomposition may be described in the homogeneous coordinates as follows. Let

$$e^q = \{(x_0 : x_1 : \cdots : x_n) \mid x_q \neq 0, x_{q+1} = 0, \ldots x_n = 0\}.$$

Exercise 4.6. *Prove that e^q is homeomorphic to $\mathbf{RP}^q \setminus \mathbf{RP}^{q-1}$.*

Exercise 4.7. *Construct cell decompositions of $\mathbf{CP}^n$ and $\mathbf{HP}^n$.*

Exercise 4.8. *Represent as CW-complex every 2-dimensional manifold. Try to find a CW-strucute with a minimal number of cells.*

Exercise 4.9. *Prove that a finite CW-complex (with finite number of cells) may be embedded into Euclidean space of finite dimension.*

4.5. *CW-structure of the Grassmannian manifolds.* We describe here the *Schubert decomposition*, and the cells of this decomposition are known as the *Schubert cells.* We consider the space $G_k(\mathbf{R}^n)$. We choose the standard basis $e_1,\dots,e_n$ of $\mathbf{R}^n$. Let $\mathbf{R}^q = \langle e_1,\dots,e_q\rangle$. It is convenient to denote $\mathbf{R}^0=\{0\}$. We have the inclusions:

$$\mathbf{R}^0 \subset \mathbf{R}^1 \subset \mathbf{R}^2 \subset \cdots \subset \mathbf{R}^n.$$

Let $\pi\in G_k(\mathbf{R}^n)$. Clearly π determines a collection of nonnegative numbers

$$0 \le \dim(\mathbf{R}^1\cap\pi) \le \dim(\mathbf{R}^2\cap\pi) \le \cdots \le \dim(\mathbf{R}^n\cap\pi) = k.$$

We note that $\dim(\mathbf{R}^j\cap\pi)\le \dim(\mathbf{R}^{j-1}\cap\pi)+1$. Indeed, we have linear maps

$$0 \longrightarrow \mathbf{R}^{j-1}\cap\pi \xrightarrow{i} \mathbf{R}^j\cap\pi \xrightarrow{j\text{-th coordinate}} \mathbf{R} \tag{11}$$

where the first one, $i:\mathbf{R}^{j-1}\cap\pi \longrightarrow \mathbf{R}^j\cap\pi$, is an embedding, and the map

$$j\text{-th coordinate} : \mathbf{R}^j\cap\pi \longrightarrow \mathbf{R}$$

is either onto or zero. In the first case $\dim(\mathbf{R}^j\cap\pi)=\dim(\mathbf{R}^{j-1}\cap\pi)+1$, and in the second case $\dim(\mathbf{R}^j\cap\pi)=\dim(\mathbf{R}^{j-1}\cap\pi)$. Thus there are exactly k “jumps” in the sequence

$$(0,\dim(\mathbf{R}^1\cap\pi),\dots,\dim(\mathbf{R}^n\cap\pi)).$$

A *Schubert* [6] *symbol* $\sigma=(\sigma_1,\dots,\sigma_k)$ is a collection of integers, such that

$$1\le\sigma_1<\sigma_2<\cdots<\sigma_k\le n.$$

Let $e(\sigma)\subset G_k(\mathbf{R}^n)$ be the following set of the following k-planes in $\mathbf{R}^n$

$$e(\sigma)=\left\{\pi\in G_k(\mathbf{R}^n) \mid \dim(\mathbf{R}^{\sigma_j}\cap\pi)=j \ \&\ \dim(\mathbf{R}^{\sigma_j-1}\cap\pi)=j-1,\ j=1,\dots,k\ \right\}.$$

Notice that every $\pi\in G_k(\mathbf{R}^n)$ belongs to exactly one subset $e(\sigma)$. Indeed, in the sequence of subspaces

$$\mathbf{R}^1\cap\pi\subset\mathbf{R}^2\cap\pi\subset\cdots\subset\mathbf{R}^n\cap\pi=\pi$$

their dimensions “jump” by one exactly k times. Clearly $\pi\in e(\sigma)$, where $\sigma=(\sigma_1,\dots,\sigma_n)$ and

$$\sigma_t=\min\left\{j \mid \dim(\mathbf{R}^j\cap\pi)=t\right\}.$$

Our goal is to prove that the set $e(\sigma)$ is homeomorphic to an open cell of dimension $d(\sigma)=(\sigma_1-1)+(\sigma_2-2)+\cdots+(\sigma_k-k)$. Let $H^j\subset\mathbf{R}^n$ denote an open “half j-plane of $\mathbf{R}^j$:

$$H^j=\{(x_1,\dots,x_j,0,\dots,0)\mid x_j>0\}.$$

It will be convenient to denote $\overline{H}^j=\{(x_1,\dots,x_j,0,\dots,0)\mid x_j\ge 0\}$.

Claim 4.1. *A k-plane π belongs to $e(\sigma)$ if and only if there exists its basis $v_1,\dots,v_k$, such that $v_1\in H^{\sigma_1}$, ..., $v_k\in H^{\sigma_k}$.*

[6]Hermann Schubert, 1848–1911, http://www-groups.dcs.st-and.ac.uk/~history/Biographies/Schubert.html. He is not related to famous Franz Schubert 1797–1828, a great composer, https://en.wikipedia.org/wiki/Franz_Schubert

Proof. Indeed, if there is such a basis $v_1, \dots, v_k$ then

$$\dim(\mathbf{R}^{\sigma_j} \cap \pi) > \dim(\mathbf{R}^{\sigma_j - 1} \cap \pi)$$

for $j = 1, \dots, k$. Thus $\pi \in e(\sigma)$. The following lemma proves Claim 4.1 in the other direction. $\square$

Lemma 4.2. *Let $\pi \in e(\sigma)$, where $\sigma = (\sigma_1, \dots, \sigma_n)$. Then there exists a unique orthonormal basis $v_1, \dots, v_k$ of π, so that $v_1 \in H^{\sigma_1}$, ..., $v_k \in H^{\sigma_k}$.*

Proof. We choose v_1 to be a unit vector which generates the line $\mathbf{R}^{\sigma_1} \cap \pi$. There are only two choices here, and the condition that the σ_1-th coordinate is positive determines v_1 uniquely. Then the unit vector $v_2 \in \mathbf{R}^{\sigma_2} \cap \pi$ should be chosen so that $v_2 \perp v_1$. There are two choices like that, and again the positivity of the σ_2-th coordinate determines v_2 uniquely. By induction one obtains the required basis. This completes proof of Lemma 4.2 and Claim 4.1. $\square$

We define the following subset of the Stiefel manifold $V_k(\mathbf{R}^n)$:

$$E(\sigma) = \{(v_1, \dots, v_k) \in V_k(\mathbf{R}^n) \mid v_1 \in H^{\sigma_1}, \dots, v_k \in H^{\sigma_k} \}.$$

Lemma 4.2 gives a well-defined map $q : e(\sigma) \longrightarrow E(\sigma)$. It is convenient to denote

$$\overline{E}(\sigma) = \{(v_1, \dots, v_k) \in V_k(\mathbf{R}^n) \mid v_1 \in \overline{H}^{\sigma_1}, \dots, v_k \in \overline{H}^{\sigma_k} \}.$$

Claim 4.2. *The set $\overline{E}(\sigma) \subset V_k(\mathbf{R}^n)$ is homeomorphic to the closed cell of dimension $d(\sigma) = (\sigma_1 - 1) + (\sigma_2 - 2) + \dots + (\sigma_k - k)$. Furthermore the map $q : e(\sigma) \longrightarrow E(\sigma)$ is a homeomorphism.*

Proof. Induction on k. If $k = 1$ the set $\overline{E}(\sigma_1)$ consists of the vectors

$$v_1 = (x_{11}, \dots, x_{1\sigma_1}, 0, \dots, 0), \quad \text{such that} \quad \sum x_{1j}^2 = 1, \quad \text{and} \ \ x_{1\sigma_1} \geq 0.$$

Clearly $\overline{E}(\sigma_1)$ is a closed hemisphere of dimension $(\sigma_1 - 1)$, i.e. $\overline{E}(\sigma_1)$ is homeomorphic to the disk $D^{\sigma_1 - 1}$.

To make an induction step, consider the following construction. Let $u, v \in \mathbf{R}^n$ be two unit vectors such that $u \neq -v$. Let $T_{u,v}$ an orthogonal transformation $\mathbf{R}^n \longrightarrow \mathbf{R}^n$ such that

- **(1)** $T_{u,v}(u) = v$;
- **(2)** $T_{u,v}(w) = w$ if $w \in \langle u, v \rangle^{\perp}$.

In other words, $T_{u,v}$ is a rotation in the plane $\langle u, v \rangle$ taking the vector u to v, and is identity on the orthogonal complement to the plane $\langle u, v \rangle$ generated by u and v.

Claim 4.3. *The transformation $T_{u,v}$ (where $u, v \in \mathbf{R}^n$, $u \neq -v$) has the following properties:*

- **(a)** $T_{u,u} = Id$;
- **(b)** $T_{v,u} = T_{u,v}^{-1}$;

(c) $T_{v,u} : \mathbf{R}^n \longrightarrow \mathbf{R}^n$ *is be given by*

$$T_{u,v}(x) = x - \frac{\langle u+v, x\rangle}{1+\langle u,v\rangle}(u+v) + 2\langle u,x\rangle v;$$

(d) *a vector* $T_{u,v}(x)$ *depends continuously on* u, v, x*;*

(e) $T_{u,v}(x) = x$ *(mod* $\mathbf{R}^j$ *) if* $u, v \in \mathbf{R}^j$*.*

The properties (a), (b), (e) follow from the definition.

Exercise 4.10. *Prove (c), (d) from Claim 4.3.*

Let $\epsilon_i \in H^{\sigma_i}$ be a vector which has σ_i-coordinate equal to 1, and all others are zeros. Thus $(\epsilon_1, \ldots, \epsilon_k) \in E(\sigma)$. For each k-frame $(v_1, \ldots, v_k) \in \overline{E}(\sigma)$ consider the transformation:

$$T = T_{\epsilon_k, v_k} \circ T_{\epsilon_{k-1}, v_{k-1}} \circ \cdots\cdots \circ T_{\epsilon_1, v_1} : \mathbf{R}^n \longrightarrow \mathbf{R}^n \tag{12}$$

First we notice that $v_i \neq -\epsilon_i$ since $v_i \in \overline{H}^{\sigma_i}$. Thus the transformations T_{ϵ_i, v_i} are well-defined.

Exercise 4.11. *Prove that the transformation* T *takes the* k*-frame* $(\epsilon_1, \ldots, \epsilon_k)$ *to the frame* $(v_1, \ldots, v_k)$*.*

Consider the following subspace $D \subset \overline{H}^{\sigma_{k+1}}$:

$$D = \left\{ u \in \overline{H}^{\sigma_{k+1}} \mid |u| = 1, \quad \langle \epsilon_j, u\rangle = 0, \ j = 1, \ldots, k \right\}.$$

Exercise 4.12. *Prove that* D *is homeomorphic to the hemisphere of the dimension* $\sigma_{k+1} - k - 1$*.*

Thus D is a closed cell of dimension $\sigma_{k+1} - k - 1$. Now we make an induction step to complete a proof of Claim 4.2. We define the map

$$f : \overline{E}(\sigma_1, \ldots, \sigma_k) \times D \longrightarrow \overline{E}(\sigma_1, \ldots, \sigma_k, \sigma_{k+1})$$

by the formula $f((v_1, \ldots, v_k), u) = (v_1, \ldots, v_k, Tu)$ where T is given by (12). We notice that

$$\langle v_i, Tu\rangle = \langle T\epsilon_i, Tu\rangle = \langle \epsilon_i, u\rangle = 0, \quad i = 1, \ldots, k,$$

and $\langle Tu, Tu\rangle = \langle u, u\rangle = 1$ by definition of T and since $T \in O(n)$.

Exercise 4.13. *Recall that* $\sigma_k < \sigma_{k+1}$*. Prove that* $Tu \in \overline{H}^{\sigma_{k+1}}$ *if* $u \in D$*.*

The inverse map $f^{-1} : \overline{E}(\sigma_1, \ldots, \sigma_k, \sigma_{k+1}) \longrightarrow \overline{E}(\sigma_1, \ldots, \sigma_k) \times D$ is defined by

$$v_j = f^{-1} v_j, j = 1, \ldots, k,$$

$$u = f^{-1} v_{k+1} = (T^{-1} v_{k+1}) = T_{v_1, \epsilon_1} \circ T_{v_2, \epsilon_2} \circ \cdots\cdots \circ T_{v_k, \epsilon_k}(v_{k+1}) \in D.$$

Both maps f and f^{-1} are continuous, thus f is a homeomorphism. This concludes induction step in the proof of Claim 4.2. Lemma 4.2 implies that $e(\sigma_1, \ldots, \sigma_k)$ is homeomorphic to an open cell of dimension $d(\sigma)=(\sigma_1-1)+(\sigma_2-2)+\cdots+(\sigma_k-k)$.

$\square$

Remark. Let $(v_1,\ldots,v_k)\in\overline{E}(\sigma)\setminus E(\sigma)$, then the k-plane $\pi=\langle v_1,\ldots,v_k\rangle$ does not belong to $e(\sigma)$. Indeed, it means that at least one vector $v_j\in\mathbf{R}^{\sigma_j-1}=\partial\left(\overline{H}^{\sigma_j}\right)$. Thus $\dim(\mathbf{R}^{\sigma_j-1}\cap\pi)\geq j$, hence $\pi\notin e(\sigma)$. $\square$

Theorem 4.3. *A collection of* $\begin{pmatrix} n \\ k \end{pmatrix}$ *cells* $e(\sigma)$ *gives* $G_k(\mathbf{R}^n)$ *a cell-decomposition.*

Proof. We should show that any point x of the boundary of the cell $e(\sigma)$ belongs to some cell $e(\tau)$ of dimension less than $d(\sigma)$. We use the map $q: e(\sigma)\longrightarrow E(\sigma)$ to see that $q(\overline{e}(\sigma))=\overline{E}(\sigma)$. Thus we can describe $\pi\in\overline{e}(\sigma)\setminus e(\sigma)$ as a k-plane $\langle v_1,\ldots,v_k\rangle$, where $v_j\in\overline{H}^{\sigma_j}$. Clearly $v_j\in\mathbf{R}^{\sigma_j}$, thus $\dim(\mathbf{R}^{\sigma_j}\cap\pi)\geq j$ for each $j=1,\ldots,k$. Hence $\tau_1\leq\sigma_1$, $\ldots$, $\tau_k\leq\sigma_k$. However, at least one vector v_j belongs to the subspace $\mathbf{R}^{\sigma_j-1}=\partial\left(\overline{H}^{\sigma_j}\right)$, and corresponding $\tau_j<\sigma_j$. Thus $d(\tau)<d(\sigma)$. The number of all cells is equal to $\begin{pmatrix} k \\ n \end{pmatrix}$ by counting. $\square$

Now we count a number of cells of dimension r in the cell decomposition of $G_k(\mathbf{R}^n)$. Recall that a *partition* of an integer r is an unordered collection $(i_1,\ldots,i_s)$ such that $i_1+\cdots+i_s=r$. Let $\rho(r)$ be a number of partitions of r. This are values of $\rho(r)$ for $r\leq 10$:

r	0	1	2	3	4	5	6	7	8	9	10
$\rho(r)$	1	1	2	3	5	7	11	15	22	30	42

Each Schubert symbol $\sigma=(\sigma_1,\ldots,\sigma_k)$ of dimension $d(\sigma)=(\sigma_1-1)+(\sigma_2-2)+\cdots+(\sigma_k-k)=r$ gives a partition $(i_1,\ldots,i_s)$ of r which is given by deleting zeros from the sequence $(\sigma_1-1),(\sigma_2-2),\ldots,(\sigma_k-k)$.

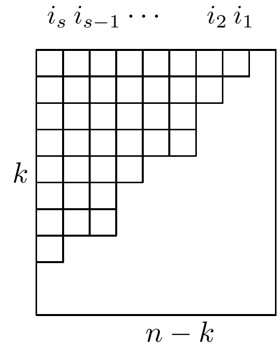
Exercise 4.14. *Show that*

$$1\leq i_1\leq i_2\leq\cdots\leq i_s\leq k,\quad \textit{and}\quad s\leq n-k.$$

Prove that a number of r*-dimensional cells of* $G_k(\mathbf{R}^n)$ *is equal to a number of partitions* $(i_1,\ldots,i_s)$ *of* r *such that* $s\leq n-k$ *and* $i_t\leq k$.

Remark. There is a natural chain of embeddings $G_k(\mathbf{R}^n)\longrightarrow G_k(R^{n+1})\longrightarrow\cdots\longrightarrow G_k(R^\infty)$. It is easy to notice that these embeddings preserve the Schubert cell decomposition, and if l and k are large enough, the number of cells of dimension r is equal to $\rho(r)$. In particular, the Schubert cells give a cell decomposition of $G(\infty,k)$ and $G(\infty,\infty)$. $\square$

Remark. Let $\iota=(i_1,\ldots,i_s)$ be a partition of r as above (i.e. $s\leq n-k$ and $1\leq i_1\leq\cdots\leq i_s\leq k$). The partition ι may be represented as a *Young tableau.*



This Young tableau gives a parametrization of the corresponding cell $e(\sigma)$. Clearly the Schubert symbols σ are in one-to-one correspondence with the Yaung tableaux corresponding to the partitions $(i_1, \ldots, i_s)$ as above. The Young tableaux were invented in the representation theory of the symmetric group S_n. This is not an accident, it turns out that there is a deep relationship between the Grassmannian manifolds and the representation theory of the symmetric groups.

Exercise 4.15. *Construct a similar CW-decomposition for the complex Grassmannian* $G_k(\mathbf{C}^n)$.

5. CW-complexes and homotopy

5.1. **Borsuk's Theorem on extension of homotopy.** We call a pair (of topological spaces) (X,A) a *Borsuk pair*, if for any map $F: X \longrightarrow Y$ a homotopy $f_t : A \longrightarrow Y$, $0 \leq t \leq 1$, such that $f_0 = F|_A$ may be extended up to homotopy $F_t : X \longrightarrow Y$, $0 \leq t \leq 1$, such that $F_t|_A = f_t$ and $F_0 = F$.

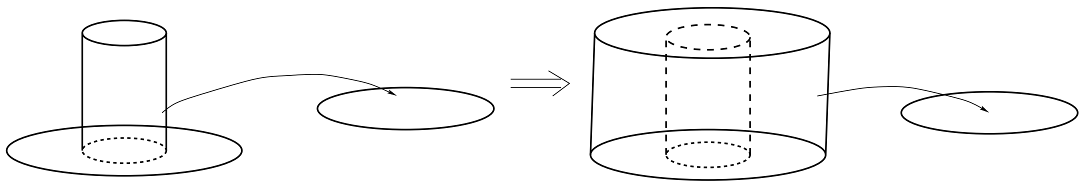


Figure 13

A major technical result of this subsection is the following theorem.

Theorem 5.1. (Borsuk) *A pair (X,A) of CW-complexes is a Borsuk pair.*

Proof. We are given a map $\Phi : A \times I \longrightarrow Y$ (a homotopy f_t) and a map $F : X \times \{0\} \longrightarrow Y$, such that $F|_{A\times\{0\}} = \Phi|_{A\times\{0\}}$. We combine the maps F and Φ to obtain a map

$$F' : X \cup (A \times I) \longrightarrow Y,$$

where we identify $A \subset X$ and $A \times \{0\} \subset A \times I$. To extend a homotopy f_t up to homotopy F_t is the same as to construct a map $\widehat{F} : X \times I \longrightarrow Y$ such that $\widehat{F}|_{X\cup(A\times I)} = F'$. We construct $\widehat{F}$ by induction on dimension of cells of $X \setminus A$. In more detail, we will construct maps

$$\widehat{F}^{(n)} : X \cup ((A \cup X^{(n)}) \times I) \longrightarrow Y$$

fo each $n = 0, 1, \ldots$ such that $F^{(n)}|_{X\cup(A\times I)} = F'$. Furthermore, the following diagram will commute

$$\begin{tikzcd}
X \cup ((A \cup X^{(n+1)}) \times I) \arrow[r, "\widehat{F}^{(n+1)}"] & Y \\
X \cup ((A \cup X^{(n)}) \times I) \arrow[u, "\iota"] \arrow[ur, "\widehat{F}^{(n)}"'] &
\end{tikzcd}$$

where ι is induced by the imbedding $X^{(n)} \subset X^{(n+1)}$.

The first step is to extend F' to the space $X \cup (A \cup X^{(0)}) \times I$ as follows:

$$\widehat{F}^{(0)}(x,t) = \begin{cases} F(x), & \text{if } x \text{ is a 0-cell from } X \text{ and if } x \notin A, \\ \Phi(x,t), & \text{if } x \in A. \end{cases}$$

Now assume by induction that $\widehat{F}^{(n)}$ is defined on $X\cup((A\cup X^{(n)})\times I)$. We notice that it is enough to define a map

$$\widehat{F}_1^{(n+1)}: X\cup((A\cup X^{(n)}\cup e^{n+1})\times I)\longrightarrow Y$$

extending $\widehat{F}^{(n)}$ to a single cell e^{n+1}. Let e^{n+1} be a $(n+1)$-cell such that $e^{n+1}\subset X\setminus A$.

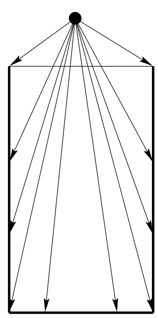
By induction, the map $\widehat{F}^{(n)}$ is already given on the cylinder $(\bar{e}^{n+1}\setminus e^{n+1})\times I$ since the boundary $\partial e^{n+1}=\bar{e}^{n+1}\setminus e^{n+1}\subset X^{(n)}$. Let $g: D^{n+1}\longrightarrow X^{(n+1)}$ be a characteristic map corresponding to the cell e^{n+1}. We have to define an extension of $\widehat{F}_1^{(n)}$ from the side $g(S^n)\times I$ and the bottom base $g(D^{n+1})\times\{0\}$ to the cylinder $g(D^{n+1})\times I$. By definition of CW-complex, it is the same as to construct an extension of the map

$$\psi=\widehat{F}^{(n)}\circ g:(D^{n+1}\times\{0\})\cup(S^n\times I)\longrightarrow Y$$

to a map of the cylinder $\psi': D^{n+1}\times I\longrightarrow Y$. Let

$$\eta: D^{n+1}\times I\longrightarrow(D^{n+1}\times\{0\})\cup(S^n\times I)$$

be a projection map of the cylinder $D^{n+1}\times I$ from a point s which is near and a bit above of the top side $D^{n+1}\times\{1\}$ of the cylinder $D^{n+1}\times I$, see the Figure below.



The map η is an identical map on $(D^{n+1}\times\{0\})\cup(S^n\times I)$. We define an extension ψ' as follows:

$$\psi': D^{n+1}\times I\xrightarrow{\eta}(D^{n+1}\times\{0\})\cup(S^n\times I)\xrightarrow{\psi} Y.$$

This procedure may be carried out independently for all $(n+1)$-cells of X, so we obtain an extension

$$\widehat{F}^{(n+1)}: X\cup((A\cup X^{(n+1)})\times I)\longrightarrow Y.$$

Exercise 5.1. *Let $D^{n+1}\times I\subset\mathbf{R}^{n+1}$ given by:*

$$D^{n+1}\times I=\left\{(x_1,\dots,x_{n+1},x_{n+2})\mid x_1^2+\cdots+x_{n+1}^2\le 1,\quad x_{n+2}\in[0,1]\right\}.$$

Give a formula for the above map η.

Thus, going from the skeleton $X^{(n)}$ to the skeleton $X^{(n+1)}$, we construct an extension $\widehat{F}: X\times I\longrightarrow Y$ of the map $F': X\cup(A\times I)\longrightarrow Y$.

We should emphasize that if X is an infinite-dimensional complex, then our construction consists of infinite number of steps; in that case the axiom **(W)** implies that $\widehat{F}$ is a continuous map. $\square$

Corollary 5.2. *Let X be a CW-complex and $A\subset X$ be its contractible subcomplex. Then X is homotopy equivalent to the complex X/A.*

Proof. Let $p : X \longrightarrow X/A$ be the projection map. Since A is a contractible there exists a homotopy $f_t : A \longrightarrow A$ such that $f_0 : A \longrightarrow A$ is an identity map, and $f_1(A) = x_0 \in A$. By Theorem 5.1 there exists a homotopy $F_t : X \longrightarrow X$, $0 \leq t \leq 1$, such that $F_0 = Id_X$ and $F_t|_A = f_t$. In particular, $F_1(A) = x_0$. It means that F_1 may be considered as a map given on X/A, (by definition of the quotient topology), i.e.

$$F_1 = q \circ p : X \xrightarrow{p} X/A \xrightarrow{q} X,$$

where $q : X/A \longrightarrow X$ is some continuous map. By construction, $F_1 \sim F_0$, i.e. $q \circ p \sim Id_X$.

Now, $F_t(A) \subset A$ for any t, i.e. $p \circ F_t(A) = x_0$. It follows that $p \circ F_t = h_t \circ p$, where $h_t : X/A \longrightarrow X/A$ is some homotopy, such that $h_0 = Id_{X/A}$ and $h_1 = p \circ q$; it means that $p \circ q \sim Id_{X/A}$. □

Corollary 5.3. *Let X be a CW-complex and $A \subset X$ be its subcomplex. Then X/A is homotopy equivalent to the complex $X \cup C(A)$, where $C(A)$ is a cone over A.*

Exercise 5.2. *Prove Corollary 5.3.*

5.2. **Cellular Approximation Theorem.** Let X and Y be CW-complexes. Recall that a map $f : X \longrightarrow Y$ is a *cellular map* if $f(X^{(n)}) \subset Y^{(n)}$ for every $n = 0, 1, \ldots$. We emphasize that it is not required that the image of n-cell belongs to a union of n-cells. For example, a constant map $* : X \longrightarrow x_0 = e^0$ is a celluar map. The following theorem provides very important tool in algebraic topology.

Theorem 5.4. *Any continuous map $f : X \longrightarrow Y$ of CW-complexes is homotopic to a cellular map.*

We shall prove the following stronger statement:

Theorem 5.5. *Let $f : X \longrightarrow Y$ be a continuous map of CW-complexes, such that a restriction $f|_A$ is a cellular map on a CW-subcomplex $A \subset X$. Then there exists a cell map $g : X \longrightarrow Y$ such that $g|_A = f|_A$ and, moreover, $f \sim g$ rel A.*

First of all, we should explain the notation $f \sim g$ rel A which we are using. Assume that we are given two maps $f, g : X \longrightarrow Y$ such that $f|_A = g|_A$. A notation $f \sim g$ rel A means that there exists a homotopy $h_t : X \longrightarrow Y$ such that $h_t(a)$ does not depend on t for any $a \in A$. Certainly $f \sim g$ rel A implies $f \sim g$, but $f \sim g$ does not imply $f \sim g$ rel A.

Exercise 5.3. *Give an example of a map $f : [0,1] \longrightarrow S^1$ which is homotopic to a constant map, and, at the same time f is not homotopic to a constant map relatively to $A = \{0\} \cup \{1\} \subset I$.*

Proof of Theorem 5.5. We assume that f is already a cellular map not only on A, but also on all cells of X of dimension less or equal to $(p-1)$. Consider a cell $e^p \subset X \setminus A$. The image $f(e^p)$ has nonempty intersection only with a finite number of cells of Y: this is because $f(\bar{e}^p)$ is a compact. We choose a cell of maximal dimension ϵ^q of Y such that it has nonempty intersection with $f(e^p)$.

If $q \le p$, then we are done with the cell e^p and we move to another p-cell. Consider the case when $q > p$. Here we need the following lemma.

Lemma 5.6. (Free-point-Lemma) *Let U be an open subset of $\mathbf{R}^p$, and $\varphi : U \to \overset{\circ}{D}{}^q$ be a continuous map such that the set $V = \varphi^{-1}(\mathbf{d}^q) \subset U$ is compact for some closed disk $\mathbf{d}^q \subset \overset{\circ}{D}{}^q$. If $q > p$ there exists a continuous map $\psi : U \longrightarrow \overset{\circ}{D}{}^q$ such that*

1. $\psi|_{U\setminus V} = \varphi|_{U\setminus V}$;
2. *the image $\psi(V)$ does not cover all disk $\mathbf{d}^q$, i.e. there exists a point $y_0 \in \mathbf{d}^q \setminus \psi(U)$.*

We postpone a proof of this Lemma for a while.

Remark. The maps φ and ψ from Lemma 5.6 are homotopic relatively to $U \setminus V$: it is enough to make a linear homotopy: $h_t(x) = (1-t)\varphi(x) + t\psi(x)$ since the disk $\overset{\circ}{D}{}^q$ is a convex set.

Claim 5.1. *Lemma 5.6 implies the following statement: The map*

$$f|_{A\cup X^{(p-1)}\cup e^p} \quad \textit{is homotopic rel } (A \cup X^{(p-1)}) \textit{ to a map} \quad f' : A \cup X^{(p-1)} \cup e^p \longrightarrow Y,$$

such that the image $f'(e^p)$ does not cover all cell ϵ^q.

Proof. Indeed, let $h : D^p \longrightarrow X$, $k : D^q \longrightarrow Y$ be the characteristic maps of the cells e^p and ϵ^q respectively. Let

$$U = h^{-1}(e^p \cap f^{-1}(\epsilon^q)),$$

and let $\varphi : U \longrightarrow \overset{\circ}{D}{}^q$ be the composition:

$$U \xrightarrow{h} e^p \cap f^{-1}\epsilon^q \xrightarrow{f} \epsilon^q \xrightarrow{k^{-1}} \overset{\circ}{D}{}^q.$$

Let $\mathbf{d}^q$ be a small disk inside $\overset{\circ}{D}{}^q$ (with the same center as $\overset{\circ}{D}{}^q$). The set $V = \varphi^{-1}(\mathbf{d}^q)$ is compact (as a closed subset of the disk $\overset{\circ}{D}{}^p$). Let $\psi : U \longrightarrow \overset{\circ}{D}{}^q$ be a map from Lemma 5.6. We define a map f' on $h(U)$ as the composition:

$$h(U) \xrightarrow{h^{-1}} U \xrightarrow{\psi} \overset{\circ}{D}{}^q \longrightarrow \epsilon^q \subset Y,$$

and $f'(x) = f(x)$ for $x \notin h(U)$. Clearly the map

$$f' : A \cup X^{(p-1)} \cup e^p \longrightarrow Y$$

is continuous (since it coincides with f on $h(U \setminus V)$) and

$$f' : A \cup X^{(p-1)} \cup e^p \longrightarrow Y \sim f|_{A\cup X^{(p-1)}\cup e^p} \ \text{ rel } \ (A \cup X^{(p-1)}),$$

moreover,

$$f' : A \cup X^{(p-1)} \cup e^p \longrightarrow Y \sim f|_{A\cup X^{(p-1)}\cup e^p} \ \text{ rel } \ (A \cup X^{(p-1)} \cup (e^p \setminus h(V)))$$

(the latter follows from a homotopy $\varphi \sim \psi$ rel $(U \setminus V)$). Also it is clear that $f'(e^p)$ does not cover all cell ϵ^q.

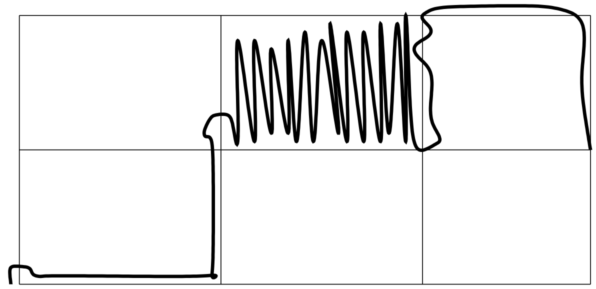


FIGURE 14

5.3. **Completion of the proof of Theorem 5.5.** Now the argument is simple. Firstly, a homotopy between the maps

$$f|_{A\cup X^{(p-1)}\cup e^p} \quad \text{and} \quad f' \quad \text{rel} \quad (A\cup X^{(p-1)})$$

can be extend to all X by Borsuk Theorem. In particular, we can assume that f' with all above properties is defined on all X.

Secondly, we consider a point $y_0\in \epsilon^q\subset Y$ which does not belong to the image $f'(e^p)$, and "blow away" the map $f'|_{e^p}$ out of that point as it is shown at Fig. 15. This is a homotopy which may be

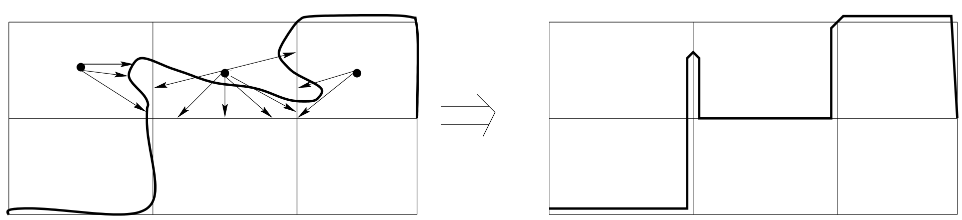


FIGURE 15

described as follows:

> If $x\in e^p$, and $x\notin (f')^{-1}(\epsilon^q)$, then $H_t(x)=f'(x)$ for all t.
> If $x\in e^p$, and $x\in (f')^{-1}(\epsilon^q)$, then $f'(x)$ moves along the ray connecting y_0 and the boundary of ϵ^q to a point on the boundary of ϵ^q.

We extend this homotopy to a homotopy of the map $f'|_{A\cup X^{(p-1)}\cup e^p}$ up to homotopy the map $f' : X\longrightarrow Y$. The resulting map f'' is homotopic to f' (and f), and $f''(e^p)$ does not touch the cell ϵ^q and any other cell of dimension $>q$. Now we can apply the procedure just described several times and we obtain a map f_1 homotopic to f, such that f_1 is a cellular map on the subcomplex $A\cup X^{(p-1)}\cup e^p$. Note that each time we applied homotopy it was fixed on (relative to) $A\cup X^{(p-1)}$. It justifies the induction step, and proves the theorem. $\square$

Exercise 5.4. *Find all points in the argument from "Completion of the proof of Theorem 5.5" where we have used Borsuk Theorem.*

Remark. Again, if the CW-complex X is infinite, then the axiom **(W)** takes care for the resulting cellular map to be continuous.

5.4. **Fighting a phantom: Proof of Lemma 5.6.** There are two well-known ways to prove our Lemma. The first one is to approximate our map by a smooth one, and then apply so called Sard Theorem. The second way is to use a *simplicial approximation* of continuous maps. The first way is more elegant, but the second is elementary, so we prove our Lemma following the second idea. First we need some new "standard spaces" which live happily inside the Euclidian space $\mathbf{R}^n$.

Let $q \leq n+1$, and $\vec{v}_1, \ldots, \vec{v}_{q+1}$ be vectors those endpoints do not belong to any $(q-1)$-dimensional subspace. We call the set

$$\Delta^q(\vec{v}_1, \ldots, \vec{v}_{q+1}) = \{ \ t_1\vec{v}_1 + \ldots + t_q\vec{v}_{q+1} \mid t_1 + \ldots + t_{q+1} = 1, \quad t_1 \geq 0, \ldots, t_{q+1} \geq 0 \ \}$$

a q-dimensional simplex.

Exercise 5.5. $\Delta(\vec{v}_1, \ldots, \vec{v}_{q+1})$ *is homeomorphic (moreover, by means of a linear map) to the standard simplex*

$$\Delta^q = \left\{ (x_1, \ldots, x_{q+1}) \in \mathbf{R}^{q+1} \mid x_1 \geq 0, \ldots, x_{q+1} \geq 0, \ \textstyle\sum_{i=1}^{q+1} x_i = 1 \ \right\}.$$

Example. A 0-simplex is a point; a simplex Δ^1 is the interval connecting two points; a simplex δ^2 is a nondegenerated triangle in the space $\mathbf{R}^n$; a simplex Δ^3 is a pyramid in $\mathbf{R}^n$ with the vertices $\vec{v}_0, \vec{v}_1, \vec{v}_2, \vec{v}_3$, see the picture below:

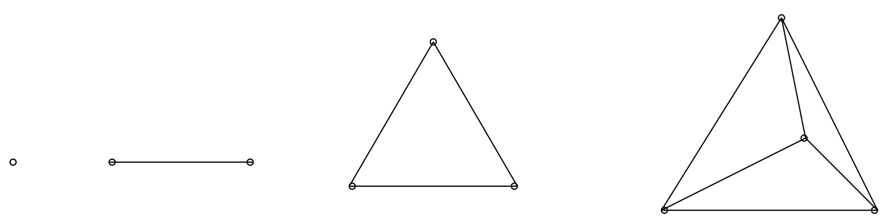


FIGURE 16

A j-th face of the simplex $\Delta^q(\vec{v}_1, \ldots, \vec{v}_{q+1})$ is the following $(q-1)$-simplex:

$$\Delta^{q-1}(\vec{v}_1, \ldots, \vec{v}_{j-1}, \vec{v}_{j+1}, \ldots, \vec{v}_q)_j = \{t_1\vec{v}_1 + \ldots + t_{q+1}\vec{v}_{q+1} \in \Delta^q(\vec{v}_1, \ldots, \vec{v}_{q+1}) \mid t_j = 0 \ \}.$$

We are not going to develop a theory of *simplicial complexes* (this theory is parallel to the theory of CW-complexes), however we need the following definition

Definition 5.7. *A finite triangulation of a subset $X \subset \mathbf{R}^n$ is a finite covering of X by simplices $\{\Delta^n(i)\}$ such that each intersection $\Delta^n(i) \cap \Delta^n(j)$ either empty, or*

$$\Delta^n(i) \cap \Delta^n(j) = \Delta^{n-1}(i)_k = \Delta^{n-1}(j)_\ell$$

for some k, ℓ.

Exercise 5.5. Let $\Delta_1^n, \ldots, \Delta_s^n$ be a finite set of n-dimensional simplexes in $\mathbf{R}^n$. Prove that the union $K = \Delta_1^n \cup \Delta_2^n \cup \cdots \cup \Delta_s^n$ is a finite simplicial complex.

Exercise 5.6. Let Δ_1^p, Δ_2^q be two simplices. Prove that $K = \Delta_1^p \times \Delta_2^q$ is a finite simplicial complex.

A *barycentric subdivision* of a q-simplex Δ^q is a subdivision of this simplex on $(q+1)!$ smaller simplices as follows. First let us look at the example:

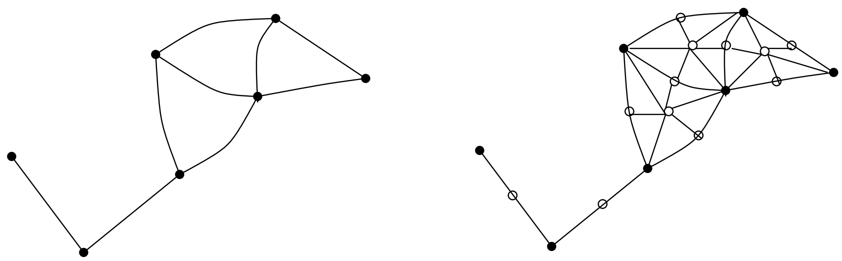


FIGURE 17

In general, we can proceed by induction. The picture above shows a barycentric subdivision of the simplices Δ^1, and Δ^2. Assume by induction that we have defined a barycentric subdivision of the simplices Δ^j for $j \leq q-1$. Now let x^* be a weight center of the simplex Δ^q. We already have a barycentric subdivision of each j-the side Δ_j^q by $(q-1)$-simplices $\Delta_j^{(1)}, \ldots, \Delta_j^{(n)}$, $n = q!$. The cones over these simplices, $j = 0, \ldots, q$, with a vertex x^* constitute a barycentric subdivision of Δ^q. Now we will prove the following "Approximation Lemma":

Lemma 5.8. *Let $V \subset U$ be two open sets of $\mathbf{R}^n$ such that their closure $\bar{V}$, $\bar{U}$ are compact sets and $\bar{V} \subset U$. Then there exists a finite triangulation of V by n-simplices $\{\Delta^n(i)\}$ such that $\Delta^n(i) \subset U$.*

Proof. For each point $x \in \bar{V}$ there exists a simplex $\Delta^n(x)$ with a center at x and $\Delta^n(x) \subset U$. By compactness of $\bar{V}$ there exist a finite number of simplices $\Delta^n(x_i)$ covering $\bar{V}$. It remains to use Exercise 5.6 to conclude that a union of finite number of $\Delta^n(x_i)$ has a finite triangulation. $\square$

5.5. **Back to the Proof of Lemma 5.6.** We consider carefully our map $\varphi : U \longrightarrow \overset{\circ}{D}{}^q$. First we construct the disks $\mathbf{d}_1$, $\mathbf{d}_2$, $\mathbf{d}_3$, $\mathbf{d}_4$ inside the disk $\mathbf{d}$ with the same center and of radii $r/5$, $2r/5$, $3r/5$, $4r/5$ respectively, where r is a radius of $\mathbf{d}$. Then we cover $V = \varphi^{-1}(\mathbf{d})$ by finite number of p-simplexes $\Delta^p(j)$, such that $\Delta^p(j) \subset U$. Making, if necessary, a barycentric subdivision (a finite number of times) of these simplices, we can assume that each simplex Δ^p has a diameter $\mathrm{d}(\varphi(\Delta^p(j))) < r/5$. Let K_1 be a union of all simplices $\Delta^p(j)$ such that the intersection $\varphi(\Delta^q(j)) \cap \mathbf{d}_4$ is not empty. Then

$$\mathbf{d}_4 \cap \varphi(U) \subset \varphi(K_1) \subset \mathbf{d}.$$

Now we consider a map $\varphi' : K_1 \longrightarrow \mathbf{d}_4$ which coincides with φ on all vertices of our triangulation, and is linear on each simplex $\Delta \subset K_1$. The maps $\varphi|_{K_1}$ and φ' are homotopic, i.e. there is a homotopy $\varphi_t : K_1 \longrightarrow \mathbf{d}_4$, such that $\varphi_0 = \varphi|_{K_1}$ and $\varphi_1 = \varphi'$.

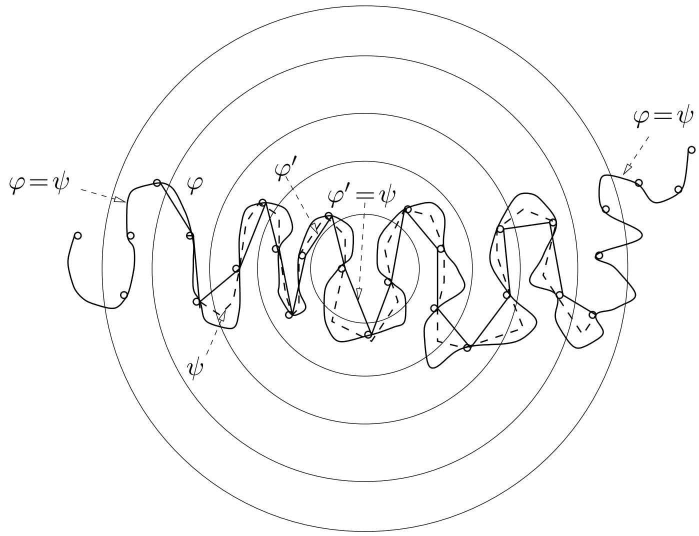


FIGURE 18

Exercise 5.7. Construct a homotopy φ_t as above.

Now we construct a map $\psi : U \longrightarrow \overset{\circ}{D}{}^q$ out of maps φ, φ_t and φ' as follows:

$$\psi(u) = \begin{cases} \varphi(u) & \text{if} \quad \varphi(u) \notin \mathbf{d}_3, \\ \varphi'(u) & \text{if} \quad \varphi(u) \in \mathbf{d}_2, \\ \varphi_{3-\frac{5r(u)}{r}}(u) & \text{if} \quad \varphi(u) \in \mathbf{d}_3 \setminus \mathbf{d}_2. \end{cases}$$

Here $r(u)$ is a distance from $\varphi(u)$ to a center of the disk $\mathbf{d}$, see Fig. 5.7.

Now we notice that ψ is a continuous map, and it coincides with φ on $U \setminus V$. Furthermore, the intersection of its image with $\mathbf{d}_1$, the set $\psi(U) \cap \mathbf{d}_1$, is a union of finite number of pieces of p-dimensional planes, i.e. there is a point $y \in \mathbf{d}_1$ which $y \notin \psi(U)$.

Exercise 5.8. Let $\pi_1, \ldots, \pi_s$ be a finite number of p-dimensional planes in $\mathbf{R}^q$, where $p < q$. Prove that the union $\pi_1 \cup \cdots \cup \pi_s$ does cover any open subset $U \subset \mathbf{R}^n$.

Thus Cellular Approximation Theorem proved. □

5.6. **First applications of Cellular Approximation Theorem.** We start with the following important result.

Theorem 5.9. *Let X be a CW-complex with only one zero-cell and without q-cells for $0 < q < n$, and Y be a CW-complex of dimension $< n$, i.e. $Y = Y^{(k)}$, where $k < n$. Then any map $Y \longrightarrow X$ is homotopic to a constant map. The same statement holds for "pointed" spaces and "pointed" maps.*

Exercise 5.9. Prove Theorem 5.9 using the Cellular Approximation Theorem.

Remark. For each pointed space (X, x_0) define $\pi_k(X, x_0) = [S^k, X]$ (where we consider homotopy classes of maps $f : (S^k, s_0) \longrightarrow (X, x_0)$). Very soon we will learn a lot about $\pi_k(X, x_0)$, in particular, that there is a natural group structure on $\pi_k(X, x_0)$ which are called *homotopy groups* of X.

The following statement is a particular case of Theorem 5.9:

Corollary 5.10. *The homotopy groups $\pi_k(S^n)$ are trivial for $1 \leq k < n$.*

We call a space X *n-connected* if it is path-connected and $\pi_k(X) = 0$ for $k = 1, \dots, n$.

Exercise 5.10. Prove that a space X is 0-connected if and only if it is path-connected.

Theorem 5.11. *Let $n \geq 1$. Any n-connected CW-complex homotopy equivalent to a CW-complex with a single zero cell and no cells of dimensions $1, 2, \dots, n$.*

Proof. Let us choose a cell e^0 and for each zero cell e^0_i choose a path s_i connecting e^0_i and e^0 (these paths may have nonempty intersections). By Cellular Approximation Theorem we can choose these paths inside 1-skeleton. Now for each path s_i we glue a 2-disk, identifying a half-circle with s_i, see the picture:

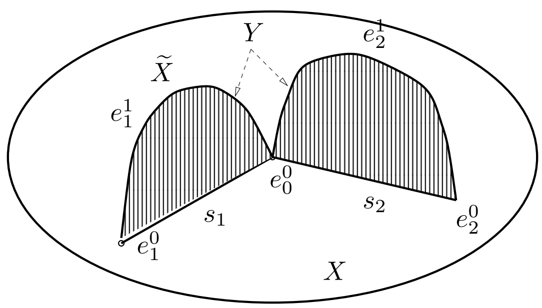


FIGURE 19

We denote the resulting CW-complex by $\widetilde{X}$. The CW-complex $\widetilde{X}$ has the same cells as X and new cells e^1_i, e^2_i (the top half-circles and interior of 2-disks). A boundary of each cell e^2_i belongs to the first skeleton since the paths s_i are in the first skeleton.

Clearly the complex $\widetilde{X}$ is a deformational retract of X (one can deform each cell e^2_i to the path s_i). Let Y be a closure of the union $\bigcup_i e^1_i$. Obviously Y is contractible. Now note that $\widetilde{X}/Y \sim \widetilde{X} \sim X$, and the complex $\widetilde{X}/Y$ has only one zero cell.

Now we use induction. Let us assume that we already have constructed the CW-complex X' such that $X' \sim X$ and X' has a single zero cell, and it does not have cells of dimensions $1, 2, \dots, k-1$, where $k \leq n$. Note that a closure of each k-cell of X' is a sphere S^k by induction. Indeed, an attaching map for every k-cell has to go to $X'^{(0)}$. Since X' is still k-connected, then the embedding

$S^k \longrightarrow X'$ (corresponding to a cell e_i^k) may be extended to a map $D^{k+1} \longrightarrow X'$. Again, Cellular Approximation Theorem implies that we can choose such extention that the image of D^{k+1} belongs to the $(k+1)$-skeleton of X'. Now we glue the disk D^{k+2} to X' using the map $D^{k+1} \longrightarrow X'^{(k+1)}$ (we identify the disk D^{k+1} with a bottom half-sphere S_-^{k+1} of the boundary sphere $S^{k+1} = \partial D^{k+2}$). We denote this $(k+2)$-cell e_i^{k+2} and the $(k+1)$-cell given by the top half-sphere S_+^{k+1}, by e_i^{k+1}. We do this procedure for each k-cell e_i^k of the complex X' and construct the complex $\widetilde{X}'$. Certainly $\widetilde{X}' \sim X' \sim X$. Now let Y' be a closure of the union $\bigcup_i e_i^{k+1}$, where, as above, e_i^{k+1} are the top half-spheres of the cells e_i^{k+2}. Clearly Y' is contractible, and we obtain a chain of homotopy equivalences:

$$\widetilde{X}'/Y' \sim \widetilde{X}' \sim X' \sim X,$$

where $\widetilde{X}'/Y'$ has no k-cells. This proves Theorem 5.11. $\square$

Corollary 5.12. *Let Y be n-connected CW-complex, and X be an n-dimensional CW-complex. Then the set $[X,Y]$ consists of a single element.*

A pair of spaces (X,A) is n-connected if for any $k \leq n$ and any map of pairs

$$f : (D^k, S^{k-1}) \longrightarrow (X, A)$$

homotopic to a map $g : (D^k, S^{k-1}) \longrightarrow (X,A)$ (as a map of pairs) so that $g(D^k) \subset A$.

Exercise 5.11. What does it mean geometrically that a pair (X,A) is 0-connected? 1-connected? Give some alternative description.

Exercise 5.12. Let (X,A) be an n-connected pair of CW-complexes. Prove that (X,A) is homotopy equivalent to a CW-pair (Y,B) so that $B \subset Y^{(n)}$.

6. Fundamental group

6.1. **General definitions.** Here we define the homotopy groups $\pi_n(X)$ for all $n \geq 1$ and examine their basic properties. Let (X, x_0) be a pointed space, and (S^n, s_0) be a pointed sphere. We have defined the set $[S^n, X]$ as a set of homotopy classes of maps $f : S^n \longrightarrow X$, such that $f(s_0) = x_0$, and homotopy between maps should preserve this property. In different terms we can think of a representative of $[S^n, X]$ as a map $I^n \longrightarrow X$ such that the image of the boundary ∂I^n of the cube I^n maps to the point x_0.

The sum of two spheres $f, g : S^n \longrightarrow X$ is defined as the map

$$f + g : S^n \longrightarrow X,$$

constructing as follows. First we identify the equator of the sphere S^n (which contains the point s_0) to a single point, so we obtain a wedge of two spheres $S^n \wedge S^n$, and then we map the "top sphere" S^n with the map f, and the "bottom sphere" S^n with the map g, see the picture below:

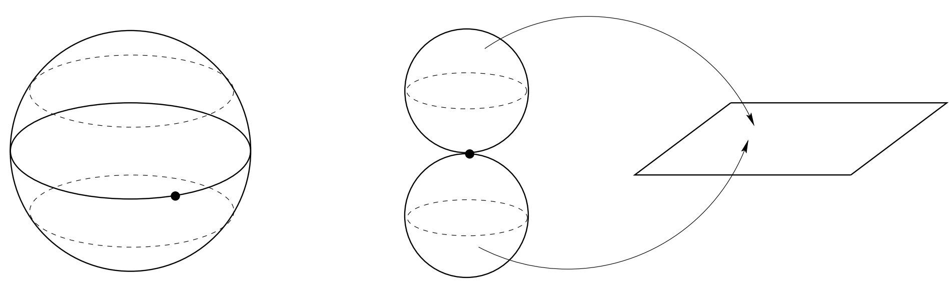


Figure 20

Exercise 6.1. *Prove that this operation is well-defined and induces a group structure on the set* $\pi_n(X) = [S^n, X]$. *In particular check associativity and existence of the unit.*

Lemma 6.1. *For* $n \geq 2$ *the homotopy group* $\pi_n(X)$ *is a commutative group.*

Proof. The corresonding homotopy is given below, where the black parts of the cube map to the point x_0:

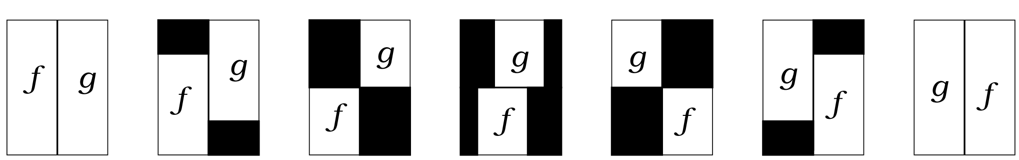


Figure 21

Remark. We note that the first homotopy group[7] $\pi_1(X)$ is not commutative in general. We will use "+" for the operation in the homotopy groups $\pi_n(X)$ for $n \geq 2$ and product sign "$\cdot$" for the fundamental group.

Now let $f : X \longrightarrow Y$ be a map; it induces a homomorphism $f_* : \pi_n(X) \longrightarrow \pi_n(Y)$.

Exercise 6.2. *Prove that if $f, g : X \longrightarrow Y$ are homotopic maps of pointed spaces, than the homomorphisms $f_*, g_* : \pi_n(X) \longrightarrow \pi_n(Y)$ coincide.*

Exercise 6.3. *Prove that $\pi_n(X \times Y) \cong \pi_n(X) \times \pi_n(Y)$ for any spaces X, Y.*

6.2. **One more definition of the fundamental group.** The definition above was two general, we repeat it in more suitable terms again.

We consider loops of the space X, i.e. such maps $\varphi : I \longrightarrow X$ that $\varphi(0) = \varphi(1) = x_0$. The loops φ, φ' are homotopic if there is a homotopy $\varphi_t : I \longrightarrow X$, $(0 \leq t \leq 1)$ such that $\varphi_0 = \varphi$, $\varphi_1 = \varphi'$. A "product" of the loops φ, ψ is the loop ω, difined by the formula:

$$\omega(t) = \begin{cases} \varphi(2t), & \text{for } 0 \leq t \leq 1/2, \\ \psi(2t-1), & \text{for } 1/2 \leq t \leq 1. \end{cases}$$

This product operation induces a group structure of $\pi_1(X)$. It is easy to check that a group operation is well-defined. Note that the loop $\bar{\varphi}(t) = \varphi(1-t)$ defines a homotopy class $[\varphi]^{-1} = [\bar{\varphi}]$

Exercise 6.4. *Write an explicit formula givinig a null-homotopy for the composition $\bar{\varphi} \cdot \varphi$.*

6.3. **Dependence of the fundamental group on the base point.**

Theorem 6.2. *Let X be a path-connected space, then $\pi_1(X, x_0) \cong \pi_1(X, x_1)$ for any two points $x_0, x_1 \in X$.*

Proof. Since X is path-connected, there exist a path $\alpha : I \longrightarrow X$, such that $\alpha(0) = x_0$, $\alpha(1) = x_1$. We define a homomorphism $\alpha_\# : \pi_1(X, x_0) \longrightarrow \pi_1(X, x_1)$ as follows. Let $[\varphi] \in \pi_1(X, x_0)$. We define $\alpha_\#([\varphi]) = (\alpha\varphi)\alpha^{-1}$. [8] It is very easy to check that $\alpha_\#$ is well-defined and is a homomorphism. Moreover, the homomorphism $\alpha_\#^{-1} : \pi_1(X, x_1) \longrightarrow \pi_1(X, x_0)$ defined by the formula $\alpha_\#^{-1}([\psi]) = [(\alpha^{-1}\psi)\alpha]$, gives a homomorphism which is inverse to $\alpha_\#$. The rest of the proof is left to you. $\square$

Perhaps the isomorphism $\alpha_\#$ depends on α. Let β be the other path, $\beta(0) = x_0$, $\beta(1) = x_1$. Let $\gamma = \beta\alpha^{-1}$ which defines an element $[\gamma] \in \pi_1(X, x_1)$.

Exercise 6.5. *Prove that $\beta_\# = [\gamma]\alpha_\#[\gamma]^{-1}$.*

Exercise 6.6. *Let $f : X \longrightarrow Y$ be a homotopy equivalence, and $x_0 \in X$. Prove that $f_* : \pi_1(X) \longrightarrow \pi_1(Y, f(x_0))$ is an isomorphism.*

---

[7] There is a special name for the group $\pi_1(X)$: *the fundamental group of X*.

[8] here we "multiply" not just loops, but paths as well: we can always do that if the second path starts at the same point where the first ends

6.4. **Fundamental group of circle.** Here we will compute the fundamental group of the circle. In fact, we will be using a "universal covering space" of the circle which we did not defined yet.

Theorem 6.3. $\pi_1 S^1 \cong \mathbf{Z}$.

Proof. Consider the map $\exp : \mathbf{R} \longrightarrow S^1$ defined by the formula: $x \longrightarrow e^{ix}$. We can think about the circle S^1 as the quotient group $\mathbf{R}/\mathbf{Z}$ (where $\mathbf{Z}$ is embedded $\mathbf{R}$ as the set of the numbers $2\pi k$, $k = 0, \pm 1, \pm 2, \dots$). A loop $\varphi : I \longrightarrow S^1$ ($\varphi(0) = \varphi(1) = e^0$) may be lifted to a map $\widetilde{\varphi} : I \longrightarrow \mathbf{R}$. It means that φ is decomposed as

$$\varphi : I \xrightarrow{\widetilde{\varphi}} \mathbf{R} \xrightarrow{\exp} \mathbf{R}/\mathbf{Z} = S^1,$$

where $\widetilde{\varphi}(0) = 0$ and $\widetilde{\varphi}(1) = 2\pi k$ for some integer k. Note that a lifting $\widetilde{\varphi} : I \longrightarrow \mathbf{R}$ with the above properties is unique.

Note that if the loops $\varphi, \varphi' : I \longrightarrow S^1$ are homotopic, then the paths $\widetilde{\varphi}$, $\widetilde{\varphi}'$ have the same end point $2\pi k$ (since we cannot "jump" from $2\pi k$ to $2\pi l$ if $l \neq k$ by means of continuous homotopy!). Now the isomorphism $\pi_1 \cong \mathbf{Z}$ becomes almost obvious: $[\varphi] \longrightarrow k \in \mathbf{Z}$. It remains to see that the loop $\varphi : I \longrightarrow S^1$ ($\widetilde{\varphi} : I \longrightarrow \mathbf{R}$, where $\widetilde{\varphi}(0) = 0$ and $\widetilde{\varphi}(1) = 2\pi k$) is homotopic to the "standatrd loop" $\widetilde{h}_k$ going from 0 to $2\pi k$, see picture below:

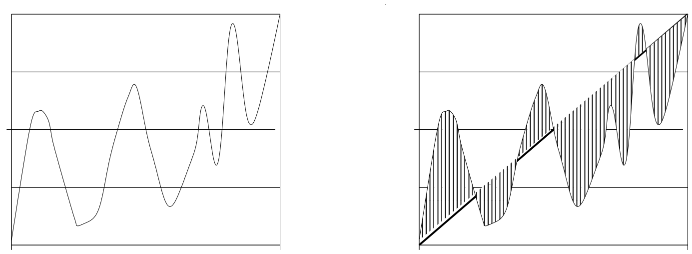


FIGURE 22

It remains to observe that $\widetilde{h}_k \widetilde{h}_l \sim \widetilde{h}_{k+l}$. ☐

Theorem 6.4. *Let* $X_A = \bigvee_{\alpha \in A} S^1_\alpha$. *Then* $\pi_1(X_A)$ *is a free group with generators* η_α, $\alpha \in A$.

Proof. Let $i_\alpha : S^1 \longrightarrow X_A$ be an embedding of the corresponding circle. Let $\eta_\alpha \in \pi_1(X_A)$ be the element given by i_α. We prove the following statement.

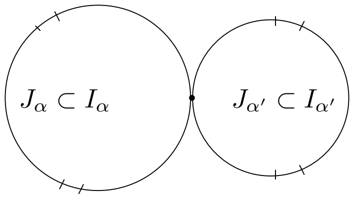
Claim 6.1. **1**^{o} *Any element* $\beta \in \pi_1(X_A)$ *may be represented as a finite product of elements* η_α, η_α^{-1}, $\alpha \in A$:

$$\beta = \eta_{\alpha_1}^{\epsilon_1} \cdots \eta_{\alpha_s}^{\epsilon_s}, \qquad \epsilon_j = \pm 1. \tag{13}$$

2^o *The presentation* (13) *is unique up to cancelation of the elements* $\eta_\alpha \eta_\alpha^{-1}$ *or* $\eta_\alpha^{-1} \eta_\alpha$.

Claim 6.1 is equivalent to Theorem 6.4. Now we prove $\mathbf{1}^o$, and we postpone $\mathbf{2}^o$ to the next section.

Proof of $\mathbf{1}^o$. Let I_α, J_α be two closed intervals in the circle, $J_\alpha \subset \mathrm{Int} I_\alpha$, and I_α does not contain the base point.

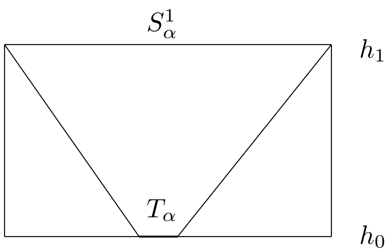


Now let $\varphi : I \longrightarrow X_A$ be a loop. We find n such that for any interval J of the length $1/n$ if the intersection $\varphi(J) \cap J_\alpha \neq \emptyset$, then $\varphi(J) \subset I_\alpha$. Let K be the following union:

$$K = \bigcup_{\varphi([k/n,(k+1)/n]) \cap (\cup_\alpha J_\alpha) \neq \emptyset} [k/n, (k+1)/n].$$

Now we construct a map $\varphi_1 : I \longrightarrow X_A$ which coincides with φ outside of K and in all points with the coordinates k/n, and it is linear on each interval $[k/n, (k+1)/n] \subset K$.[9]

Exercise 6.7. *Give a formula for the map φ_1.*



Homotopy h_t

Clearly the loop φ_1 is homotopic to φ. Now we find an interval $T_\alpha \subset J_\alpha$, so that T_α does not contain points $\varphi_1(k/n)$. We can do this since there is only finite number of points like that inside of each J_α. We notice that $\varphi_1^{-1}(T_\alpha) \subset I$ is a finite number of disjoint intervals $S^{(1)}_\alpha, \ldots, S^{(r_\alpha)}_\alpha$ so that the map $\varphi_1|_{S^{(j)}_\alpha} : S^{(j)}_\alpha \longrightarrow T_\alpha$ is linear for each j. The last step: we define a homotopy $h_t : X_A \longrightarrow X_A$ which stretches linearly each interval T_α on the circle S^1_α and taking $S^1_\alpha \setminus T_\alpha$ to the base point.

Exercise 6.8. *Give a formula for the homotopy h_t.*

Exercise 6.9. *Prove that the inverse image $\varphi_1^{-1}(\cup_\alpha T_\alpha) \subset I$ consists of finite number of disjoint intervals.*

Then the map $\psi = h_1 \circ \varphi_1$ gives a loop which maps I as follows. For each $\alpha \in A$ there is finite number of disjoint intervals $S^{(j)}_\alpha \subset I$ so that $S^{(j)}_\alpha$ maps linearly on the circle S^1_α. The restriction $\psi|_{S^{(j)}_\alpha}$ maps the interval $S^{(j)}_\alpha$ clock-wise or counterclock-wise; this corresponds to either element η_α or η_α^{-1}. Then the rest of the interval I, a complement to the union

$$\bigcup_{\alpha \in A} (S^{(1)}_\alpha \sqcup \cdots \sqcup S^{(r_\alpha)}_\alpha)$$

maps to the base point. $\square$

6.5. **Fundamental group of a finite CW-complex.** Here we prove a general result showing how to compute the fundamental group $\pi_1(X)$ for arbitrary CW-complex X.

Remark. Let X be a path-connected. If a map $S^1 \longrightarrow X$ sends a base point s_0 to a base point x_0 then it determines an element of $\pi_1(X, x_0)$; if f sends s_0 somethere else, then it defines an element

[9] A linear map $I \longrightarrow S^1$ is given by $t \mapsto (\cos(\lambda t + \mu), \sin(\lambda t + \nu))$ for some constants λ, μ, ν.

of the group $\pi(X, f(s_0))$, which is isomorphic to $\pi_1(X, x_0)$ with an isomorphism $\alpha_\#$. The images of the element $[f] \in \pi(X, f(s_0))$ in the group $\pi_1(X, x_0)$ under all possible isomorphisms $\alpha_\#$ define a class of conjugated elements. So we can say that a map $S^1 \longrightarrow X$ to a path-connected space X determines an element of $\pi_1(X, x_0)$ up to conjugation.

Let X be a CW-complex with a single zero-cell $e^0 = x_0$, one-cells e^1_i, $i \in I$, and two-cells e^2_j, $j \in J$. Then we identify the first skeleton $X^{(1)}$ with $\bigvee_{i\in I} S^1_i$. The inclusion map $S^1_i \to \bigvee_{i\in I} S^1_i$ determines an element $\alpha_i \in \pi_1(X^{(1)}, x_0)$. By Theorem 6.4 $\pi_1(X^{(1)}, x_0)$ is a free group on generators α_i, $i \in I$. The characteristic map $g_j : D^2 \longrightarrow X$ of the cell e^2_j determines attaching map $f_j : S^1 \longrightarrow X^{(1)}$ which determines an element $\beta_j \in \pi_1(X^{(1)}, x_0)$ up to conjugation.

Theorem 6.5. *Let X be a CW-complex with a single zero cell e^0, one-cells e^1_i ($i \in I$), and two-cells e^2_j ($j \in J$). Let α_i be the generators of $\pi_1(X^{(1)}, x_0)$ corresponding to the the cells e^1_i, and $\beta_j \in \pi_1(X^{(1)}, x_0) = F(\alpha_i \mid i \in I)$ be elements determined by the attaching maps $f_j : S^1 \longrightarrow X^1$ of the cells e^2_j. Then*

1. $\pi_1(X, x_0) \cong \pi_1(X^{(2)}, x_0)$*;*
2. $\pi_1(X, x_0)$ *is a group on generators α_i, $i \in I$, and relations $\beta_j = 1$, $j \in J$.*

Proof. We consider the circle S^1 as 1-dimensional CW-complex. Cellular Approximation Theorem implies then that any loop $S^1 \longrightarrow X$ homotopic to a loop in the first skeleton, i.e. the homomorphism

$$\iota_* : \pi_1(X^{(1)}, x_0) \longrightarrow \pi_1(X, x_0)$$

induced by the inclusion $\iota : X^{(1)} \longrightarrow X$, is an epimorphism. It is enough to prove that Ker i_* is generated by β_j, $j \in J$. It is clear that $\beta_j \in \text{Ker } \iota_*$. Indeed, the attaching map $f_j : S^1 \longrightarrow X^{(1)}$ is extended to the characteristic map $g_j : D^2 \longrightarrow X$, and determines a trivial element in the group $\pi_1(X, f_j(s_0))$; and this element corresponds to β_j under some isomorphism $\pi_1(X, f_j(s_0)) \cong \pi_1(X, x_0)$.

It is more difficult to prove that if $\gamma \in \text{Ker } \iota_*$ then γ may be presented (up to conjugations) as a product of elements $\beta_j^{\pm}$. Here we will apply again the technique we used to prove Cellular Approximation Theorem. We identify each cell e^2_j with the open disk D^2_j in $\mathbf{R}^2$, so we can construct disks $d^{(j)} \subset D^2_j$ of radius $r^{(j)}$, and disks $d^{(j)}_1$, $d^{(j)}_2$, $d^{(j)}_3$ and $d^{(j)}_4$ (with the same center) of radius $r^{(j)}/5$, $2r^{(j)}/5$, $3r^{(j)}/5$ and $4r^{(j)}/5$ respectively. Now let $\varphi : S^1 \longrightarrow X^{(1)}$ be a representative of an element $\gamma \in \text{Ker } \iota_*$. Clearly there is an extension $\Phi : D^2 \longrightarrow X$ of the map φ. By the Cellular Approximation Theorem we can assume that $\Phi(D^2) \subset X^{(2)}$. We triangulate D^2 in such way that if Δ is a triangle from this triangulation such that $\Phi(\Delta) \cap d^{(j)}_4 \neq \emptyset$, then

(a) $\Phi(\Delta) \subset d^{(j)}$ and

(b) $\text{diam}(\Phi(\Delta) < r^{(j)}/5$.

Let K be a union of all triangles Δ of our triangulation such that

$$\Phi(\Delta) \cap \left(\bigcup_{j \in J} d_4^{(j)} \right) \neq \emptyset.$$

Now we make the map $\Phi' : K \longrightarrow X^{(2)}$ which concides with Φ on the vertexes of each simplex and is linear on each simplex Δ. The maps Φ' and $\Phi|_K$ are homotopic (inside the cell e_j^2) by means of a homotopy $\Phi_t = (1-t)\Phi + t\Phi'$, with $\Phi_0 = \Phi|_K$, $\Phi_1 = \Phi'$. Now we use a familiar formula

$$\Phi''(u) = \begin{cases} \Phi(u), & \text{if} \quad \Phi(u) \notin \bigcup_j d_3^{(j)} \\ \Phi'(u), & \text{if} \quad \Phi(u) \in \bigcup_j d_2^{(j)} \\ \Phi_{3 - \frac{5r(u)}{r(j)}}(u), & \text{if} \quad \Phi(u) \in d_3^{(j)} \setminus d_2^{(j)} \end{cases}$$

to define a map Φ'', which is a piece-wise linear on the inverse image of $\bigcup_j d_1^{(j)}$.

Now we choose a small disk $\delta^{(j)} \subset d_1^{(j)}$ which does not intersect with images of all vertices and 1-faces of all simlexes Δ. There are two possibilities:

1. $\delta^{(j)} \subset \Phi''(\Delta)$ for some simplex Δ;

2. $(\Phi'')^{-1}(\delta^{(j)}) = \emptyset$.

Let $\omega : X^{(2)} \longrightarrow X^{(2)}$ be a map identical on $X^{(1)}$ and mapping each disk $\delta^{(j)}$ on the cell e_j^2 (by pushing $e^2 \setminus \delta^{(j)}$ to the boundary of e_j^2). The map

$$\Psi : D^2 \xrightarrow{\Phi''} X^{(2)} \xrightarrow{\omega} X^{(2)},$$

extends the same map $\varphi : S^1 \longrightarrow X^{(1)}$.

Note that in the case **1.** the inverse image of $\delta^{(j)}$ under the map Φ'' is a finite number of ovals $E_1, \ldots, E_s$ (bounded by an ellips), and in the case **2.** the inverse image of $\delta^{(j)}$ is empty. We see that the map Ψ maps the complement $D^2 \setminus (\bigcup_s E_s)$ to $X^{(1)}$, and maps each oval $E_1 \ldots, E_k$ linearly on one of the cells e_j^2.

We join now a point $s_0 \in S^1 \subset D^2$ with each oval $E_1, \ldots, E_k$ by paths $s_1, \ldots s_k$, which do not intersect with each other, see the picture below:

We denote by $\sigma_1, \ldots, \sigma_k$ the loops, going clock-wise around each oval. Then the loop σ going clock-wise along the circle $S^1 \subset D^2$ is homotopic in $D^2 \setminus \bigcup_t \mathrm{Int}(E_t)$ to the loop:

$$(s_k \sigma_k s_k^{-1}) \cdots (s_2 \sigma_2 s_2^{-1})(s_1 \sigma_1 s_1^{-1}),$$

see Fig. 6.6.

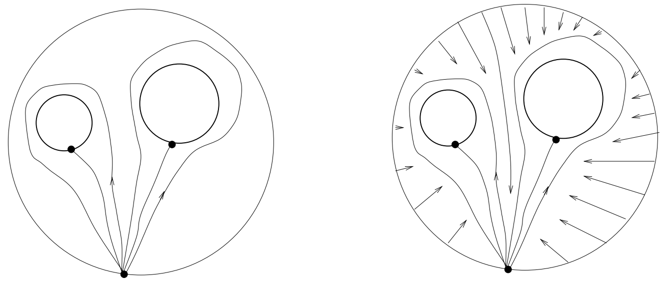


FIGURE 23

It means that the loop $\varphi : S^1 \longrightarrow X^{(2)}$ is homotopic (in $X^{(1)}$) to the loop

$$[\Psi \circ (s_k \sigma_k s_k^{-1})] \cdots [\Psi \circ (s_2 \sigma_2 s_2^{-1})][\Psi \circ (s_1 \sigma_1 s_1^{-1})].$$

It remains to observe that the loop $[\Psi \circ (s_j \sigma_j s_j^{-1})]$ determines an element in $\pi_1(X, x_0)$, conjugate to $\beta_j^{\pm 1}$.

We see now that the element γ belongs to a normal subgroup of $F(\alpha_i \mid i \in I)$, generated by β_j. $\square$

Exercise 6.10. *Finish the proof in the case* **2**, *i.e. when* $(\Phi'')^{-1}(\delta^{(j)}) = \emptyset$.

Theorem 6.5 helps to compute fundamental groups of all classic spaces. In the case of S^n $(n \geq 2)$ and $\mathbf{CP}^n$, $n \geq 1$ we see that the fundamental group is trivial. However, there are several interesting cases:

Theorem 6.6. *Let M_g^2 be a two-dimensional manifold, the sphere with g handles (oriented manifold of genus g). Then $\pi_1(M_g^2)$ is generated by $2g$ generators $a_1, \dots a_g, b_1, \dots, b_g$ with a single relation:*

$$a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} = 1.$$

Exercise 6.11. *Prove Theorem 6.6.*

Exercise 6.12. *For a group π, we let $[\pi, \pi]$ be its commutator. Compute the group $\pi/[\pi, \pi]$ for $\pi = \pi_1(M_g)$.*

Remark. We note that in particular $\pi_1(T^2) \cong \mathbf{Z} \oplus \mathbf{Z}$, which is obvious from the product formula $\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$.

Recall that a non-oriented two-dimensional manifold of genus g is heomeomorphic either to $M_g^2(1)$, a connective sum of a projective plane $\mathbf{RP}^2$ and g tori $T^2 \# \cdots \# T^2$, or to $M_g^2(2)$, a connective sum of the Klein bottle Kl^2 and g tori $T^2 \# \cdots \# T^2$.

Theorem 6.7. 1. *The group $\pi_1(M_g^2(1))$ is isomorphic to a group on generators $c_1, \dots, c_{2g+1}$ wit a single relation*

$$c_1^2 \cdots c_{2g+1}^2 = 1.$$

2. *The group $\pi_1(M_g^2(2))$ is isomorphic to a group on generators $c_1, \dots, c_{2g+2}$ wit a single relation*

$$c_1^2 \cdots c_{2g+1}^2 c_{2g+2}^2 = 1.$$

Exercise 6.13. *Prove Theorem 6.7.*

Exercise 6.14. *Compute $\pi_1(\mathbf{RP}^n)$, $\pi_1(Kl^2)$.*

Exercise 6.15. *Compute the group $\pi/[\pi,\pi]$ for the groups $\pi = \pi_1(M_g^2(1)), \pi_1(M_g^2(2))$.*

Exercise 6.16. *Prove that the fundamental groups computed in Theorems 6.6, 6.7 are pair-wise non-isomorphic. Prove that any two manifolds above are not homeomorphic and even are not homotopy equivalent to each other.*

6.6. **Theorem of Seifert and Van Kampen.** Here we will need some algebraic material, we give only basic definition and refer to [Massey, Chapter 3] and [Hatcher, 1.2] for detailes.

Let G_1, G_2 be two groups with system of generators A_1, A_2 and relations R_1, R_2 respectively. A group with a system of generators $A_1 \cup A_2$ (disjoint union) and system of relations $R_1 \cup R_2$ is called a *free product* of G_1 and G_2 and is denoted as $G_1 * G_2$.

Exercise 6.17. *Prove that the group $\mathbf{Z}_2 * \mathbf{Z}_2$ contains a subgroup isomorphic to $\mathbf{Z}$ and $(\mathbf{Z}_2 * \mathbf{Z}_2)/\mathbf{Z} \cong \mathbf{Z}_2$.*

Exercise 6.18. *Let X, Y be two CW-complexes. Prove that $\pi_1(X \vee Y) = \pi_1(X) * \pi_1(Y)$, where the base points $x_0 \in X$ and $y_0 \in Y$ are identified with a base point in $X \vee Y$.*

Remark. As it is defined in [Massey, Ch. 3], the group $G = G_1 * G_2$ may be characterized as follows. Let $\varphi_1 : G_1 \longrightarrow G$ and $\varphi_2 : G_2 \longrightarrow G$ be natural homomorphisms and let L be a group and $\psi_1 : G_1 \longrightarrow L$, $\psi_2 : G_2 \longrightarrow L$, then there exist a unique homomorphism $\psi : G \longrightarrow L$, such that the diagram

$$\begin{tikzcd}
& G \arrow[dd, "\psi"] & \\
G_1 \arrow[ur, "\varphi_1"] \arrow[dr, "\psi_i"'] & & G_2 \arrow[ul, "\varphi_2"'] \arrow[dl, "\psi_2"] \\
& L &
\end{tikzcd} \tag{14}$$

is commutative. The above definition may be generalized as follows. Assume that we also are given two homomorphisms $\rho_1 : H \longrightarrow G_1$, $\rho_2 : H \longrightarrow G_2$. Let us choose generators $\{h_\alpha\}$ of H and define a group $G_1 *_H G_2$ by adding the relations $\rho_1(h_\alpha) = \rho_2(h_\alpha)$ to relations of $G_1 * G_2$.

In different terms we may define the group $G_1 *_H G_2$ as follows. Assume that we are given a commutative diagram:

$$
\begin{tikzcd}
 & G_1 \arrow[dr, "\psi_1"] & \\
H \arrow[ur, "\rho_1"] \arrow[rr, "\psi_{1,2}"] \arrow[dr, "\rho_2"'] & & L \\
 & G_2 \arrow[ur, "\psi_2"'] &
\end{tikzcd}
\tag{15}
$$

The group $G_1 *_H G_2$ is characterized by the following property: There are such homomorphisms $\sigma : H \longrightarrow G_1 *_H G_2$, $\sigma_1 : G_1 \longrightarrow G_1 *_H G_2$ and $\sigma_2 : G_2 \longrightarrow G_1 *_H G_2$ that for each homomorphisms $\psi_1 : G_1 \longrightarrow L$, $\psi_2 : G_2 \longrightarrow L$ and $\psi_{1,2} : H \longrightarrow L$ such that the diagram (15) is commutative, there exists a unique homomorphism $\psi : G_1 *_H G_2 \longrightarrow L$ such that the following diagram is commutative:

$$
\begin{tikzcd}
 & & G_1 *_H G_2 \arrow[dd, "\psi"'] & & \\
H \arrow[urr, "\sigma"] \arrow[r, "\rho_1"] \arrow[drr, "\psi_{1,2}"'] & G_1 \arrow[ur, "\sigma_1"] \arrow[dr, "\psi_1"] & & G_2 \arrow[ul, "\sigma_2"'] \arrow[dl, "\psi_2"'] & H \arrow[l, "\rho_2"'] \arrow[ull, "\sigma"'] \arrow[dll, "\psi_{1,2}"] \\
 & & L & &
\end{tikzcd}
\tag{16}
$$

Exercise 6.19.* *Prove that the group $SL_2(\mathbf{Z})$ of unimodular 2×2-matrices is isomorphic to $\mathbf{Z}_4 *_{\mathbf{Z}_2} \mathbf{Z}_6$.*

Theorem 6.8. (Seifert, Van Kampen) *Let $X = Y_1 \cup Y_2$ be a connected CW-complex, where Y_1, Y_2 and $Z = Y_1 \cap Y_2$ are connected CW-subcomplexes of X. Let a base point $x_0 \in Y_1 \cap Y_2 \subset X$, and $\rho_1 : \pi_1(Y_1) \longrightarrow \pi_1(X)$, $\rho_2 : \pi_1(Y_2) \longrightarrow \pi_1(X)$. Then*

$$\pi_1(X) \cong \pi_1(Y_1) *_{\pi_1(Z)} \pi_1(Y_2).$$

Exercise 6.20. *Prove Theorem 6.8 in the case of finite CW-complexes using induction on the number of cells of $Y_1 \cap Y_2$.*

Remark. There is more general version of Van Kampen Theorem, see [Massey, Ch. 4] and [Hatcher, 1.2].

7. Covering spaces

7.1. **Definition and examples.** A path-connected space T is a *covering space* over a path-connected space X, if there is a map $p : T \longrightarrow X$ such that for any point $x \in X$ there exists a path-connected neighbourhood $U \subset X$, such that $p^{-1}(U)$ is homeomorphic to $U \times \Gamma$ (where Γ is a discrete set), futhermore the following diagram commutes

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "\cong"] \arrow[dr, "p"'] & & U \times \Gamma \arrow[dl, "pr"] \\ & U & \end{tikzcd} \tag{17}$$

The neighbourhood U from the above definition is called *elementary neighborhood.*

Examples. 1. $p : \mathbf{R} \longrightarrow S^1$, where $S^1 = \{z \in \mathbf{C} \mid |z| = 1\}$, and $p(\varphi) = e^{i\varphi}$.

2. $p : S^1 \longrightarrow S^1$, where $p(z) = z^k$, $k \in \mathbf{Z}$, and $S^1 = \{z \in \mathbf{C} \mid |z| = 1\}$.

3. $p : S^n \longrightarrow \mathbf{RP}^n$, where p maps a point $x \in S^n$ to the line in $\mathbf{R}^{n+1}$ going through the origin and x.

7.2. **Theorem on covering homotopy.** The following result is a key fact allowing to classify coverings.

Theorem 7.1. *Let $p : T \to X$ be a covering space and Z be a CW-complex, and $f : Z \to X$, $\widetilde{f} : Z \to T$ such that the diagram*

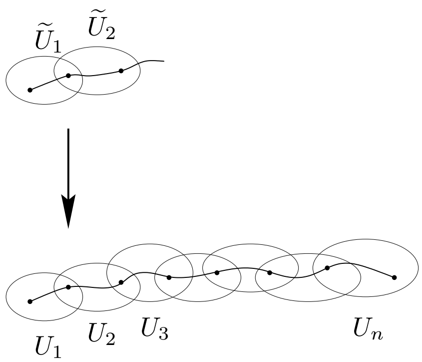
$$\begin{tikzcd} & T \arrow[d, "p"] \\ Z \arrow[ur, "\widetilde{f}"] \arrow[r, "f"] & X \end{tikzcd} \tag{18}$$

commutes; futhermore it is given a homotopy $F : Z \times I \longrightarrow X$ such that $F|_{Z\times\{0\}} = f$. Then there exists a unique homotopy $\widetilde{F} : Z \longrightarrow T$ such that $\widetilde{F}|Z \times \{0\} = \widetilde{f}$ and the following diagram commutes:

$$\begin{tikzcd} & T \arrow[d, "p"] \\ Z \times I \arrow[ur, "\widetilde{F}"] \arrow[r, "F"] & X \end{tikzcd}$$

We prove first the following lemma:

Lemma 7.2. *For any path $s : I \longrightarrow X$ and any point $\widetilde{x}_0 \in T$, such that $p(\widetilde{x}_0) = x_0 = s(0)$ there exists a unique path $\widetilde{s} : I \longrightarrow T$, such that $\widetilde{s}(0) = \widetilde{x}_0$ and $p \circ \widetilde{s} = s$.*



Proof. For each $t \in I$ we find an elementary neighbourhood U_t of the point $s(t)$. A compactness of $I = [0,1]$ implies that there exists a finite number of points

$$0 = t_1 < t_2 < \ldots < t_n = 1,$$

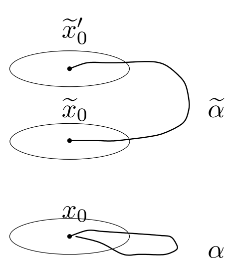
such that $U_j \supset s([t_j, t_{j+1}])$. The inverse image $p^{-1}(U_1)$ is homeomorphic to $U_1 \times \Gamma$ Let $\widetilde{U}_1$ be such that $\widetilde{x}_0 \in \widetilde{U}_1$. Then the path $s|_{[0,t_2]} : [0, t_2] \longrightarrow X$ has a unique lifting $\widetilde{s} : [0, t_2] \longrightarrow T$ covering the path $s|_{[0,t_2]}$. Then we do the same in the neighbourhood U_2 and so on. Note that we have a finite number of U_j, and in each neighbourhood U_j a "lifting" is unique, see Figure to the left.

Proof of Theorem 7.1. Let $z \in Z$ be any point. The formula $t \longrightarrow F(z,t)$ defines a path in X. Lemma 7.2 gives a unique lifting of this path to T, such that it starts at $\widetilde{f}(z)$. It gives a map $Z \times I \longrightarrow T$. This is our homotopy $\widetilde{F}$. □

7.3. **Covering spaces and fundamental group.**

Theorem 7.3. *Let $p : T \longrightarrow X$ be a covering space, then $p_* : \pi_1(T, \widetilde{x}_0) \longrightarrow \pi_1(X, x_0)$ is a monomorphism (injective).*

Proof: Let $\widetilde{s} : I \longrightarrow T$ be a loop, where $\widetilde{s}(0) = \widetilde{s}(1) = \widetilde{x}_0$. Denote $x_0 = p(\widetilde{x}_0)$. Assume that the loop $s = p \circ \widetilde{s} : I \longrightarrow X$ is homotopic to zero. Let $s_t : I \longrightarrow X$ be such a homotopy: $s_0 = s$, $s_t(0) = s_t(1) = x_0$, and $s_1(I) = x_0$.



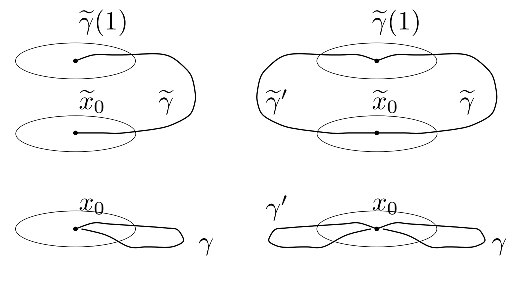
Theorem 7.1 implies that there is a homotopy $\widetilde{s}_t : I \longrightarrow T$ covering the homotopy s_t. Since the inverse image $p^{-1}(x_0)$ is a discrete set, then $\widetilde{s}_t(0) = \widetilde{s}_t(1) = \widetilde{x}_0$. □

The subgroup $p_*(\pi_1(T, \widetilde{x}_0)) \subset \pi_1(X, x_0)$ is called the *covering group of* $T \xrightarrow{p} X$. Let $\widetilde{x}'_0 \neq \widetilde{x}_0$, $p(\widetilde{x}'_0) = p(\widetilde{x}_0) = x_0$. Consider a path $\widetilde{\alpha} : I \longrightarrow T$ such that $\widetilde{\alpha}(0) = \widetilde{x}_0$, $\widetilde{\alpha}(1) = \widetilde{x}'_0$. Then the projection $\alpha = p(\widetilde{\alpha})$ is a loop in X, see Figure to the left. Clearly $\alpha_\# : p_*(\pi_1(T, \widetilde{x}_0)) \longrightarrow p_*(\pi_1(T, \widetilde{x}'_0))$ given by $\alpha_\#(g) = \alpha g \alpha^{-1}$ is an isomorphism.

Consider the coset $\pi_1(X, x_0)/p_*(\pi_1(T, \widetilde{x}_0))$ (the subgroup $p_*(\pi_1(T, \widetilde{x}_0)) \subset \pi_1(X, x_0)$ is not normal subgroup in general).

Claim 7.1. *There is one-to-one correspondence $p^{-1}(x_0) \longleftrightarrow \pi_1(X, x_0)/p_*(\pi_1(T, \widetilde{x}_0))$.*

Proof. Let $[\gamma] \in \pi_1(X, x_0)$, where $\gamma : I \longrightarrow X$, $\gamma(0) = \gamma(1) = x_0$. There exists a unique lifting $\widetilde{\gamma} : I \longrightarrow T$ of γ, so that $\widetilde{\gamma}(0) = \widetilde{x}_0$. We define $A([\gamma]) = \widetilde{\gamma}(1) \in p^{-1}(x_0)$, see Fig. (a). The homotopy lifting property implies that if $\gamma \sim \gamma'$ then $\widetilde{\gamma} \sim \widetilde{\gamma}'$ and $\widetilde{\gamma}(1) = \widetilde{\gamma}'(1)$. Now let $A([\gamma]) = A([\gamma'])$. Then the loop $\beta = (\gamma')^{-1}\gamma$ is covered by the loop $\widetilde{\beta} = (\widetilde{\gamma}')^{-1}\widetilde{\gamma}$, see Fig. (b). Thus $[\beta] = [(\gamma')^{-1}\gamma] \in p_*(\pi_1(T, \widetilde{x}_0))$.



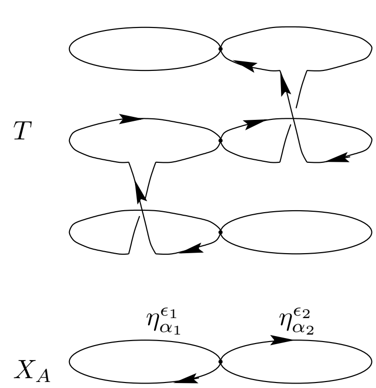
(a) **(b)**

This proves that $A : \pi_1(X, x_0)/p_*(\pi_1(T, \widetilde{x}_0)) \longrightarrow p^{-1}(x_0)$ is an injection. Clearly A is onto since T is path-connected, and if $\widetilde{x} \in p^{-1}(x_0)$ there exists a path connecting x_0 and x which projects to a loop in X. □

Claim 7.2. *Let $p : T \longrightarrow X$ be a covering and $x_0, x_1 \in X$. There is one-to-one correspondence $p^{-1}(x_0) \longleftrightarrow p^{-1}(x_1)$.*

Exercise 7.1. *Prove Claim 7.2.* **Hint:** Consider a path connecting x_0 and x_1.

7.4. **Observation.** Let γ be a loop in X, $\gamma(0) = \gamma(1) = x_0$, and $\widetilde{\gamma} : I \longrightarrow T$ be its lifting with $\widetilde{\gamma}(0) = \widetilde{x}_0$. Then if $\widetilde{\gamma}(1) \neq \widetilde{\gamma}(0)$ then the loop γ is not homotopic to zero. Indeed, if such homotopy would exist, then necessarily it implies that $\widetilde{\gamma}(1) = \widetilde{\gamma}(0)$. □



We use this observation to complete the proof of Theorem 6.4, or, to be precise, the proof of Claim 6.1, 2^o. Indeed, let $\beta = \eta_{\alpha_1}^{\epsilon_1} \cdots \eta_{\alpha_s}^{\epsilon_s}$, $\epsilon_j = \pm 1$, where all elements $\eta_\alpha \eta_\alpha^{-1}$, $\eta_\alpha^{-1}\eta_\alpha$ are canceled It is enough to show that $\beta \neq e$, where e is the identity elelement. Recall that η_α is given by the inclusion $S^1_\alpha \longrightarrow \bigvee_{\alpha \in A} S^1_\alpha = X_A$. It is enough to construct a covering space $p : T \longrightarrow X_A$ so that the loop β is covered by a loop $\widetilde{\beta}$ with the property that $\widetilde{\beta}(0) \neq \widetilde{\beta}(1)$. Consider $s+1$ copies of the wedge X_A placed over X_A, see Fig. 7.5. We assume that these copies of X_A project vertically on X_A. Consider the word $\beta = \eta_{\alpha_1}^{\epsilon_1} \cdots \eta_{\alpha_s}^{\epsilon_s}$. Then we delete small intervals of the circles $S^1_{\alpha_1}$ at the first and second levels and "braid" these two circles together as it is shown at Figure above. We extend the verical projection to the "braid" in the obvious way. Then we join by a braid the circles $S^1_{\alpha_2}$ at the second and the third levels, and so on. In this way we construct a covering space T so that the loop $\beta = \eta_{\alpha_1}^{\epsilon_1} \cdots \eta_{\alpha_s}^{\epsilon_s}$ is covered by $\widetilde{\beta}$ which starts at the first level and ends at the last level. Thus $\beta = \eta_{\alpha_1}^{\epsilon_1} \cdots \eta_{\alpha_s}^{\epsilon_s} \neq 0$. □

7.5. **Lifting to a covering space.** Consider the following situation. Let $p : T \longrightarrow X$ be a covering space, $x_0 \in X$, $\widetilde{x}_0 \in p^{-1}(x_0) \in T$. Let $f : Z \longrightarrow X$ be a map, so that $f(z_0) = x_0$. There is a natural question:

Question: Does there exist a map $\widetilde{f} : Z \longrightarrow T$ covering the map $f : Z \longrightarrow X$, such that $\widetilde{f}(z_0) = \widetilde{x}_0$? In other words, the lifting map $\widetilde{f}$ should make the following diagram commutative:

(19)
$$\begin{tikzcd} & T \arrow[d, "p"] \\ Z \arrow[ur, "\widetilde{f}"] \arrow[r, "f"] & X \end{tikzcd}$$

where $f(z_0) = x_0$, $\widetilde{f}(z_0) = \widetilde{x}_0$. Clearly the diagram (19) gives the following commutative diagram of groups:

(20)
$$\begin{tikzcd} & \pi_1(T, \widetilde{x}_0) \arrow[d, "p_*"] \\ \pi_1(Z, z_0) \arrow[ur, "\widetilde{f}_*"] \arrow[r, "f_*"] & \pi_1(X, x_0) \end{tikzcd}$$

It is clear that commutativity of the diagram (20) implies that

$$f_*(\pi_1(Z, z_0)) \subset p_*(\pi_1(T, \widetilde{x}_0)). \tag{21}$$

Thus (21) is a necessary condition for the existence of the map $\widetilde{f}$. It turns out that (21) is also a sufficient condition.

Theorem 7.4. *Let $p : T \longrightarrow X$ be a covering space, and Z be a path-connected space, $x_0 \in X$, $\widetilde{x}_0 \in T$, $p(\widetilde{x}_0) = x_0$. Given a map $f : (Z, z_0) \longrightarrow (X, x_0)$ there exists a lifting $\widetilde{f} : (Z, z_0) \longrightarrow (T, \widetilde{x}_0)$ if and only if $f_*(\pi_1(Z, z_0)) \subset p_*(\pi_1(T, \widetilde{x}_0))$.*

Proof (outline). We have to define a map $\widetilde{f} : (Z, z_0) \longrightarrow (T, \widetilde{x}_0)$. Let $z \in Z$. Consider a path $\omega : I \longrightarrow Z$, so that $\omega(0) = z_0$, $\omega(1) = z$. Then the path $f(\omega) = \gamma$ has a unique lift $\widetilde{\gamma}$ so that $\widetilde{\gamma}(0) = \widetilde{x}_0$. We define $\widetilde{f}(z) = \widetilde{\gamma}(1) \in T$. We have to check that the construction does not depend on the choice of ω. Let ω' be another path such that $\omega'(0) = z_0$, $\omega'(1) = z$, see Fig. 7.6.

Let $\gamma' = f(\omega')$. Then we have a loop $\beta = (\gamma')^{-1}\gamma$, and $[\beta] \in f_*(\pi_1(Z, z_0))$. Since $f_*(\pi_1(Z, z_0)) \subset p_*(\pi_1(T, \widetilde{x}_0))$, the loop β may be lifted to the loop $\widetilde{\beta}$ in T. In particular, it follows that $\widetilde{\gamma}(1) = \widetilde{\gamma}'(1)$ because of uniqueness of the liftings $\widetilde{\gamma}$ and $\widetilde{\gamma}'$ and $\widetilde{\gamma}$.

Exercise 7.2. *Prove that the map $\widetilde{f}$ we constructed is continuous.*

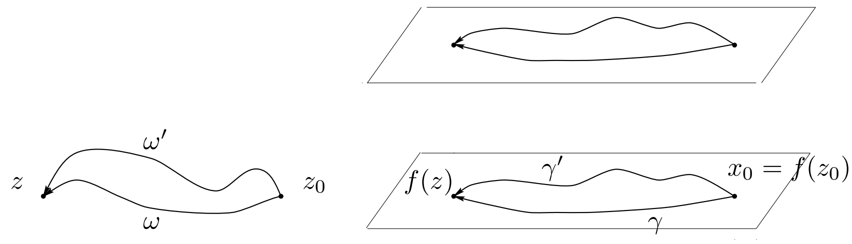


FIGURE 24

Exercise 7.3. *Let $p : T \longrightarrow X$ be a covering, and $\widetilde{f}, \widetilde{f}' : Y \longrightarrow T$ be two maps so that $p \circ \widetilde{f} = p \circ \widetilde{f}' = f$, where Y is path-connected. Assume that $\widetilde{f}(y) = \widetilde{f}'(y)$ for some point $y \in Y$. Prove that $\widetilde{f} = \widetilde{f}'$.*

Hint: Consider the set $V = \left\{ y \in Y \mid \widetilde{f}(y) = \widetilde{f}'(y) \right\}$ and prove that V is open and closed in Y.

Exercise 7.2 completes the proof. Exercise 7.3 implies that the lifting $\widetilde{f}$ is unique. □

7.6. **Classification of coverings over given space.** Consider a category of covering over a space X. The objects of this category are covering spaces $T \xrightarrow{p} X$, and a morphism of covering $T_1 \xrightarrow{p_1} X$ to $T_2 \xrightarrow{p_2} X$ is a map $\varphi : T_1 \longrightarrow T_2$ so that the following diagram commutes:

$$\begin{tikzcd} T_1 \arrow[rr, "\varphi"] \arrow[dr, "p_1"'] & & T_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd} \tag{22}$$

Claim 7.3. *Let $\varphi, \varphi' : T_1 \longrightarrow T_2$ be two morphisms, and $\varphi(t) = \varphi'(t)$ for some $t \in T_1$. Then $\varphi = \varphi'$.*

Exercise 7.4. *Use Exercise 7.3 to prove Claim 7.3.*

Claim 7.4. *Let $T_1 \xrightarrow{p_1} X$, $T_2 \xrightarrow{p_2} X$ be two coverings, $x_0 \in X$, $\widetilde{x}_0^{(1)} \in p_1(x_0)$, $\widetilde{x}_0^{(2)} \in p_2(x_0)$. There exists a morphism $\varphi : T_1 \longrightarrow T_2$ such that $\varphi(\widetilde{x}_0^{(1)}) = \widetilde{x}_0^{(2)}$ if and only if $(p_1)_*(\pi_1(T_1, \widetilde{x}_0^{(1)})) \subset (p_2)_*(\pi_1(T_2, \widetilde{x}_0^{(2)}))$.*

Exercise 7.5. *Prove Claim 7.4.*

A morphism $\varphi : T \longrightarrow T$ is *automorphism* if there exists a morphism $\psi : T \longrightarrow T$ so that $\psi \circ \varphi = Id$ and $\varphi \circ \psi = Id$. Now consider the group $\mathrm{Aut}(T \xrightarrow{p} X)$ of automorphisms of a given covering $p : T \longrightarrow X$. The group operation is a composition and the identity element is the identity map $Id : T \longrightarrow T$. An element $\varphi \in \mathrm{Aut}(T \xrightarrow{p} X)$ acts on the space T.

Claim 7.5. *The group $Aut(T \xrightarrow{p} X)$ acts on the space T without fixed points.*

Exercise 7.6. *Prove Claim 7.5.*

Hint: A point $t \in T$ is a fixed point if $\varphi(t) = t$.

Claim 7.6. *Let $T \xrightarrow{p} X$ be a covering, $x_0 \in X$, $\widetilde{x}_0, \widetilde{x}'_0 \in p^{-1}(x_0)$. Then there exists an automorphism $\varphi \in Aut(T \xrightarrow{p} X)$ such that $\varphi(\widetilde{x}_0) = \widetilde{x}'_0$ if and only if $p_*(\pi_1(T, \widetilde{x}_0)) = p_*(\pi_1(T, \widetilde{x}'_0))$.*

Exercise 7.7. *Prove Claim 7.6.*

Theorem 7.5. *Two coverings $T_1 \xrightarrow{p_1} X$ and $T_2 \xrightarrow{p} X$ are isomorphic if and only if for any two points $\widetilde{x}_0^{(1)} \in p_1^{-1}(x_0)$, $\widetilde{x}_0^{(2)} \in p_2^{-1}(x_0)$ the subgroups $(p_1)_*(\pi_1(T_1, \widetilde{x}_0^{(1)}))$ $(p_1)_*(\pi_1(T_2, \widetilde{x}_0^{(2)}))$ belong to the same conjugation class.*

Exercise 7.8. *Prove Theorem 7.5.*

Let $H \subset G$ be a subgroup. Recall that a normalizer $N(H)$ of H is a maximal subgroup of G such that H is a normal subgroup of that group. The subgroup $N(H)$ of the group G may be described as follows:

$$N(H) = \left\{ g \in G \mid gHg^{-1} = H \right\}.$$

Recall also that the group $\pi_1(X, x_0)$ acts on the set $\Gamma = p^{-1}(x_0)$, and Γ may be considered as a *right* $\pi_1(X, x_0)$*-set*; the subgroup $p_*(\pi_1(T, \widetilde{x}_0))$ is the "isotropy group" of the point $\widetilde{x}_0 \in p^{-1}(x_0)$. Again, we have seen that coset $\pi_1(X, x_0)/p_*(\pi_1(T, \widetilde{x}_0))$ is isomorphic to $p^{-1}(x_0)$.

Corollary 7.6. *The group of automorphisms $Aut(T \xrightarrow{p} X)$ is isomorphic to the group $N(H)/H$, where $H = p_*(\pi_1(T, \widetilde{x}_0)) \subset \pi_1(X, x_0)$ for any points $x_0 \in X$, $\widetilde{x}_0 \in p^{-1}(x_0)$.*

Exercise 7.9. *Prove Corollary 7.6.*

Now remind that a covering space $p : T \longrightarrow X$ is a *regular covering space* if the group $p_*(\pi_1(T, \widetilde{x}_0))$ is a normal subgroup of the group $\pi_1(X, x_0)$.

Exercise 7.10. *Prove that a covering space $p : T \longrightarrow X$ is regular if and only if there is no loop in X which is covered by a loop and a path (starting and ending in different points) in the same time.*

Exercise 7.11. *Prove that if a covering space $p : T \longrightarrow X$ is regular then there exists a free action of the group $G = \pi_1(X, x_0)/\pi_1(T, \widetilde{x}_0)$ on the space T such that $X \cong T/G$.*

Exercise 7.12. *Prove that a two-folded covering space $p : T \longrightarrow X$ is always a regular one.*

We complete this section with the classification theorem:

Theorem 7.7. *Let X be a "good" path-connected space (in particular, CW-complexes are "good" spaces), $x_0 \in X$. Then for any subgroup $G \subset \pi_1(X, x_0)$ there exist a covering space $p : T \longrightarrow X$ and a point $\widetilde{x}_0 \in T$, such that $p_*(\pi_1(T, \widetilde{x}_0)) = G$.*

The idea of the proof: We consider the following equivalence relation on the space of paths $\mathcal{E}(X, x_0)$: two paths $s \sim s_1$ if $s(1) = s_1(1)$ and a homotopy class of the loop ss_1^{-1} belongs to G. We define $T = \mathcal{E}(X, x_0)/\sim$. The projection $p : T \longrightarrow X$ maps a path s to a point $s(1)$. The details are left to you. □

Exercise 7.13. *Prove that in the above construction* $p_*(\pi_1(T, \widetilde{x}_0)) = G$.

In particular, Theorem 7.7 claims the existence of the *universal covering space* $\widehat{T} \xrightarrow{\widehat{p}} X$ (i.e. such that $\pi_1(\widehat{T}, \widetilde{x}_0) = 0$).

Exercise 7.14. *Let* $\widehat{T} \xrightarrow{\widehat{p}} X$ *be a universal covering over* X*, and* $T \longrightarrow X$ *be a covering. Prove that there exists a morphism* $\varphi : \widehat{T} \longrightarrow T$ *so that it is a covering over* T.

7.7. **Homotopy groups and covering spaces.** First, we have the following result:

Theorem 7.8. *Let* $p : T \longrightarrow X$ *be a covering space and* $n \geq 2$. *Then the homomorphism* $p_* : \pi_n(T, \widetilde{x}_0) \longrightarrow \pi_n(X, x_0)$ *is an isomorphism.*

Exercise 7.15. *Prove Theorem 7.8.*

Theorem 7.8 allows us to compute homotopy groups of several important spaces. Actually there are only few spaces where all homotopy groups are known. Believe me or not, here we have at least half of those examples.

Theorem 7.9. $\pi_n(S^1) = \begin{cases} \mathbf{Z} & \textit{if } n = 1, \\ 0 & \textit{if } n \geq 2. \end{cases}$

One may prove Theorem 7.9 by applying Theorem 7.8 to the covering space $\mathbf{R} \xrightarrow{\exp} S^1$; of course, one should be able to prove $\pi_n(\mathbf{R}) = 0$ for all $n \geq 0$.

Corollary 7.10. *Let* $X = \bigvee S^1$. *Then* $\pi_n(X) = 0$ *for* $n \geq 2$.

Exercise 7.16. *Prove Theorem 7.9 and Corollary 7.10.*

Hint: Construct a universal covering space over $\bigvee S^1$; see the pictures given in [Hatcher, p.59].

The next example is T^2: here we have a universal covering $\mathbf{R}^2 \longrightarrow T^2$, so it follows from Theorem 7.7 that $\pi_n(T^2) = 0$ for $n \geq 2$.

Exercise 7.17. *Let* Kl^2 *be the Klein bottle. Construct two-folded covering space* $T^2 \longrightarrow Kl^2$. *Compute* $\pi_n(Kl^2)$ *for all* n.

Theorem 7.11. *Let* M^2 *be a two-dimensional manifold without boundary,* $M^2 \neq S^2, \mathbf{RP}^2$. *Then* $\pi_n(M^2) = 0$ *for* $n \geq 2$.

Exercise 7.18. *Prove Theorem 7.11.*

Hint: One way is to construct a universal covering space over M^2; this universal covering space turns our to be $\mathbf{R}^2$. The second way may be as follows: Let M^2 be a sphere with two handles, and $X \longrightarrow M^2$ be the covering space pictured below:

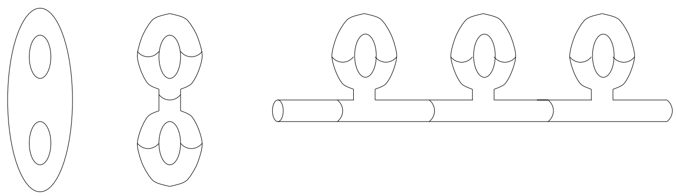


FIGURE 25

Theorem 7.8 shows that $\pi_n(X) = \pi_n(M^2)$. Now let $f : S^n \longrightarrow X$, you may observe that $f(S^n)$ lies in the compact part of X; after cutting down the rest of X it becomes two-dimensional manifold with boundary and homotopy equivalent to its one-skeleton (Prove it!). Now it remains to make an argument in a general case.

7.8. **Lens spaces.** We conclude with important examples. Let $S^1 = \{z \in \mathbf{C} \mid |z| = 1\}$. The group S^1 acts freely on the sphere $S^{2n-1} \subset \mathbf{C}^n$ by $(z_1, \ldots, z_n) \mapsto (e^{i\varphi} z_1, \ldots, e^{i\varphi} z_n)$. The group $\mathbf{Z}/m$ may be thought as a subgroup of S^1:

$$\mathbf{Z}/m = \left\{ e^{2i\pi m/k} \mid k = 0, \ldots, m-1 \right\} \subset S^1.$$

Thus $\mathbf{Z}/m$ acts freely on the sphere S^{2n-1}. The space $L^{2n-1}(\mathbf{Z}/m) = S^{2n-1}/(\mathbf{Z}/m)$ is called a *lens space*. Thus S^{2n-1} is a universal covering space over the lens space $L^{2n-1}(\mathbf{Z}/m)$. Clearly $\pi_1(L^{2n-1}(\mathbf{Z}/m)) \cong \mathbf{Z}/m$, and $\pi_j(L^{2n-1}(\mathbf{Z}/m)) \cong \pi_j(S^{2n-1})$ for $j \geq 2$. The case $m = 2$ is well-known to us: $L^{2n-1}(\mathbf{Z}/2) = \mathbf{RP}^{2n-1}$.

Exercise 7.19. *Describe a cell decomposition of the lens space* $L^{2n-1}(\mathbf{Z}/m)$.

Consider the sphere $S^3 \subset \mathbf{C}^2$. Let p be a prime number, and $q \neq 0 \mod p$. We define the lens spaces $L^3(p,q)$ as follows. We consider the action of $\mathbf{Z}/p$ on $S^3 \subset \mathbf{C}^2$ given by the formula: $T : (z_1, z_2) \mapsto (e^{2\pi i/p} z_1, e^{2\pi i q/p} z_2)$. Let $L^3(p,q) = S^3/T$.

Exercise 7.20. *Prove that* $\pi_1(L^3(p,q)) \cong \mathbf{Z}/p$.

Certainly the lens spaces $L^3(p,q)$ are 3-dimensional manifolds, and for given p they all have the same fundamental group and the same higher homotopy groups $\pi_j(L^3(p,q))$ for $j \geq 2$ since S^3 is a universal covering space for all of them. Clearly one may suspect that some of these spaces are homeomorphic or at least homotopy equivalent. The following theorem gives classification of the lens spaces $L^3(p,q)$ up to homotopy equivalence. The result is rather surprising.

Theorem 7.12. *The lens spaces $L^3(p,q)$ and $L^3(p,q')$ are homotopy equivalent if and only if $q' \equiv \pm kq$ mod p for some integer k.*

We are not ready to prove Theorem 7.12. For instance the lenses $L^3(5,1)$ and $L^3(5,2)$ are not homotopy equivalent, and $L^3(7,1)$ and $L^3(7,2)$ are homotopy equivalent. However it is known that the lenses $L^3(7,1)$ and $L^3(7,2)$ are not homeomorphic, and the classification of the lenses $L^3(p,q)$ is completely resolved.

8. Higher homotopy groups

8.1. **More about homotopy groups.** Let X be a space with a base point $x_0 \in X$. We have defined the homotopy groups $\pi_n(X, x_0)$ for all $n \geq 1$ and even noticed that the groups $\pi_n(X, x_0)$ are commutative for $n \geq 2$ (see the begining of Section 6). Now it is a good time to give more details. First we have defined $\pi_n(X, x_0) = [(S^n, s_0), (X, x_0)]$, where $s_0 \in S^n$ is a based point. Alternatively an element $\alpha \in \pi_n(X, x_0)$ could be represented by a map

$$f : (D^n, S^{n-1}) \longrightarrow (X, x_0) \quad \text{or a map}$$

$$f : (I^n, \partial I^n) \longrightarrow (X, x_0).$$

We already defined the group operation in $\pi_n(X, x_0)$, where the unit element is represented my constant map $S^n \longrightarrow \{x_0\} \subset X$. It is convenient to construct a canonical inverse $-\alpha$ for any element $\alpha \in \pi_n(X, x_0)$. Let $f \in \alpha$ be a map

$$f : (D^n, S^{n-1}) \longrightarrow (X, x_0)$$

representing α. We construct the map $(-f) : (D^n, S^{n-1}) \longrightarrow (X, x_0)$ as follows. Consider the sphere $S^n = D^n_+ \cup_{S^{n-1}} D^n_-$, where the hemisphere D^n_+ is identified with the above disk D^n, as a domain of the map f, see Fig. 26.

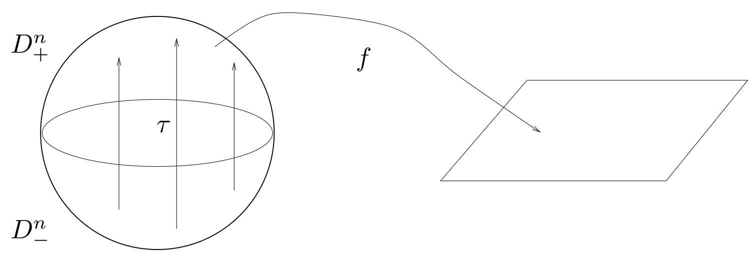


Figure 26

Let $\tau : S^n \longrightarrow S^n$ be a map which is identical on D^n_+ and which maps D^n_- to D^n_+ by the formula $(x_1, \ldots, x_{n+1}) \mapsto (x_1, \ldots, -x_{n+1})$. Then $-f = f \circ \tau : D^n_- \longrightarrow X$.

Exercise 8.1. *Prove that the map $f + (-f) : S^n \longrightarrow X$ is null-homotopic.* Hint: It is enough to show that the map $f + (-f) : S^n \longrightarrow X$ exends to a map $g : D^{n+1} \longrightarrow X$.

Exercise 8.2. *Prove that $\pi_n(X \times Y, x_0 \times y_0) \cong \pi_n(X, x_0) \times \pi_n(Y, y_0)$. Compute $\pi_n(T^k)$ for all n and k.*

8.2. **Dependence on the base point.** Let X be a path-connected space, and $x_0, x_1 \in X$ be two different points. Choose a path $\gamma : I \longrightarrow X$ so that $\gamma(0) = x_0$ and $\gamma(1) = x_1$. We define a homomorphism

$$\gamma_\# : \pi_n(X, x_0) \longrightarrow \pi_n(X, x_1)$$

as follows. Consider the sphere S^n with a base point $s_0 \in S^n$ and the map $\omega : S^n \longrightarrow S^n \vee I$ (Fig 8.2 below shows how to construct the map ω). Indeed, the map ω takes the base point $s_0 \in S^n$ to the point $\{1\} \in I \subset S^n \vee I$. Then for any map $f : (S^n, s_0) \longrightarrow (X, x_0)$ we define $\gamma_{\#}(f)$ to be the composition

$$\gamma_{\#}(f) : S^n \xrightarrow{\omega} S^n \vee I \xrightarrow{f \vee \bar{\gamma}} X,$$

where $\bar{\gamma}(t) = \gamma(1-t)$.

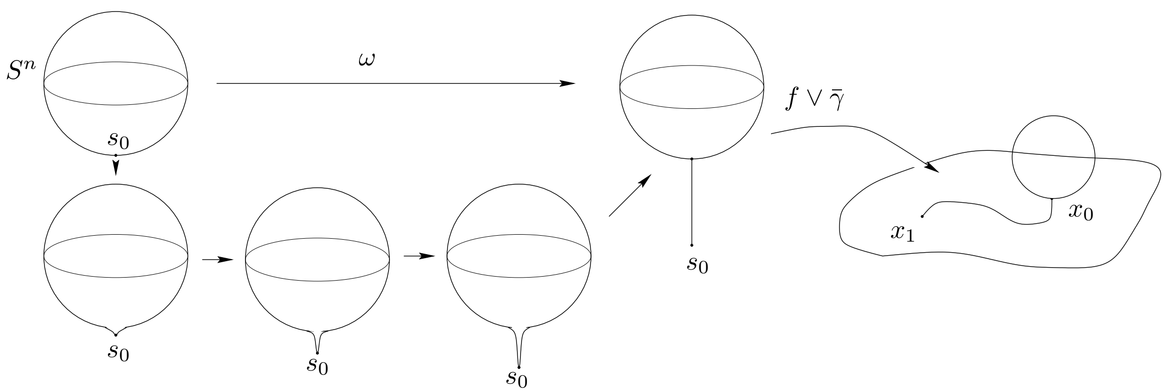


FIGURE 27

It is easy to check that $\gamma_{\#}(f+g) \sim \gamma_{\#}(f) + \gamma_{\#}(g)$ and that $(\gamma^{-1})_{\#} = (\gamma_{\#})^{-1}$.

Exercise 8.3. *Prove that $\gamma_{\#}$ is an isomorphism.*

A path-connected space X is called n*-simple* if the isomorphism

$$\gamma_{\#} : \pi_k(X, x_0) \longrightarrow \pi_k(X, x_1)$$

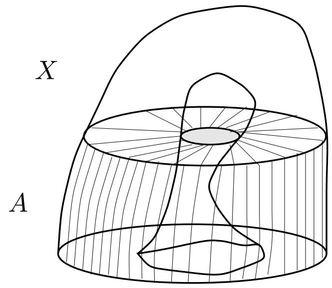
does not depend on the choice of a path γ conecting any two points for $k \leq n$.

Consider the case when $x_0 = x_1$. We have that any element $\sigma \in \pi_1(X, x_0) = \pi$ acts on the group $\pi_n(X, x_0)$ for each $n = 1, 2, \ldots$ by isomorphisms, i.e. any element $\sigma \in \pi$ determines an isomorphism $\sigma_{\#} : \pi_n(X, x_0) \longrightarrow \pi_n(X, x_0)$. We consider the case $n \geq 2$. This action turns the group $\pi_n(X, x_0)$ into $\mathbf{Z}[\pi]$-module as follows. Let $\sigma = \sum_i^N k_i \sigma_i \in \mathbf{Z}[\pi]$, where $\sigma_i \in \pi$, and $k_i \in \mathbf{Z}$. Then the module map

$$\mathbf{Z}[\pi] \otimes \pi_n(X, x_0) \longrightarrow \pi_n(X, x_0)$$

is defined by $\sigma(\alpha) = \sum_i^N k_i \sigma_i(\alpha) \in \pi_n(X, x_0)$. The above definition may be rephrased as follows. A path-connected space X is n-simple if the $\mathbf{Z}[\pi]$-modules $\pi_k(X, x_0)$ are trivial for $k \leq n$ (i.e. each element $\sigma \in \pi$ acts on $\pi_k(X, x_0)$ identically).

8.3. **Relative homotopy groups.** Let (X, A) be a pair of spaces and $x_0 \in A$ be a base point. A *relative homotopy group* $\pi_n(X, A; x_0)$ is a set of homotopy classes of maps $(D^n, S^{n-1}; s_0) \xrightarrow{f} (X, A; x_0)$, i.e. $f(S^{n-1}) \subset A$, $f(s_0) = x_0$, where a base point $s_0 \in S^{n-1}$, see Figure below.



The other convenient geometric representation is to map cubes: $f : (I^n, \partial I^n) \longrightarrow (X, A)$, so that the base point $s_0 \in \partial I^n$ maps to x_0. We shall use both geometric interpretations. Let $\alpha, \beta \in \pi_n(X, A; x_0)$ be represented by maps $f, g : (D^n, S^{n-1}) \longrightarrow (X, A)$ respecively. To define the sum $\alpha + \beta$ we construct a map $f + g$ as follows. First we define a map $c : D^n \longrightarrow D^n \vee D^n$ collapsing the equator disk to the base point, and the we compose c with the map $f \vee g$.

Let $\alpha, \beta \in \pi_n(X, A; x_0)$ be represented by maps $f, g : (D^n, S^{n-1}) \longrightarrow (X, A)$ respecively. To define the sum $\alpha + \beta$ we construct a map $f + g$ as follows. First we define a map $c : D^n \longrightarrow D^n \vee D^n$ collapsing the equator disk to the base point, and the we compose c with the map $f \vee g$. Thus $f + g = (f \vee g) \circ c$, and $\alpha + \beta = [f + g]$, see Fig. 28.

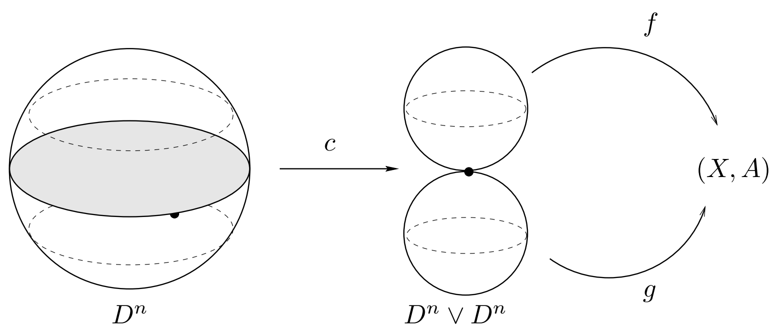


FIGURE 28

Again it is convenient to describe precisely the inverse element $-\alpha$. Let

$$f : (D^n, S^{n-1}) \longrightarrow (X, A)$$

represent $\alpha \in \pi_n(X, A; x_0)$. We define a map $-f$ as follows. We consider the disk $D^n = D^n_- \cup_{D^{n-1}} D^n_+ \subset \mathbf{R}^n \subset \mathbf{R}^{n+1}$, see Fig. 29.

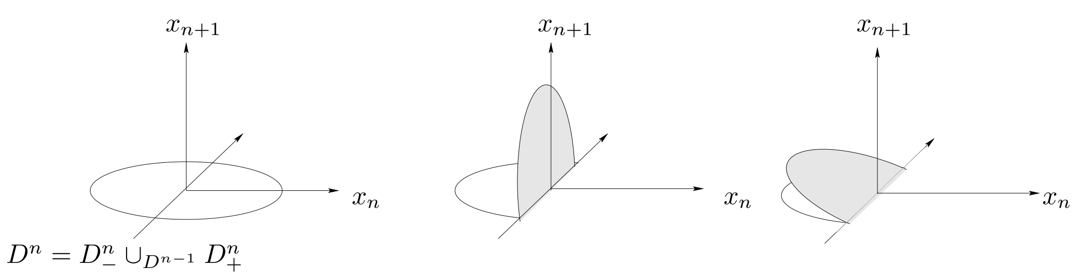


FIGURE 29

The disks D^n_- and D^n_+ are defined by the unequalities $\pm x_n \geq 0$. We consider a map $\varphi : D^n_- \cup_{D^{n-1}} D^n_+ \longrightarrow D^n_-$ flipping over the disk D^n_+ onto D^n_-, see Fig. 29. We may assume that the map

$f:(D^n,S^{n-1})\longrightarrow (X,A)$ is defined on the disk D^n_- so that $f|_{D^{n-1}}$ sends D^{n-1} to the base point x_0. Now we difine

$$-f: D^n_+ \xrightarrow{\varphi|_{D^n_+}} D^n_- \xrightarrow{f} X.$$

Exercise 8.4. *Prove that $f+(-f)\sim *$, where $-f$ as above.*

Exercise 8.5. *Prove that the group $\pi_n(X,A;x_0)$ is commutative for $n\geq 3$.*

Note that if we have a map of pairs $(X,A)\xrightarrow{f}(Y,B)$, such that $f(x_0)=y_0$, then there is a homomorphism

$$f_*:\pi_n(X,A;x_0)\longrightarrow \pi_n(Y,B;y_0).$$

Exercise 8.6. *Prove that if $f,g:(X,A)\xrightarrow{f}(Y,B)$ are homotopic maps, than $f_*=g_*$.*

Remark: Note the homotopy groups $\pi_n(X,x_0)$ may be interpreted as "relative homotopy groups": $\pi_n(X,x_0)=\pi_n(X,\{x_0\};x_0)$. Moreover, one may construct a space Y such that $\pi_n(X,A;x_0)\cong \pi_{n-1}(Y,y_0)$. We will see this construction later.

The maps of pairs

$$(A,x_0)\xrightarrow{i}(X,x_0),\qquad (X,x_0)\xrightarrow{j}(X,A)$$

give the homomorphisms:

$$\pi_n(A,x_0)\xrightarrow{i_*}\pi_n(X,x_0),\qquad \pi_n(X,x_0)\xrightarrow{j_*}\pi_n(X,A;x_0).$$

Exercise 8.7. *Let π be a group. Give definition of the center of π. Prove that the image of the homomorphism $j_*:\pi_2(X,x_0)\xrightarrow{j_*}\pi_2(X,A;x_0)$ belongs to the center of the group $\pi_2(X,A;x_0)$.*

Also we have a "connective homomorphism":

$$\partial:\pi_n(X,A;x_0)\longrightarrow \pi_{n-1}(A,x_0)$$

which maps the relative spheroid $f:(D^n,S^{n-1})\longrightarrow (X,A)$, $f(s_0)=x_0$ to the spheroid $f|_{S^{n-1}}:(S^{n-1},s_0)\longrightarrow (A,x_0)$.

Theorem 8.1. *The following sequence of groups is exact:*

$$\cdots\longrightarrow \pi_n(A,x_0)\xrightarrow{i_*}\pi_n(X,x_0)\xrightarrow{j_*}\pi_n(X,A;x_0)\xrightarrow{\partial}\pi_{n-1}(A,x_0)\longrightarrow\cdots \tag{23}$$

First we remind that the sequence of groups and homomorphisms

$$\cdots\longrightarrow A_1\xrightarrow{\alpha_1}A_2\xrightarrow{\alpha_2}A_3\xrightarrow{\alpha_3}\cdots$$

is exact if $\mathrm{Ker}\ \alpha_{i+1}=\mathrm{Im}\ \alpha_i$.

Exercise 8.8. *Prove that the sequence* (23) *is exact*

(a) *in the term* $\pi_n(A, x_0)$,

(b) *in the term* $\pi_n(X, x_0)$,

(c) *in the term* $\pi_n(X, A; x_0)$.

In the following exercises all groups are assumed to be abelian.

Exercise 8.9. *Prove the following statements*

(a) *The sequence* $0 \longrightarrow A \longrightarrow B$ *is exact if and only if* $A \longrightarrow B$ *is a monomorphism; and the sequence* $A \longrightarrow B \longrightarrow 0$ *is exact if and only if* $A \longrightarrow B$ *is an epimorphism.*

(b) *The sequence* $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ *is exact if and only if* $C \cong B/A$.

Corollary 8.2. 1. *Let* $A \subset X$ *be a contractible subspace. Then* $\pi_n(X, x_0) \cong \pi_n(X, A; x_0)$ *for* $n \geq 1$.

2. *Let* X *be contractible, and* $A \subset X$. *Then* $\pi_n(X, A; x_0) \cong \pi_{n-1}(A, x_0)$ *for* $n \geq 1$.

3. *Let* $A \subset X$ *be a deformational retract of* X. *Then* $\pi_n(X, A; x_0) = 0$.

Exercise 8.10. *Prove Corollary 8.2.*

Exercise 8.11. *Let* $A \subset X$ *be a retract. Prove that*

- $i_* : \pi_n(A, x_0) \longrightarrow \pi_n(X, x_0)$ *is monomorphism,*
- $j_* : \pi_n(X, x_0) \longrightarrow \pi_n(X, A; x_0)$ *is epimorphism,*
- $\partial : \pi_n(X, A; x_0) \longrightarrow \pi_{n-1}(A, x_0)$ *is zero homomorphism.*

Exercise 8.12. *Let* A *be contractible in* X. *Prove that*

- $i_* : \pi_n(A, x_0) \longrightarrow \pi_n(X, x_0)$ *is zero homomorphism,*
- $j_* : \pi_n(X, x_0) \longrightarrow \pi_n(X, A; x_0)$ *is monomorphism,*
- $\partial : \pi_n(X, A; x_0) \longrightarrow \pi_{n-1}(A, x_0)$ *is epimorphism.*

Exercise 8.13. *Let* $f_t : X \longrightarrow X$ *be a homotopy such that* $f_0 = Id_X$, *and* $f_1(X) \subset A$. *Prove that*

- $i_* : \pi_n(A, x_0) \longrightarrow \pi_n(X, x_0)$ *is epimorphism.,*
- $j_* : \pi_n(X, x_0) \longrightarrow \pi_n(X, A; x_0)$ *is zero homomorphism,*
- $\partial : \pi_n(X, A; x_0) \longrightarrow \pi_{n-1}(A, x_0)$ *is monomorphism.*

Lemma 8.3. (Five-Lemma) *Let the following diagram be commutative:*

$$
\begin{tikzcd}
A_1 \arrow[r] \arrow[d, "\varphi_1"] & A_2 \arrow[r] \arrow[d, "\varphi_2"] & A_3 \arrow[r] \arrow[d, "\varphi_3"] & A_4 \arrow[r] \arrow[d, "\varphi_4"] & A_5 \arrow[d, "\varphi_5"] \\
B_1 \arrow[r] & B_2 \arrow[r] & B_3 \arrow[r] & B_4 \arrow[r] & B_5
\end{tikzcd} \tag{24}
$$

Furthermore, let the rows be exact and the homomorphisms $\varphi_1, \varphi_2, \varphi_4, \varphi_5$ be isomorphisms. Then φ_3 is isomorphism.

Exercise 8.14. *Prove Lemma 8.3.*

Exercise 8.15. *Let us exclude the homomorphism φ_3 from the diagram (24) and keep all other conditions of Lemma 8.3 the same. Does it follow then that $A_3 \cong B_3$? If not, give a counter example.*

Exercise 8.16. *Let $0 \longrightarrow A_1 \longrightarrow A_2 \longrightarrow \cdots \longrightarrow A_n \longrightarrow 0$ be an exact sequence of finitely generated abelian groups, then $\sum_{i=1}^{n}(-1)^i \mathit{rank}\ A_i = 0$.*

Exercise 8.17. *Let $1 \longrightarrow G_1 \longrightarrow G_2 \longrightarrow \cdots \longrightarrow G_n \longrightarrow 1$ be an exact sequence of finite groups (not necessarily abelian), then $\sum_{i=1}^{n}(-1)^i |G_i| = 0$, where $|G_i|$ is the order of G_i.*

Corollary 8.4. *Let $f : (X, A) \longrightarrow (Y, B)$ be a map of pairs, $f(x_0) = y_0$, where $x_0 \in A$, $y_0 \in B$. Then any two following statements imply the third one:*

- $f_* : \pi_n(X, x_0) \longrightarrow \pi_n(Y, y_0)$ *is an isomorphism for all* n.
- $f_* : \pi_n(A, x_0) \longrightarrow \pi_n(A, y_0)$ *is an isomorphism for all* n.
- $f_* : \pi_n(X, A; x_0) \longrightarrow \pi_n(Y, B; y_0)$ *is an isomorphism for all* n.

Exercise 8.18. *Prove Corollary 8.4.*

9. Fiber bundles

9.1. **First steps toward fiber bundles.** Covering spaces may be considered as a perfect tool to study the fundamental group. Fiber bundles provide the same kind of tool to study the higher homotopy groups, as we shall see soon.

Definition 9.1. A *locally trivial fiber bundle* is a four-tuple (E, B, F, p), where E, B, F are spaces, and $p : E \longrightarrow B$ is a map with the following property:

- *For each point $x \in B$ there exists a neighborhood U of x such that $p^{-1}(U)$ is homeomorphic to $U \times F$, moreover the homeomorphism $\varphi_U : p^{-1}(U) \longrightarrow U \times F$ should make the diagram*

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "\varphi_U"] \arrow[dr, "p"'] & & U \times F \arrow[dl, "pr"] \\ & U & \end{tikzcd}$$

  commute. Here $pr : U \times F \longrightarrow U$ is a projection on the first factor.

The spaces E, B, F have their special names: E is a *total space*, B is a *base*, and F is a *fiber*. The inverse image $F_x \cong p^{-1}(x)$ is clearly homeomorphic to the fiber F for each point $x \in B$. However, these homemorphisms depend on x. As in the case of covering spaces, the following commutative diagram

$$\begin{tikzcd} E_1 \arrow[rr, "f"] \arrow[dr, "p_1"'] & & E_2 \arrow[dl, "p_2"] \\ & B & \end{tikzcd}$$

gives a *morphism* of fiber bundles (E_1, B, F_1, p_1) to (E_2, B, F_2, p_2). Two fiber bundles (E_1, B, F_1, p_1) and (E_2, B, F_2, p_2) are *equivalent* if there exist morphisms

$$f : (E_1, B, F_1, p_1) \longrightarrow (E_2, B, F_2, p_2), \quad g : (E_2, B, F_2, p_2) \longrightarrow (E_1, B, F_1, p_1)$$

such that $f \circ g = Id$ and $g \circ f = Id$. In particular, a fiber bundle $p : E \longrightarrow B$ is *trivial* if it is equivalent to the bundle $B \times F \longrightarrow B$:

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "p"'] & & B \times F \arrow[dl, "pr"] \\ & B & \end{tikzcd}$$

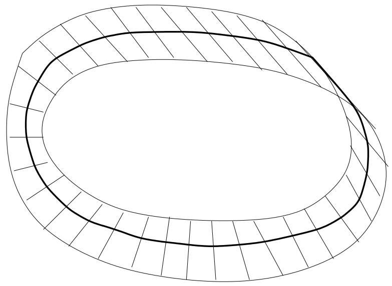
Examples. 1. Trivial bundle $B \times F \longrightarrow B$.

2. Covering spaces.

3. A projection $M \longrightarrow S^1$ of the Möbius band on the middle circle, see Fig. 9.1. The fiber is the interval I.

4. The Hopf bundle $h : S^{2n+1} \longrightarrow \mathbf{CP}^n$ with a circle S^1 as a fiber.

5. Let G be a Lie group and H be its compact subgroup. Then the space of cosets G/H is a base space of the bundle $G \longrightarrow G/H$ with a fiber H.



Möbius band.

6. Let G be a Lie group. Assume that G acts freely on a smooth manifold M. We denote by M/G the space of orbits, then the projection $M \longrightarrow M/G$ is a fiber bundle with the fiber G.

It is not so difficult to verify that the examples above are indeed locally trivial fiber bundles. To give a sample of such verification, we consider the Example 4 in more detail:

Lemma 9.2. *The Hopf map* $h : S^{2n+1} \longrightarrow \mathbf{CP}^n$ *is a locally trivial fiber with a fiber* S^1.

Proof. We use the construction given in the proof of Lemma 2.1 (Section 2). Again, we take a close look at the Hopf map $h : S^{2n+1} \longrightarrow \mathbf{CP}^n$: we take a point $(z_1, \cdots, z_{n+1}) \in S^{2n+1}$, (where $|z_1|^2 + \cdots + |z_{n+1}|^2 = 1$), then h maps the point $(z_1, \cdots, z_{n+1})$ to the line $(z_1 : \cdots : z_{n+1}) \in \mathbf{CP}^n$. Moreover $h(z_1, \cdots, z_{n+1}) = h(z'_1, \cdots, z'_{n+1})$ if and only if $z'_j = e^{i\varphi} z_j$. Thus we can identify $\mathbf{CP}^n$ with the following quotient space:

$$\mathbf{CP}^n = S^{2n+1}/\sim, \quad \text{where } (z_1, \cdots, z_{n+1}) \sim (e^{i\varphi} z_1, \cdots, e^{i\varphi} z_{n+1}). \tag{25}$$

For each $j = 1, \ldots, n+1$, consider the following open subset in $\mathbf{CP}^n$

$$U_j = \left\{ (z_1, \cdots, z_{n+1}) \in S^{2n+1} \mid z_j \neq 0 \text{ and } (z_1, \cdots, z_{n+1}) \sim (e^{i\varphi} z_1, \cdots, e^{i\varphi} z_{n+1}) \right\}.$$

Since $z_j \neq 0$, we may write $z_j = re^{i\alpha}$, where $0 < r \leq 1$. Then the map $g : U_j \longrightarrow \overset{\circ}{D}{}^{2n}$ is given by

$$(z_1, \cdots, z_{j-1}, re^{i\alpha}, z_{j+1}, \cdots, z_{n+1}) \mapsto (e^{-i\alpha} z_1, \cdots, e^{-i\alpha} z_{j-1}, r, e^{-i\alpha} z_{j+1}, \cdots, e^{-i\alpha} z_{n+1})$$

is a homeomorphism. Indeed, we have:

$$|z_1|^2 + \cdots + |z_{j-1}|^2 + |z_{j+1}|^2 + \cdots + |z_{n+1}|^2 = 1 - r^2,$$

and for a given r, $0 < r \leq 1$, a point $(e^{-i\alpha} z_1, \cdots, e^{-i\alpha} z_{j-1}, e^{-i\alpha} z_{j+1}, \cdots, e^{-i\alpha} z_{n+1}) \in \mathbf{C}^n$ belongs to the sphere S^{2n-1} of radius $\sqrt{1 - r^2}$. Since $0 < r \leq 1$, this gives parametrization of open disk $\overset{\circ}{D}{}^{2n}$ of radius 1.

Now let $\ell \in \mathbf{CP}^n$. In order to prove that the Hopf map $h : S^{2n+1} \longrightarrow \mathbf{CP}^n$ is a locally trivial fiber bundle, we have to find a neighborhood U of ℓ such that $h^{-1}(U)$ is homemorphic to the product $U \times S^1$. Notice that there exists j, $1 \leq j \leq n+1$, such that $\ell \in U_j$. Hence it is enough to show that

$h^{-1}(U_j)$ is homeomorphic to $U_j \times S^1$, and that a projection on the first factor $pr : U_j \times S^1 \longrightarrow U_j$ coincides with the Hopf map.

Now we see that $h^{-1}(U_j) \subset S^{2n+1}$ is given as

$$\begin{array}{rcl} h^{-1}(U_j) & = & \left\{(z_1, \cdots, z_{n+1}) \in S^{2n+1} \mid z_j = re^{i\alpha} \neq 0\right\} \\ & \cong & \left\{((e^{-i\alpha}z_1, \cdots, e^{-i\alpha}z_{j-1}, r, e^{-i\alpha}z_{j+1}, \cdots, e^{-i\alpha}z_{n+1}), e^{-i\alpha})\right\} = U_j \times S^1. \end{array}$$

Clearly the projection on the first factor coincides with the Hopf map. □

Exercise 9.1. *Prove that the Hopf map $S^{4n+3} \longrightarrow \mathbf{HP}^n$ is locally trivial fiber bundle.*

Exercise 9.2. *Here we specify the example 5. Note that $Sp(1) = S^3$, and S^1 is a subgroup of $Sp(1)$. Prove that the fiber bundle $Sp(1) \longrightarrow Sp(1)/S^1$ is equivalent to the Hopf bundle $S^3 \longrightarrow \mathbf{CP}^1$.*[10]

Exercise 9.3. *Here we specify the example 6. Let S^{2n+1} be a unit sphere in $\mathbf{C}^{n+1}$, $S^{2n+1} = \left\{|z_1|^2 + \ldots + |z_{n+1}| = 1|\right\}$. The group $S^1 = \left\{e^{i\varphi}\right\}$ acts on S^{2n+1} by the formula $(z_1, \ldots, z_{n+1}) \longrightarrow (e^{i\varphi}z_1, \ldots, e^{i\varphi}z_{n+1})$. Prove that this action is free, and that a fiber bundle $S^{2n+1} \longrightarrow S^{2n+1}/S^1$ is equivalent to the Hopf bundle $S^{2n+1} \longrightarrow \mathbf{CP}^n$.*

Exercise 9.4. *Let $f : M \longrightarrow N$ be a smooth map, where M, N are smooth manifolds. Assume that the map f is a submersion, i.e. f is onto and the differential $df_x : TM_x \longrightarrow TM_{f(x)}$ is an epimorphism for any $x \in M$. Prove that $(M, N, f^{-1}(x), f)$ is a locally-trivial fiber bundle.*

Exercise 9.5. *Prove that the fiber bundles from the examples 3–6 are nontrivial fiber bundles.*

9.2. **Constructions of new fiber bundles.** There are two important ways to construct new fiber bundles.

1. Restriction. Let $E \xrightarrow{p} B$ be a fiber bundle with a fiber F, and let $B' \subset B$ be a subset. Let $E' = p^{-1}(B')$. The bundle $E' \xrightarrow{p'} B'$, where $p' = p|_{E'}$ is a *restriction of the bundle $E \xrightarrow{p} B$ on the subspace $B' \subset B$.*

2. Induced fiber bundle. Let $E \xrightarrow{p} B$ be a fiber bundle with a fiber F, and $X \xrightarrow{f} B$ be a map. Let $f^*(E) \subset X \times E$ be the following subspace:

$$f^*(E) = \{(x, e) \in X \times E \mid f(x) = p(e) \}.$$

There are two natural maps: $f^*(E) \xrightarrow{f^*} E$ (where $f^*(x, e) = e$) and $f^*(E) \xrightarrow{p'} X$ (where $p'(x, e) = x$). It is easy to check that the map $f^*(E) \xrightarrow{p'} X$ is a locally–trivial bundle over X with the same

[10] It may be helpfull to remember that $Sp(1)$ is the set of all unit quaternions $\alpha = \alpha_1 + i\alpha_2 + j\alpha_3 + k\alpha_4 \in Sp(1)$, and $S^1 = \{\alpha_1 + i\alpha_2\}$.

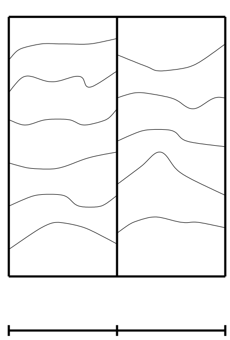
fiber F and that the diagram

$$\begin{tikzcd} f^*(E) \arrow[r, "f^*"] \arrow[d, "p'"'] & E \arrow[d, "p"] \\ X \arrow[r, "f"] & B \end{tikzcd}$$

commutes. The bundle $f^*(E) \xrightarrow{p'} X$ is called *induced fiber bundle.*

Lemma 9.3. *Any locally–trivial fiber bundle over the cube I^q is trivial.*

Proof. Let $E \xrightarrow{p} I^q$ be a locally–trivial fiber bundle. We prove the statement in two steps.



STEP 1. First we assume that the restriction of the bundle $E \xrightarrow{p} I^q$ on each of the cubes

$$I_1^q = \{(x_1, \ldots, x_q) \in I^q \mid x_q \leq 1/2\ \},$$

$$I_2^q = \{(x_1, \ldots, x_q) \in I^q \mid x_q \geq 1/2\ \}.$$

is a trivial fiber bundle. Let $p_1 : E_1 \longrightarrow I_1^q$, $p_2 : E_2 \longrightarrow I_2^q$ be these restrictions. Since these bundles are trivial, we can assume that $E_1 = I_1^q \times F$, $E_1 = I_1^q \times F$, so a point of E_1 has coordinates (x, y), $x \in I_1^q$, $y \in F$, and, analogously, a point of E_2 has coordinates (x, y'), $x \in I_2^q$, $y' \in F$.

In particular, if $x \in I_1^q \cap I_2^q$, then the map $f_x : y \mapsto y'$ is well-defined, and is a homeomorphism of the fiber F. We define a projection $\pi : I_1^q \longrightarrow I_1^q \cap I_2^q$ by the formula: $\pi(x_1, \ldots, x_q) = (x_1, \ldots, x_{q-1}, 1/2)$. Define new map $\varphi : E_1 \longrightarrow I_1^q \times F$ by the formula $\varphi(x, y) = (x, f_{\pi(x)}(y))$. It gives a homeomorphism $E_1 \cong I_1^q \times F$ which coincide with the chosen trivialization over I_2^q, i.e. we obtain a homeomorphism $E \longrightarrow I^q \times F$.

STEP 2. Now we prove the general case. Since the bundle $p : E \longrightarrow I^q$ is a locally–trivial bundle, we may cut the cube I^q into finite number of small cubes I_i^q, $i = 1, 2, \ldots$, such that a restriction of the bundle $p : E \longrightarrow I^q$ on each of these small cubes is trivial, and each space $J_k = \bigcup_{i=1}^k I_i^q$ is homeomorphic to a cube I_1^q. Assume that we have constructed a trivialization of the bundle over J_k, then $J_{k+1} = J_k \cup I_{k+1}^q$ homeomorphic to a cube. Choose homeomorphisms $J_k \cong I_1^q$ and I_2^q and then Step 1 completes the proof. $\square$

We will say that a *covering homotopy property* (CHP) holds for a map $p : E \longrightarrow B$, if for any CW-complex Z and commutative diagram

$$\begin{tikzcd} & E \arrow[d, "p"] \\ Z \arrow[ur, "\widetilde{g}"] \arrow[r, "g"] & B \end{tikzcd}$$

and a homotopy $G : Z \times I \longrightarrow B$, such that $G|_{Z\times\{0\}} = g$ there exists a homotopy $\widetilde{G} : Z \times I \longrightarrow E$ such that $\widetilde{G}|_{Z\times\{0\}} = \widetilde{g}$ and the diagram

$$\begin{tikzcd} & E \arrow[d, "p"] \\ Z \times I \arrow[ur, "\widetilde{G}"] \arrow[r, "G"] & B \end{tikzcd}$$

commutes.

Theorem 9.4. (Theorem on Covering Homotopy) *The covering homotopy property holds for a locally-trivial fiber bundle $E \longrightarrow B$.*

We will prove a stronger version of Theorem 9.4, namely we assume in addition the following.

- *There is a subcomplex $Z' \subset Z$ and a homotopy $\widetilde{G}' : Z' \times I \longrightarrow E$ covering the homotopy $G|_{Z'\times I}$.*

Proof. Case 1. *Let the fiber bundle $p : E \longrightarrow B$ be trivial, and Z be any CW-complex.* We identify $E \cong B \times F$, and maps to $B \times F$ with the pairs of maps to B and F. Then the map $\widetilde{g} : Z \longrightarrow E = B \times F$ is given by a pair $\widetilde{g} = (g, h)$, where $g : Z \longrightarrow B$ is the above map, and $h : Z \longrightarrow F$ be some continuous map. The homotopy $\widetilde{G}' : Z' \times I \longrightarrow E$ is given by the pair $\widetilde{G}' = (G', H')$, where $G' = G|_{Z'\times I} : Z' \times I \longrightarrow B$ is determined by the homotopy G and and the homotopy $H' : Z' \times I \longrightarrow F$ is such that $H'|_{Z'\times\{0\}} = h|_{Z'}$. Thus the Borsuk Theorem gives us that there exists a homotopy $H : Z \times I \longrightarrow F$ extending the map $h : Z \longrightarrow F$ and the homotopy $H' : Z' \times I \longrightarrow F$. The covering homotopy $\widetilde{G} : Z \times I \longrightarrow B \times F = E$ is defined by $\widetilde{G}(z, t) = (G(z, t), H(z, t))$.

Case 2. *The fiber bundle $p : E \longrightarrow B$ is arbitrary, $Z = D^n$, $Z' = S^{n-1}$.* Let $g : D^n \longrightarrow B$, $g' : S^{n-1} \longrightarrow B$, $\widetilde{f} : D^n \longrightarrow E$, $\widetilde{g}' : S^{n-1} \longrightarrow E$, and $G : D^n \times I \longrightarrow B$, $\widetilde{G}' : S^{n-1} \times I \longrightarrow E$ be the corresponding maps and homotopies.

The map $G : D^n \times I \longrightarrow B$ induces the bundle $G^*(E) \longrightarrow D^n \times I$, which is trivial by Lemma 9.3. Recall that the total space

$$G^*(E) = \{((x,t),e) \mid G(x,t) = p(e)\} \subset (D^n \times I) \times E.$$

Let $G^* : G^*(E) \longrightarrow E$ be a natural map (projection). We define a map $\widetilde{h} : D^n \longrightarrow G^*(E)$ by $\widetilde{h}(x) = ((x,0),\widetilde{g}(x))$. This map is well-defined since $G(x,0) = g(x) = p \circ \widetilde{g}(x)$ and $\widetilde{h}$ covers the map $h : D^n \longrightarrow D^n \times I$ given by $x \mapsto (x,0)$. The homotopy $H : D^n \times I \longrightarrow D^n \times I$ (the identity map!), and $\widetilde{H}' : S^{n-1} \times I \longrightarrow G^*(E)$, where $\widetilde{H}'(x,t) = ((x,t),\widetilde{G}'(x,t))$ satisfy the conditions of the theorem. Indeed, we have the commutative diagrams:

$$\begin{tikzcd} & & G^*(E) \arrow[d, "p'"] \\ S^{n-1} \arrow[r] & D^n \arrow[ur, "\widetilde{h}"] \arrow[r, "h"] & D^n \times I \end{tikzcd} \qquad \begin{tikzcd} & & G^*(E) \arrow[d, "p'"] \\ S^{n-1} \times I \arrow[urr, "\widetilde{H}'"] \arrow[r] & D \times I \arrow[ur, "\widetilde{H}"] \arrow[r, "H=Id"] & D^n \times I \end{tikzcd} \tag{26}$$

The map $\widetilde{H}$ from (26) exists be the Case 1. Thus the map $\widetilde{G} = G^* \circ \widetilde{H} : D^n \times I \longrightarrow E$ covers the homotopy $G : D^n \times I \longrightarrow B$ as required.

Case 3. *Now the fiber bundle $E \xrightarrow{p} B$ is arbitrary, and the CW-complex Z is finite.* By induction, we may assume that the difference $Z \setminus Z'$ is a single cell e^n. Let $\Phi : D^n \longrightarrow Z$ be a corresponding characteristic map, and $\varphi = \Phi|_{S^{n-1}}$ be an attaching map. Then the map $\widetilde{h} = \widetilde{g} \circ \Phi : D^n \longrightarrow E$ and the homotopies

$$H = G \circ (\Phi \times Id) : D^n \times I \longrightarrow B,$$

$$\widetilde{H}' = \widetilde{G}' \circ (\Phi|_{S^{n-1}} \times Id) : S^{n-1} \times I \longrightarrow E$$

satisfy the conditions of the theorem, so by Case 2 one completes the proof. $\square$

Exercise 9.6. *Prove the general case, i.e. when Z is an arbitrary CW-complex and $E \longrightarrow B$ is any locally trivial bundle.*

9.3. **Serre fiber bundles.** Serre fiber bundles generalize locally trivial fiber bundles. We start with a definition and examples.

Definition 9.5. A map $p : E \longrightarrow B$ is a *Serre fiber bundle* if the CHP holds for any CW-complex.

Remark. We emphasize that we do not assume uniqueness of the covering homotopy. A Serre fiber bundle in general is not locally trivial, see Fig. 9.3.

Examples. **1.** Locally-trivial fiber bundles.

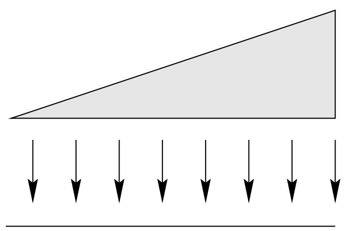


Fig. 9.3.

2. Let Y be an arbitrary path-connected space, $\mathcal{E}(Y, y_0)$ be the space of paths starting at y_0. The map $p : \mathcal{E}(Y, y_0) \longrightarrow Y$, where $p(s : I \longrightarrow Y) = s(1) \in Y$ is Serre fiber bundle. Note that $p^{-1}(y_0) = \Omega(Y, y_0)$.

Let $f : Z \longrightarrow Y$ be a map, $\widetilde{f} : Z \longrightarrow \mathcal{E}(Y, y_0)$ be a covering map, and $F : Z \times I \longrightarrow Y$ be a homotopy of f $(F|_{Z\times\{0\}} = f)$. Then a covering homotopy $\widetilde{F} : Z \times I \longrightarrow \mathcal{E}(Y, y_0)$ may be defined by the formula (see Fig. 9.4):

$$(\widetilde{F}(z,t))(\tau) = \begin{cases} (\widetilde{f}(z))(\tau(1+t)) & \text{if } \tau(1+t) \leq 1, \\ F(z, \tau(1+t)-1) & \text{if } \tau(1+t) \geq 1 \end{cases} \tag{27}$$

Exercise 9.7. *Check that the formula* (27) *indeed defines a covering homotopy as required.*

3. (A generalization of the previous example.) Let $A \subset X$, and (X, A) be a Borsuk pair (for example, a CW-pair). Let $E = \mathcal{C}(X, Y)$, $B = \mathcal{C}(A, Y)$, and the map $p : E \longrightarrow B$ be defined as $p(f : X \longrightarrow Y) = (f|_A : A \longrightarrow Y)$.

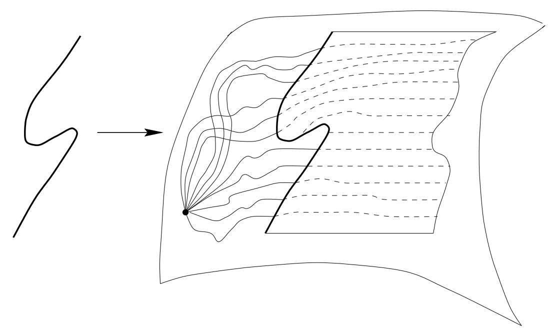


Fig. 9.4.

Exercise 9.8. *Prove that the map*

$$p : \mathcal{C}(X, Y) \longrightarrow \mathcal{C}(A, Y)$$

defined above is indeed a Serre fiber bundle.

As far as the fibers of a Serre fiber bundle $p : E \longrightarrow B$ are concerned, we cannot claim that for any two points $x_0, x_1 \in B$ the fibers $p^{-1}(x_0)$, $p^{-1}(x_1)$ are homeomorphic. However, we will prove here that the fibers are *weak homotopy equivalent.*

Definition 9.6. *Spaces X and Y are weak homotopy equivalent if there is a natural one-to-one correspondence $\varphi_K : [K, X] \xrightarrow{\cong} [K, Y]$ for any CW-complex K. Naturality means that for any map $f : K \longrightarrow K'$ the following diagram*

$$\begin{array}{ccc} [K, X] & \xrightarrow{\varphi_K} & [K, Y] \\ \uparrow{\scriptstyle f^*} & & \uparrow{\scriptstyle f^*} \\ [K', X] & \xrightarrow{\varphi_{K'}} & [K', Y] \end{array}$$

commutes. (Compare with Definitions 3.1, 3.2, 3.3.)

Exercise 9.9. *Prove that finite CW-complexes X, Y are weak homotopy equivalent if and only if they are homotopy equivalent.*

Theorem 9.7. *Let $p : E \longrightarrow B$ be Serre fiber bundle, where B is a path-connected space. Then the spaces $F_0 = p^{-1}(x_0)$ and $F_1 = p^{-1}(x_1)$ are weak homotopy equivalent for any two points $x_0, x_1 \in B$.*

Proof. Let $s : I \longrightarrow B$ be a path connecting x_0 and x_1. We have to define one-to-one correspondence $\varphi_K : [K, F_0] \longrightarrow [K, F_1]$ for any CW-complex K.

Let $h_0 : K \longrightarrow F_0$ be a map. Denote $i_0 : F_0 \longrightarrow E$ the inclusion map. We have the map:

$$\widetilde{f} : K \xrightarrow{h_0} F_0 \xrightarrow{i_0} E$$

which lifts the map $f : K \longrightarrow \{x_0\} \subset B$. Consider also the homotopy $F : K \times I \longrightarrow B$, where $F(x,t) = s(t)$ of the map f. By the CHP there exists a covering homotopy $\widetilde{F} : K \times I \longrightarrow E$ of the map $\widetilde{f}$ such that $p \circ \widetilde{F} = F$, in particular, $\widetilde{F}(K \times \{t\}) \subset p^{-1}(s(t))$, and $\widetilde{F}(K \times \{1\}) \subset F_1$.

We define $\varphi_K(h_0 : K \longrightarrow F_0) = (h_1 : K \longrightarrow F_1)$, where $h_1 = \widetilde{F}|_{K \times \{1\}}$. We should show that the map φ_K is well-defined.

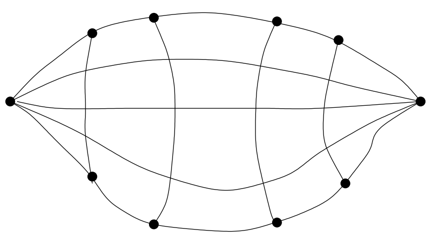


Fig. 9.5.

Let s' be a different path connecting x_0 and x_1, and $\widetilde{f}' : K \longrightarrow E$, $F' : K \times I \longrightarrow B$, $h' : K \longrightarrow F_1$ be corresponding maps and homotopies determined by s'. Assume that s and s' are homotopic, and let $S : I \times I \longrightarrow B$ be a corresponding homotopy. Denote by $T : I \times I \longrightarrow B$ a map defined by $T(t_1, t_2) = S(t_2, t_1)$, see Fig. 9.5. We are going to use the *relative version* of the CHP for the pair $Z' \subset Z$ where $Z = K \times I$ and $Z' = K \times \{0, 1\}$.

Consider the following commutative diagram:

$$\begin{tikzcd}
& & E \arrow[d, "p"] \\
(K \times \{0,1\}) \times I \arrow[r] \arrow[urr, "\tilde{G}'"] & (K \times I) \times I \arrow[r, "G"] \arrow[ur, "\tilde{G}"] & B \\
(K \times \{0,1\}) \times \{0\} \arrow[u] \arrow[r] & (K \times I) \times \{0\} \arrow[u] \arrow[uur, "\tilde{g}"] \arrow[ur, "g"] &
\end{tikzcd} \tag{28}$$

Here the map $g : (K \times I) \times \{0\} \to B$ sends everything to x_0, and $\widetilde{g} : K \times I \longrightarrow E$ defined by $\widetilde{g}(k, t_1) = h_0(k)$ (see above). The homotopy $G : (K \times I) \times I \longrightarrow B$ is defined by the formula: $G(k, t_1, t_2) = T(t_1, t_2)$. The map $\tilde{G}' : (K \times \{0,1\}) \times I \to E$ is defined by the homotopies F and F':

$$\tilde{G}'|_{K \times \{0\} \times I} = F, \quad \tilde{G}'|_{K \times \{1\} \times I} = F'.$$

The relative version of the CHP implies that there exists $\widetilde{G} : K \times I \longrightarrow E$ covering G and $\tilde{G}'$ as it is shown at in (28). The map $(k, t) \longrightarrow \widetilde{G}(k, t, 1)$ maps $K \times I$ to F_1: this is the homotopy connecting

h_1 and h_1', see Fig. 9.6. Thus a path $s : I \longrightarrow B$ defines a map $\varphi_K(s) : [K, F_0] \longrightarrow [K, F_1]$, $F_0 = p^{-1}(s(0))$, $F_1 = p^{-1}(s(1))$, which does depend only of the homotopy class of s.

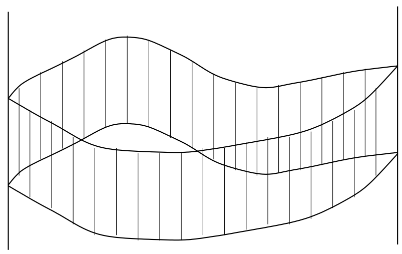


Fig. 9.6.

Clearly the map φ_K is natural with respect to K; note also that if s is a constant path, then $\varphi_K = Id_{F_0}$. Moreover, if a composition of paths $s_2 \cdot s_1$ (i.e. $s_1(1) = s_2(0)$) gives a map $\varphi_K(s_2 \cdot s_1) = \varphi_K(s_2) \circ \varphi_K(s_1)$. In particular, the map $\varphi_K(s^{-1})$ is inverse to $\varphi_K(s)$: it implies that $\varphi_K(s)$ is one-to-one. $\square$

Now let $f : X \longrightarrow Y$ be a map. We say that a map $f_1 : X_1 \longrightarrow Y_1$ is *homotopy equivalent* to f, if there are homotopy equivalences $\varphi : X \longrightarrow X_1$, $\psi : Y \longrightarrow Y_1$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\varphi"] & Y \arrow[d, "\psi"] \\ X_1 \arrow[r, "f_1"] & Y_1 \end{tikzcd}$$

Theorem 9.8. *For any continuous map $f : X \longrightarrow Y$ there exists homotopy equivalent map $f_1 : X_1 \longrightarrow Y_1$, such that $f_1 : X_1 \longrightarrow Y_1$ is Serre fiber bundle.*

Remark. It will be clear from the construction below that the space Y_1 may be chosen to be equal to Y. It is also important that the construction below is natural. It means that the commutative diagram on the left implies a commutativity of the diagram on the right:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\alpha"] & Y \arrow[d, "\beta"] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd}$$

$$\begin{tikzcd} & X \arrow[rr, "f"] \arrow[dd, "\alpha"] & & Y \arrow[dd, "\beta"] \\ X_1 \arrow[ur, "\varphi"] \arrow[rr, "f_1"] \arrow[dd, "\alpha_1"] & & Y_1 \arrow[ur, "\psi"] \arrow[dd, "\beta_1"] & \\ & X' \arrow[rr, "f'"] & & Y' \\ X_1' \arrow[ur, "\varphi'"] \arrow[rr, "f_1'"] & & Y_1' \arrow[ur, "\psi'"] & \end{tikzcd}$$

Proof of Theorem 9.8. Let $Y_1 = Y$, and

$$X_1 = \{(x, s) \in X \times \mathcal{E}(Y) \mid s(0) = f(x)\}.$$

Then $p : X_1 \longrightarrow Y$ is defined by $p(x,s) = s(1)$. Clearly X and X_1 are homotopy equivalent. $\square$

The following statement is “dual” to Theorem 9.8:

Claim 9.1. *Let $f : X \longrightarrow Y$ be a continuous map. Then there exists a homotopy equivalent map $g : X \longrightarrow Y'$, so that g is an inclusion.*

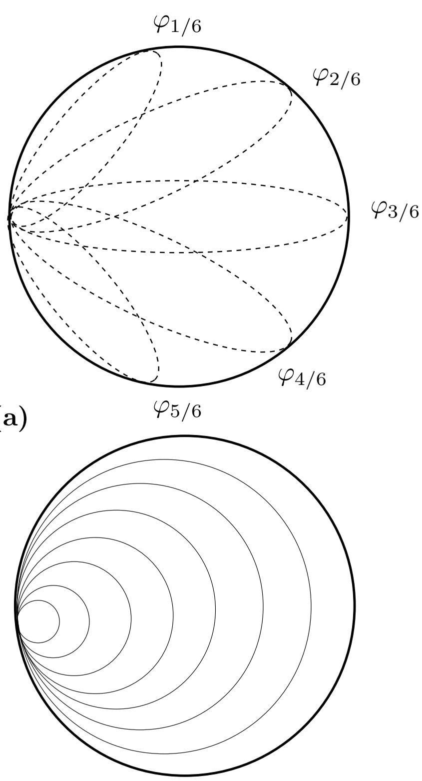
Proof. Let $Y' = (X \times I) \cup_f Y$ be the cylinder of the map f. Clearly $Y' \sim Y$, and $g : X \longrightarrow Y'$ is an embedding of X into the top base of $X \times I$. $\square$

9.4. **Homotopy exact sequence of a fiber bundle.** First we prove the following important fact:

Lemma 9.9. *Let $p : E \longrightarrow B$ be Serre fiber bundle, $y \in E$ be any point, $x = p(y)$, $F = p^{-1}(x)$. The homomorphism*

$$p_* : \pi_n(E, F; y) \longrightarrow \pi_n(B, x)$$

is an isomorphism for all $n \geq 1$.



(a) (b) **Fig. 9.7.**

Proof. p_* IS A MONOMORPHISM. Let $\alpha \in \pi_n(E, F; y)$ be represented by a map $\tilde{f} : D^n \longrightarrow E$ (where $\tilde{f}|_{S^{n-1}} : S^{n-1} \longrightarrow F$, and $\tilde{f}(s_0) = y$). Then the map

$$f = p \circ \tilde{f} : D^n \longrightarrow B$$

has the property that $f(S^{n-1}) = x$ and $[f] = p_*(\alpha) \in \pi_n(B, x)$. Assume $\alpha \in \text{Ker } p_*$, then there exists a homotopy $f_t : D^n \longrightarrow B$, so that $f_0 = f$ and $f_1(D^n) = x$. The covering homotopy property (the strong version) implies that there exists a homotopy $\tilde{f}_t : D^n \longrightarrow E$ covering the homotopy f_t. In particular $\tilde{f}_1(D^n) \subset F = p^{-1}(x)$ since $p \circ \tilde{f}_1(D^n) = f_1(D^n) = x$.

p_* IS AN EPIMORPHISM. Consider the homotopy $\varphi_t : S^{n-1} \longrightarrow S^n$, so that $\varphi : S^{n-1} \times I \longrightarrow S^n$, $\varphi(S^{n-1} \times I) = S^n$ as it is shown at Fig. 9.7 (a). Let $f : S^n \longrightarrow B$ be a map representing $\beta \in \pi_n(B, x)$. Consider the homotopy $g_t = f \circ \varphi_t : S^{n-1} \longrightarrow B$. Then we lift the homotopy g_t up to a homotopy $\tilde{g}_t : S^{n-1} \longrightarrow E$ by applying the CHP. The homotopy $\tilde{g}_t$ may be considered as a map $\tilde{h} : D^n \longrightarrow E$, where the disk D^n is covered by $(n-1)$-spheres as it is shown, see Fig. 9.7 (b), and the map h on these spheres is given by $\tilde{g}_t$. Clearly the map $h : D^n \longrightarrow E$ gives a representative of an element $\alpha \in \pi_n(E, F)$, so that $p_*(\alpha) = \beta$. $\square$

Now the exact sequence of the pair $(E,F;y)$:

$$\cdots \longrightarrow \pi_n(F,y) \xrightarrow{i_*} \pi_n(E,y) \xrightarrow{j_*} \pi_n(E,F;y) \xrightarrow{\partial} \pi_{n-1}(F,y) \longrightarrow \cdots$$

gives the exact sequence:

$$\cdots \longrightarrow \pi_n(F,y) \xrightarrow{i_*} \pi_n(E,y) \xrightarrow{j_*} \pi_n(B,x) \xrightarrow{\partial} \pi_{n-1}(F,y) \longrightarrow \cdots \tag{29}$$

We call the sequence (29) a *homotopy exact sequence of Serre fibration.*

Exercise 9.10. *Apply the sequence (29) for the Hopf fibration* $S^3 \longrightarrow S^2$*. Prove that (a)* $\pi_2(S^2) = \pi_1(S^1) = \mathbf{Z}$*; (b)* $\pi_n(S^3) = \pi_n(S^2)$*.*

Exercise 9.11. *Let* $S^\infty \longrightarrow \mathbf{CP}^\infty$ *be the Hopf fibration. Using the fact* $S^\infty \sim *$*, prove that* $\pi_n(\mathbf{CP}^\infty) = 0$ *for* $n \neq 2$*, and* $\pi_2(\mathbf{CP}^\infty) = \mathbf{Z}$*.*

Exercise 9.12. *Prove that* $\pi_n(\Omega(X)) \cong \pi_{n+1}(X)$ *for any* X *and* $n \geq 0$*.*

Exercise 9.13. *Prove that if the groups* $\pi_*(B)$*,* $\pi_*(F)$ *are finite (finitely generated), then the groups* $\pi_*(E)$ *are finite (finitely generated) as well.*

Exercise 9.14. *Assume that a fiber bundle* $p : E \longrightarrow B$ *has a section, i.e. a map* $s : B \longrightarrow E$*, such that* $p \circ s = Id_B$*. Prove the isomorphism* $\pi_n(E) \cong \pi_n(B) \oplus \pi_n(F)$*.*

9.5. **More on the groups $\pi_n(X,A;x_0)$.** Now we construct such a space Y that $\pi_n(X,A;x_0) \cong \pi_{n-1}(Y,y_0)$. First, we construct Serre fiber bundle $A_1 \xrightarrow{p} X$ which is homotopy equivalent to the inclusion $A \xrightarrow{i} X$. Let $Y = p^{-1}(x_0)$ be a fiber of this fiber bundle. By construction above, Y is a space of loops in X which starting in A and ending at the point x_0:

$$Y = \{(a,\gamma) \mid \gamma(0) = a, \quad \gamma(1) = x_0 \ \}.$$

We construct a homomorphism $\alpha : \pi_{n-1}(Y,y_0) \longrightarrow \pi_n(X,A;x_0)$ as follows.

A map $S^{n-1} \xrightarrow{g} Y$ gives a map $G : D^n = C(S^{n-1}) \longrightarrow X$ by the formula: $G(s,t) = g(s)(t)$, $s \in S^{n-1}$, $t \in I$. Here $g(s) = \gamma(t)$. Clearly it is well-defined since $\gamma(1) = x_0$ for all paths γ such that $(\gamma(0),\gamma) \in Y$. The map α may be included to the commutative diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_n(A_1) \arrow[r, "(p_1)_*"] \arrow[d] & \pi_n(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(C)}"] & \pi_{n-1}(Y) \arrow[r] \arrow[d, "\alpha"] & \pi_{n-1}(A_1) \arrow[r] \arrow[d] & \pi_{n-1}(X) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \pi_n(A) \arrow[r, "i_*"] & \pi_n(X) \arrow[r] & \pi_n(X,A) \arrow[r] & \pi_{n-1}(A) \arrow[r] & \pi_{n-1}(X) \arrow[r] & \cdots
\end{tikzcd}$$

where rows are exact. Five–Lemma implies that $\alpha : \pi_{n-1}(Y) \longrightarrow \pi_n(X,A)$ is an isomorphism. In particular, we conclude that $\pi_n(X,A)$ is abelian group for $n \geq 3$.

Exercise 9.15. Prove that the square (C) of the above diagram commutes.

10. Suspension Theorem and Whitehead product

10.1. **The Freudenthal Theorem.** Let X be a space with a base point x_0. We construct a homomorphism

$$\Sigma : \pi_q(X) \longrightarrow \pi_{q+1}(\Sigma X) \tag{30}$$

as follows. Let $\alpha \in \pi_q(X)$, and a map $f : S^q \longrightarrow X$ be a representative of α. The map

$$\Sigma f : \Sigma S^q = S^{q+1} \longrightarrow \Sigma X$$

defined by the formula $\Sigma f(y,t) = (f(y),t) \in \Sigma X$ gives a representative for $\Sigma(a) \in \pi_{q+1}(\Sigma X)$. It is not hard to check that

1. $f \sim g$ implies that $\Sigma f \sim \Sigma g$;
2. $\Sigma f + \Sigma g \sim \Sigma(f+g)$.

The homomorphism Σ is called the *suspension homomorphism.*

Theorem 10.1. (Freudenthal Theorem) *The suspension homomorphism*

$$\Sigma : \pi_q(S^n) \longrightarrow \pi_{q+1}(S^{n+1})$$

is isomorphism for $q < 2n-1$ and epimorphism for $q = 2n-1$.

Remark. This is the "easy part" of the suspension Theorem. The "hard part" will be discussed later, see Theorem 10.11. The general *Freudenthal Theorem* goes as follows:

Theorem 10.2. *Let X be an $(n-1)$-connected CW-complex (it implies that $\pi_i(X) = 0$ for $i < n$). Then the suspension homomorphism $\Sigma : \pi_q(X) \longrightarrow \pi_{q+1}(\Sigma X)$ is isomorphism for $q < 2n-1$ and epimorphism for $q = 2n-1$.*

Proof that Σ is surjective. Let $f : S^{q+1} \longrightarrow S^{n+1}$ be an arbitrary map. We have to prove that we can perform a homotopy of this map f to a map Σh, where $h : S^q \longrightarrow S^n$. We will assume that $n > 0$, and $q \geq n$. In particular, the group $\pi_{q+1}(S^{n+1})$ is abelian, and $\pi_1(S^{n+1}) = 0$, so we can forget about particular choice of the base point.

Let a, b be the north and south poles of the sphere S^{n+1}. We identify the sphere S^{q+1} with the space $\mathbf{R}^{q+1} \cup \infty$, moreover, we choose this identification in such way that $f^{-1}(a), f^{-1}(b)$ do not contain the infinity.

First we should take care about the sets $f^{-1}(a)$ and $f^{-1}(b)$. We do not have any control over the map f, the only property we can use is that f continuous. However clearly $f^{-1}(a)$ and $f^{-1}(b)$ are compact sets in $\mathbf{R}^{q+1}$. Recall that if K is a finite simplicial complex in $\mathbf{R}^{q+1}$, then $\dim K$ is a maximal dimension of the simplices of K.

Lemma 10.3. *There exists a map $f_1 : S^{q+1} \longrightarrow S^{n+1}$ homotopic to f (and, actually as close to f as one may wish), such that $f_1^{-1}(a)$, $f_1^{-1}(b)$ are finite simplicial complexes in $\mathbf{R}^{q+1}$ of dimension less or equal to $q-n$.*

Proof. Here we apply the same constuction as we used in "free point Lemma".

Let us recall briefly the main steps:

1. Find five small disks $a \in D_1^{(a)} \subset \ldots \subset D_5^{(a)}$ centered at a, and five small disks $b \in D_1^{(b)} \subset \ldots \subset D_5^{(b)}$ centered at b. We assume that the radius of the disk $D_i^{(j)}$ is $\frac{ir}{5}$, $i = 1, \ldots, 5$, $j = a, b$. [11]
2. Find a huge simplex Δ in R^{q+1} containing $f^{-1}(D_5^{(a)}) \cup f^{-1}(D_5^{(b)})$.
3. Find fine enough barycentric triangulation $\{\Delta_\alpha\}$ of the simplex Δ,
$$\Delta = \bigcup_\alpha \Delta_\alpha,$$
such that for any simplex Δ_α satisfies the following conditions:
   - if $f(\Delta_\alpha) \cap D_i^{(j)} \neq \emptyset$, then $f(\Delta_\alpha) \subset D_{i+1}^{(j)}$ (here $i = 1, 2, 3, 4$, $j = a, b$);
   - the diameter of the image $f(\Delta_\alpha)$ is no more than $r/5$ for each α.
4. Consider the simplicial complex
$$K = \bigcup_{f(\Delta_\alpha)\cap(D_4^{(a)}\cup D_4^{(b)})\neq\emptyset} \Delta_\alpha.$$
5. Construct a map $f' : K \longrightarrow S^{n+1}$ which coincides with f on each vertex of K and extended linearly to all simplices.
6. "Glue" the maps f' and f to get a map f_1 which coincides with f' on $f^{-1}(D_2^{(a)} \cup D_2^{(b)})$ and with f outside of $f^{-1}(D_3^{(a)} \cup D_3^{(b)})$.

This gives us a map f_1 (which is homotopic to f) with the following property:

The inverse images $f_1^{-1}(D_1^{(a)})$ and $f_1^{-1}(D_1^{(b)})$ are covered by finite number of $q+1$-simplices Δ_α, such that $f_1|_{\Delta_i}$ is a linear map.

Assume for a moment that there is such a simplex $\Delta_\alpha \subset \Delta$ that the simplex $f_1(\Delta_\alpha) \subset S^{n+1}$ has dimesion less than $(n+1)$, and $a \in f_1(\Delta_\alpha)$. Then we can change a little bit the map f_1 (it is enough to change a value of f_1 at one vertex!) to get a map f_2 such that $a \notin f_1(\Delta_\alpha)$.

This observation allows us to assume that if $a \in f_2(\Delta_\alpha)$, then the simplex $f_2(\Delta_\alpha)$ has dimension $(n+1)$. Since the restriction $f_2|_{\Delta_\alpha}$ is a linear map of maximal rank, than $f_2^{-1}(a) = K$ consists of simplices of dimension at most $(q+1)-(n+1) = q-n$. This proves that the inverse images $K = f_2^{-1}(a)$, $L = f_2^{-1}(b)$ are simplicial complexes of dimension at most $n-q$. □

[11] Once again, we identify a neighborhood of a (respectively of b) with an open subset of $\mathbf{R}^{n+1}$ via, say, the stereographic projection.

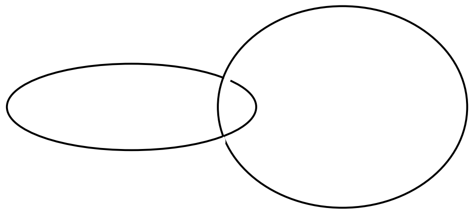


Fig. 10.1.

Now we have to introduce a couple of definitions.

A homotopy $F : \mathbf{R}^p \times I \longrightarrow \mathbf{R}^p$ is an **isotopy** if $F_t : \mathbf{R}^p \longrightarrow \mathbf{R}^p$ is a homeomorphism for eact $t \in I$. A hyperplane $\Pi \subset \mathbf{R}^p$ divides $\mathbf{R}^p \setminus \Pi$ into two half-spaces: $\mathbf{R}^p_-$ and $\mathbf{R}^p_+$.

We say that two simlicial complexes $K, L \subset \mathbf{R}^p$ **are not linked** if there exist a hyperplane $\Pi \subset \mathbf{R}^p$, and an isotopy $F_t : \mathbf{R}^p \longrightarrow \mathbf{R}^p$, so that $F_0 = Id$, and the sets $F_1(K)$ and $F_1(L)$ are separated by the hyperplane Π. Fig. 10.1 shows an example of two linked circles.

Lemma 10.4. *Let $K, L \subset \mathbf{R}^p$ be two finite simplicial complexes of dimensions k, l respectively. Let $k + l + 1 < p$. Then the simplicial complexes K and L are not linked.*

Proof. First let $\Pi \subset \mathbf{R}^p$ be a hyperplane such that $K \cap \Pi = \emptyset$, and $L \cap \Pi = \emptyset$. If K and L are in the different half-spaces, then we are done. Let K and L be in $\mathbf{R}^p_+$. We want to produce an isotopy $F_t : \mathbf{R}^p \longrightarrow \mathbf{R}^p$ such that $F_0 = Id_{\mathbf{R}^p}$ and $F_1(K)$ and $F_1(L)$ are separated by the hyperplane Π. We need the following statement.

Claim 10.1. *There exists a point $x_0 \in \mathbf{R}^p_-$ such that any line going through x_0 does not intersect both K and L.*

Proof of Claim 10.1. Let $W_1, \ldots, W_\nu \subset \mathbf{R}^p$ be planes (of minimal dimensions) containing the simplices $\Delta_1, \ldots, \Delta_\nu$ of the simplicial complex K, and let $U_1, \ldots, U_\mu \subset \mathbf{R}^p$ be the corresponding planes containing the simplices of L. Notice that $\dim W_i \leq k$ and $\dim U_j \leq l$, $i = 1, \ldots, \nu$, $j = 1, \ldots, \mu$. Let Π_{ij} be a minimal plane containing W_i and U_j. Notice that the maximal dimension of Π_{ij} is $k + l + 1$. Indeed, let $w \in W_i$, $u \in U_j$ be any points. Then a basis of W_i, a basis of U_j, and the vector $w - u$ generate Π_{ij}, see Fig. 10.2. Since $k + l + 1 < p$, there exists a point x_0 of $\mathbf{R}^p_-$, such that $x_0 \notin \bigcup \Pi_{ij}$. $\square$

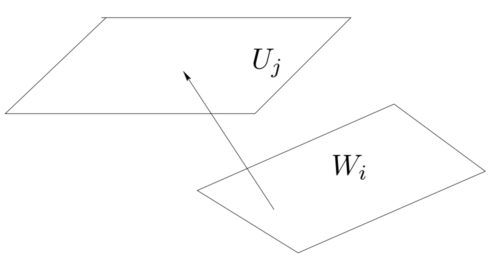


Fig. 10.2: The plane Π_{ij}.

Now we continue the proof of Lemma 10.4. The isotopy F_t may be costructed as follows. Consider the space of all lines going through the point $x_0 \in \mathbf{R}^p$. This is the projective space $\mathbf{RP}^{p-1}$. Choose a continuous nonnegative function

$$\varphi : \mathbf{RP}^{p-1} \longrightarrow \mathbf{R}$$

such that $\varphi(\lambda) = 0$ if $\lambda \cap L \neq \emptyset$, and $\varphi(\lambda) = v_0 > 0$ if $\lambda \cap K \neq \emptyset$.

Now the isotopy $F_t : \mathbf{R}^p \longrightarrow \mathbf{R}^p$ moves a point $x \in \mathbf{R}^p$ along the line λ (connecting x and x_0) toward x_0 with the velocity $\varphi(\lambda)$, where φ is as above.

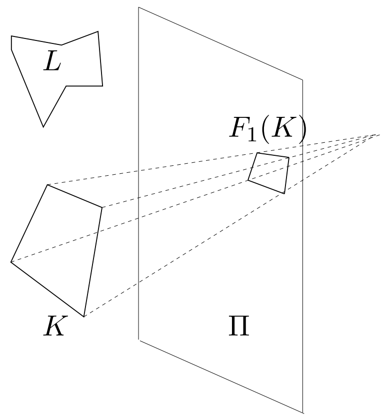


Fig. 10.3: The isotopy F_t.

Clearly at some moment the image of K will be inside of $\mathbf{R}^p_-$, see Fig. 10.3. $\square$

We complete the proof that Σ is surjective. We use Lemma 10.3 and Lemma 10.4 to construct a map $f_2 : S^{q+1} \longrightarrow S^{n+1}$ homotopic to f such that the inverse images $f_2^{-1}(a)$ and $f_2^{-1}(b)$ are located in the "northern" and respectively "southern" parts of the sphere S^{q+1} (we use here the following obvious estimation: $(q-n)+(q-n)+1 = 2q-(2n-1) \leq 2q-q < q+1$ provided $q \leq 2n-1$). Furthermore, there are two "ice caps", disks A and B centered at the poles a and b respectively, which do not touch the equator of S^{n+1}, and such that $f_2^{-1}(A)$ and $f_2^{-1}(B)$ do not touch the equator of S^{q+1} as well, see the picture below:

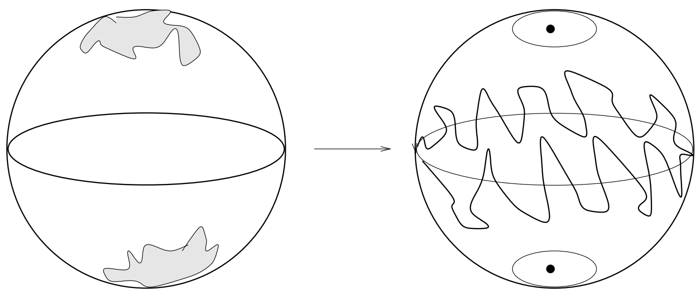


Fig. 10.4.

Now we make a homotopy $S^{n+1} \longrightarrow S^{n+1}$ which sretches A and B to the north and the south hemispheres respectively, and squeezes the remainder onto the equator sphere $S^n \subset S^{n+1}$. By composing this map with f_2, we obtain a map f_3 which sends the equator of S^{q+1} to the equator of S^{n+1}, and the north and south poles of S^{q+1} sends to the north and south poles of S^{n+1}. Now we look at the spheres S^{q+1} and S^{n+1} from the North:

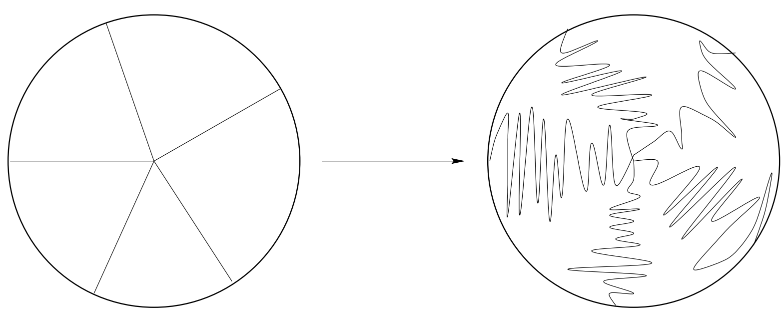


Fig. 10.5.

Here we see only the northern hemispheres. We have here all possible meridians of S^{q+1} and their images under the map f_3. The further homotopy which finally turns the map f_4 into the suspension map may be constructed as follows:

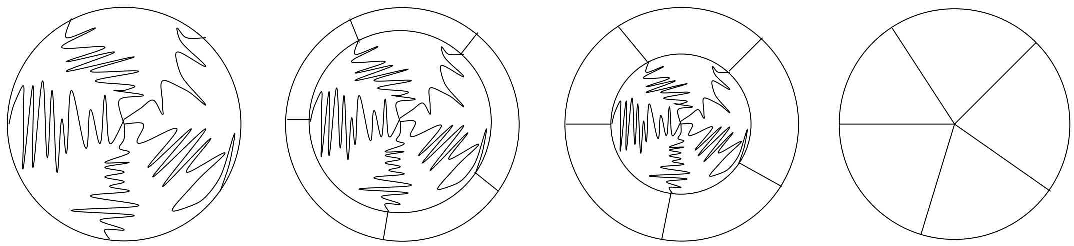


Fig. 10.6.

This construction due to J. Alexander. □

Exercise 10.1. Describe the last homotopy in more detail.

Proof that Σ is injective for $q < 2n-1$. Let $f_0 = \Sigma h_0 : S^{q+1} \longrightarrow S^{n+1}$, and $f_1 = \Sigma h_1 : S^{q+1} \longrightarrow S^{n+1}$, and $f_0 \sim f_1$. We should show that $h_0 \sim h_1$.

We consider the homotopy $F : S^{q+1} \times I \longrightarrow S^{n+1}$. Again, we examine $F^{-1}(a)$ and $F^{-1}(b)$, and by Lemma 10.3 (to be precise, its generalization) we conclude that F is homotopic to F_1 such that $F_1^{-1}(a) = K$ and $F_1^{-1}(b) = L$ are finite simplicial complexes of dimension at most $q+1-n$. The condition $q < 2n-1$ and Lemma 10.4 imply that the simplicial complexes K and L may be separated. The rest of the arguments are very similar to those applied in the above proof. □

Exercise 10.2. Prove the injectivity of Σ in detail.

10.2. **First applications.**

Theorem 10.5. (Hopf) $\pi_n(S^n) \cong \mathbf{Z}$ *for each* $n \geq 1$.

Exercise 10.3. Prove Theorem 10.5.

Exercise 10.4. Prove that $\pi_3(S^2) \cong \mathbf{Z}$, and the Hopf map $S^3 \longrightarrow S^2$ is a representative of the generator of $\pi_3(S^2)$.

Corollary 10.6. *The sphere S^n is not contractible.*

10.3. **A degree of a map $S^n \to S^n$.** A map $f : S^n \longrightarrow S^n$ gives a representative of some element $\alpha \in \pi_n(S^n) \cong \mathbf{Z}$. We choose the generator ι_n of $\pi_n(S^n)$ as a homotopy class of the identity map. Thus $[f] = \alpha = \lambda\iota_n$. The integer $\lambda \in \mathbf{Z}$ is called a degree of the map f. The notation is $\deg f$.

Exercise 10.5. Prove the following properties of the degree:

(a) Two maps $f, g : S^n \longrightarrow S^n$ are homotopic if and only if $\deg f = \deg g$.

(b) A map $f : S^n \longrightarrow S^n$, $\deg f = \lambda$ induces the homomorphism $f_* : \pi_n(S^n) \longrightarrow \pi_n(S^n)$ which is a multiplication by λ.

(c) The suspension $\Sigma f : \Sigma S^n \longrightarrow \Sigma S^n$ has degree λ if and only if the map $f : S^n \longrightarrow S^n$ has degree λ.

10.4. **Stable homotopy groups of spheres.** Consider the following chain of the suspension homomorphisms:

$$\pi_{k+1}S^1 \xrightarrow{\Sigma} \pi_{k+2}S^2 \xrightarrow{\Sigma} \cdots \xrightarrow{\Sigma} \pi_{k+n}S^n \xrightarrow{\Sigma} \pi_{k+n+1}S^{n+1} \xrightarrow{\Sigma} \cdots$$

By the Suspension Theorem the homomorphism $\Sigma : \pi_{k+n}S^n \longrightarrow \pi_{k+n+1}S^{n+1}$ is isomorphism provided that $n \geq k+2$. The group $\pi_{k+n}S^n$ with $n \geq k+2$ is called the stable homotopy group of sphere. The notation:

$$\pi_k^s(S^0) = \pi_{k+n}S^n \quad \text{where } n \geq k+2.$$

So far we proved that $\pi_0(S^0) = \pi_n S^n \cong \mathbf{Z}$. The problem to compute the stable homotopy groups of spheres is highly nontrivial. We shall return to this problem later.

10.5. **Whitehead product.** Consider the product $S^n \times S^k$ as a CW-complex. Clearly we can choose a cell decomposition of $S^n \times S^k$ into four cells of dimensions $0, n, k, n+k$. The first three cells give us the wedge $S^n \vee S^k \subset S^n \times S^k$. The last cell $e^{n+k} \subset S^n \times S^k$ has the attaching map $w : S^{n+k-1} \longrightarrow S^n \vee S^k$. This attaching map is called the *Whitehead map.* It is convenient to have a particular construction of the map w.

We can think about the sphere S^{n+k-1} as a boundary of the unit disk $D^{n+k} \subset \mathbf{R}^{n+k}$. Thus a point $x \in S^{n+k-1}$ has coordinates $(x_1, \ldots, x_{n+k})$, where $x_1^2 + \cdots + x_{n+k}^2 = 1$. We define

$$\begin{aligned} U &= \left\{(x_1, \ldots, x_{n+k}) \in S^{n+k-1} \mid x_1^2 + \cdots + x_n^2 \leq 1/2\right\}, \\ V &= \left\{(x_1, \ldots, x_{n+k}) \in S^{n+k-1} \mid x_{n+1}^2 + \cdots + x_{n+k}^2 \leq 1/2\right\} \end{aligned}$$

Exercise 10.6. Prove that U is homeomorphic to $D^n \times S^{k-1}$, V is homeomorphic to $S^{n-1} \times D^k$, and that

$$S^{n+k-1} \cong D^n \times S^{k-1} \cup_{S^{n-1}\times S^{k-1}} S^{n-1} \times D^k$$

Remark. The same decomposition may be constructed by using the homeomorphisms:

$$\begin{aligned} S^{n+k-1} = \partial(D^{n+k}) = \partial(D^n \times D^k) &= \partial(D^n) \times D^k \cup_{S^{n-1}\times S^{k-1}} D^n \times \partial(D^k) \\ &= S^{n-1} \times D^k \cup_{S^{n-1}\times S^{k-1}} D^n \times S^{k-1}. \end{aligned}$$

The map $w : S^{n+k-1} \longrightarrow S^n \vee S^k$ is defined as follows. First we construct the maps

$$\varphi_U : U \longrightarrow S^n \vee S^k \quad \text{and} \quad \varphi_V : V \longrightarrow S^n \vee S^k$$

as the compositions:

$$\varphi_U : U \xrightarrow{\cong} D^n \times S^{k-1} \xrightarrow{pr} D^n \longrightarrow D^n/S^{n-1} \xrightarrow{\cong} S^n \longrightarrow S^n \vee S^k,$$

$$\varphi_V : V \xrightarrow{\cong} S^{n-1} \times D^k \xrightarrow{pr} D^k \longrightarrow D^k/S^{k-1} \xrightarrow{\cong} S^k \longrightarrow S^n \vee S^k.$$

Clearly we have that

$$\varphi_U|_{S^{n-1}\times S^{k-1}} = * = \varphi_V|_{S^{n-1}\times S^{k-1}}$$

and hence the maps φ_U, φ_V define the map $w : S^{n+k-1} \longrightarrow S^n \vee S^k$.

Remark. It is easy to see that the above map $w : S^{n+k-1} \longrightarrow S^n \vee S^k$ is the attaching map for the cell e^{n+k} in the product $S^n \times S^k$.

Now let $\alpha \in \pi_n(X, x_0)$ and $\beta \in \pi_k(X, x_0)$ be represented by maps

$$f : S^n \longrightarrow X, \quad g : S^k \longrightarrow X.$$

We define a map $h : S^{n+k-1} \longrightarrow X$ as the composition:

$$h : S^{n+k-1} \xrightarrow{w} S^n \vee S^k \xrightarrow{f\vee g} X.$$

A homotopy class of the map h defines an element $[\alpha, \beta] \in \pi_{n+k-1}(X, x_0)$ which is called the *Whitehead product.*

Lemma 10.7. *The Whitehead product satisfies the following properties:*

- **(1)** *Naturality: Let* $f : (X, x_0) \longrightarrow (Y, y_0)$ *be a map,* $\alpha \in \pi_n(X, x_0))$ *and* $\beta \in \pi_k(X, x_0)$*. Then*

$$f_*([\alpha, \beta]) = [f_*(\alpha), f_*(\beta)],$$

  where $f_* : \pi_*(X, x_0) \longrightarrow \pi_*(Y, y_0)$ *is the homomorphism induced by the map* f*.*
- **(2)** $[\alpha + \beta, \gamma] = [\alpha, \gamma] + [\beta, \gamma]$.
- **(3)** *If* $\alpha \in \pi_n(X)$, $\beta \in \pi_k(X)$ *then* $[\alpha, \beta] = (-1)^{nk}[\beta, \alpha]$.
- **(4)** *If* $\alpha \in \pi_n(X)$, $\beta \in \pi_k(X)$, $\gamma \in \pi_\ell(X)$, *then (the Jacobi identity)*

$$(-1)^{n\ell}[[\alpha, \beta], \gamma] + (-1)^{nk}[[\beta, \gamma], \alpha] + (-1)^{k\ell}[[\gamma, \alpha], \beta] = 0.$$

- **(5)** *If* $\alpha \in \pi_1(X)$, $\beta \in \pi_1(X)$ *then* $[\alpha, \beta] = \alpha\beta\alpha^{-1}\beta^{-1}$.

Exercise 10.7. Prove the above property (3).

Exercise 10.8. Prove the above property (5).

To prove more about the Whitehead product we have to figure out several facts about the Whitehead map $w : S^{n+k-1} \longrightarrow S^n \vee S^k$. The map w defines an element $w \in \pi_{n+k-1}(S^n \vee S^k)$.

Remark. Denote $\iota_n \in \pi_n(S^n)$, $\iota_k \in \pi_k(S^k)$ the generators given by the identity maps $Id : S^n \longrightarrow S^n$, $Id : S^k \longrightarrow S^k$ respectively. We denote also by ι_n, ι_k the image of the elements ι_n,

ι_k in $\pi_n(S^n \vee S^k)$, $\pi_k(S^n \vee S^k)$ respectively. Comparing the definitions of the Whitehead map $w : S^{n+k-1} \longrightarrow S^n \vee S^k$ and of the Whitehead product gives the identity:

$$w = [\iota_n, \iota_k] \in \pi_{n+k-1}(S^n \vee S^k).$$

Theorem 10.8. *The element $w \in \pi_{n+k-1}(S^n \vee S^k)$ has infinite order. In particular, the group $\pi_{n+k-1}(S^n \vee S^k)$ is infinite.*

Proof. The map w is the attaching map of the cell e^{n+k} in the product $S^n \times S^k$. It gives us the commutative diagram:

$$\begin{array}{ccc} S^{n+k-1} & \xrightarrow{\ w\ } & S^n \vee S^k \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle i} \\ D^{n+k} & \xrightarrow{\ \Phi\ } & S^n \times S^k \end{array}$$

Clearly the map $\Phi : D^{n+k} \longrightarrow S^n \times S^k$ determines an element $\iota \in \pi_{n+k}(S^n \times S^k, S^n \vee S^k)$. Consider the map

$$\bar{j} : (S^n \times S^k, S^n \vee S^k) \longrightarrow (S^{n+k}, s_0)$$

which maps $S^n \vee S^k$ to the base point $s_0 \in S^{n+k}$. The composition $\bar{j} \circ \Phi : D^{n+k} \longrightarrow S^{n+k}$ is a representative of a generator of the group $\pi_{n+k}(S^{n+k}) \cong \mathbf{Z}$. Thus we conclude that the element $\iota \in \pi_{n+k}(S^n \times S^k, S^n \vee S^k)$ is nontrivial and has infinite order.

Next we consider the long exact sequence in homotopy for the pair $(S^n \times S^k, S^n \vee S^k)$:

$$\pi_{n+k}(S^n \vee S^k) \xrightarrow{i_*} \pi_{n+k}(S^n \times S^k) \xrightarrow{j_*} \pi_{n+k}(S^n \times S^k, S^n \vee S^k) \xrightarrow{\partial} \pi_{n+k-1}(S^n \vee S^k)$$

We claim that i_* is epimorphic since $\pi_{n+k}(S^n \times S^k) = \pi_{n+k}(S^n) \oplus \pi_{n+k}(S^k)$. Thus the homomorphism j_* is zero, and ∂ is monomorphims. Since $w = \partial(\iota)$ it follows that the group $\pi_{n+k-1}(S^n \vee S^k)$ is infinite and w has infinite order. $\square$

Excercise 10.9. Give a proof that the above homomorphism

$$i_* : \pi_{n+k}(S^n \vee S^k) \to \pi_{n+k}(S^n \times S^k)$$

is epimorphism.

Lemma 10.9. *The element $w \in \pi_{n+k-1}(S^n \vee S^k)$ is in a kernel of each of the following homomorphisms:*

(1) $i_* : \pi_{n+k-1}(S^n \vee S^k) \longrightarrow \pi_{n+k-1}(S^n \times S^k)$,
(2) $pr_*^{(n)} : \pi_{n+k-1}(S^n \vee S^k) \longrightarrow \pi_{n+k-1}(S^n)$,
(3) $pr_*^{(k)} : \pi_{n+k-1}(S^n \vee S^k) \longrightarrow \pi_{n+k-1}(S^k)$.

Proof. The exact sequence

$$\longrightarrow \pi_{n+k}(S^n \times S^k, S^n \vee S^k) \xrightarrow{\partial} \pi_{n+k-1}(S^n \vee S^k) \xrightarrow{i_*} \pi_{n+k-1}(S^n \times S^k) \longrightarrow$$

implies that $w \in \text{Ker } i_*$ since $w = \partial(\iota)$.

The commutative diagram

$$\begin{tikzcd}
\pi_{n+k-1}(S^n \vee S^k) \arrow[r, "i_*"] \arrow[dr, "pr_*^{(n)}"'] & \pi_{n+k-1}(S^n \times S^k) \arrow[d, "pr_*"] \\
 & \pi_{n+k-1}(S^n)
\end{tikzcd}$$

(where $pr : S^n \times S^k \longrightarrow S^n$ is a map collapsing S^k to the base point) implies that $w \in \text{Ker } pr_*^{(n)}$ and similarly $w \in \text{Ker } pr_*^{(n)}$. □

Now consider the suspension homomorphism

$$\Sigma : \pi_q(S^n \vee S^k) \longrightarrow \pi_{q+1}(\Sigma(S^n \vee S^k)).$$

Claim 10.2. *The element $w \in \pi_{n+k-1}(S^n \vee S^k)$ is in the kernel of the suspension homomorphism*

$$\Sigma : \pi_{n+k-1}(S^n \vee S^k) \longrightarrow \pi_{n+k}(\Sigma(S^n \vee S^k)).$$

Proof. Consider the commutative diagram:

$$\begin{tikzcd}
\pi_{n+k-1}(S^n) \arrow[d, "\Sigma"'] & \pi_{n+k-1}(S^n \vee S^k) \arrow[l, "pr_*^{(n)}"'] \arrow[r, "pr_*^{(k)}"] \arrow[d, "\Sigma"] & \pi_{n+k-1}(S^k) \arrow[d, "\Sigma"'] \\
\pi_{n+k}(S^{n+1}) & \pi_{n+k}(\Sigma(S^n \vee S^k)) \arrow[l, "\Sigma(pr_*^{(n)})"'] \arrow[r, "\Sigma(pr_*^{(k)})"] & \pi_{n+k}(S^{k+1})
\end{tikzcd} \tag{31}$$

where pr denote the collapsing maps. By Claim 10.9 $w \in \text{Ker } pr_*^{(n)}$, $w \in \text{Ker } pr_*^{(k)}$. Notice that $\Sigma(S^n \vee S^k) \sim S^{n+1} \vee S^{k+1}$. We need the following lemma.

Lemma 10.10. *There is an isomorphism*

$$\pi_{n+k}(S^{n+1} \vee S^{k+1}) \cong \pi_{n+k}(S^{n+1}) \oplus \pi_{n+k}(S^{k+1})$$

Proof. Consider the long exact sequence for the pair $(S^{n+1} \times S^{k+1}, S^{n+1} \vee S^{k+1})$:

$$\begin{aligned}
\pi_{n+k+1}(S^{n+1} \times S^{k+1}, S^{n+1} \vee S^{k+1}) &\xrightarrow{\partial} \pi_{n+k}(S^{n+1} \vee S^{k+1}) \xrightarrow{i_*} \pi_{n+k}(S^{n+1} \times S^{k+1}) \\
&\xrightarrow{j_*} \pi_{n+k}(S^{n+1} \times S^{k+1}, S^{n+1} \vee S^{k+1}) \longrightarrow
\end{aligned} \tag{32}$$

We notice that the $(n+k+1)$-skeleton of the product $S^{n+1}\times S^{k+1}$ is the wedge $S^{n+1}\vee S^{k+1}$. Thus any map $D^{k+n+1}\longrightarrow S^{n+1}\times S^{k+1}$ may be deformed to the subcomplex $S^{n+1}\vee S^{k+1}$. Thus $\pi_{n+k+1}(S^{n+1}\times S^{k+1}, S^{n+1}\vee S^{k+1})=0$. The same argument gives that

$$\pi_{n+k}(S^{n+1}\times S^{k+1}, S^{n+1}\vee S^{k+1})=0.$$

Thus the long exact sequence (32) gives the isomorphism:

$$i_*:\pi_{n+k}(S^{n+1}\vee S^{k+1})\xrightarrow{\cong}\pi_{n+k}(S^{n+1}\times S^{k+1})\cong\pi_{n+k}(S^{n+1})\oplus\pi_{n+k}(S^{k+1}). \qquad \square$$

To complete the proof of Claim 10.2 we notice that Lemma 10.10 and the diagram (31) imply that $w\in\mathrm{Ker}\,\Sigma$. $\square$

Claim 10.3. *Let $\alpha\in\pi_n(X)$, $\beta\in\pi_k(X)$. Then $[\alpha,\beta]\in Ker\ \Sigma$, where*

$$\Sigma:\pi_{n+k-1}(X)\longrightarrow\pi_{n+k}(\Sigma X)$$

is the suspension homomorphism.

Exercise 10.9. Prove Claim 10.3.

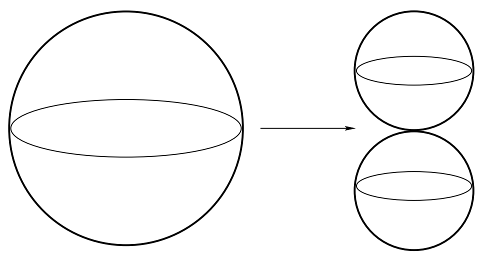


Fig. 10.7.

Now we want to study a particular case. Consider the map $\mu: S^{2q}\longrightarrow S^{2q}\vee S^{2q}$ which collapses the equator sphere, see Fig. 10.7. It induces the homomorphism

$$\mu_*:\pi_*(S^{2q})\longrightarrow\pi_*(S^{2q}\vee S^{2q}).$$

Let $\iota_{2q}\in\pi_{2q}(S^{2q})$ be the generator represented by the identity map $Id: S^{2q}\longrightarrow S^{2q}$. Let $\iota_{2q}^{(1)}\in\pi_{2q}(S^{2q}\vee S^{2q})$ be the image of the element ι_{2q} under the inclusion map $i^{(1)}: S^{2q}\longrightarrow S^{2q}\vee S^{2q}$ of the sphere S^{2q} to the first sphere in $S^{2q}\vee S^{2q}$. Let $\iota_{2q}^{(2)}$ be the corresponding element for the second sphere in $S^{2q}\vee S^{2q}$. Clearly $\mu_*(\iota_{2q})=\iota_{2q}^{(1)}+\iota_{2q}^{(2)}$.

Claim 10.4. *The Whitehead product $[\iota_{2q},\iota_{2q}]\in\pi_{4q-1}(S^{2q})$ is a nontrivial element of infinite order.*

Proof. The map $\mu: S^{2q}\vee S^{2q}\longrightarrow S^{2q}$ induces the homomorphism

$$\mu_*:\pi_{4q-1}(S^{2q})\longrightarrow\pi_{4q-1}(S^{2q}\vee S^{2q}).$$

By naturality we have that

$$\mu_*([\iota_{2q},\iota_{2q}])=[\mu_*(\iota_{2q}),\mu_*(\iota_{2q})]=[\iota_{2q}^{(1)}+\iota_{2q}^{(2)},\iota_{2q}^{(1)}+\iota_{2q}^{(2)}].$$

By additivity and commutativity (Claim 10.7 (2), (3)) we also have:

$$\begin{aligned}
\mu_*([\iota_{2q},\iota_{2q}]) &= [\iota_{2q}^{(1)}+\iota_{2q}^{(2)},\iota_{2q}^{(1)}+\iota_{2q}^{(2)}] \\
&= [\iota_{2q}^{(1)},\iota_{2q}^{(1)}]+[\iota_{2q}^{(1)},\iota_{2q}^{(2)}]+[\iota_{2q}^{(2)},\iota_{2q}^{(1)}]+[\iota_{2q}^{(2)},\iota_{2q}^{(2)}] \\
&= [\iota_{2q}^{(1)},\iota_{2q}^{(1)}]+[\iota_{2q}^{(1)},\iota_{2q}^{(2)}]+(-1)^{4q}[\iota_{2q}^{(1)},\iota_{2q}^{(2)}]+[\iota_{2q}^{(2)},\iota_{2q}^{(2)}] \\
&= [\iota_{2q}^{(1)},\iota_{2q}^{(1)}]+[\iota_{2q}^{(2)},\iota_{2q}^{(2)}]+2w_{2q}
\end{aligned}$$

where $w_{2q}=[\iota_{2q}^{(1)},\iota_{2q}^{(2)}]$. Notice that we used the fact that the sphere S^{2q} is even-dimensional. Now assume that the element $[\iota_{2q},\iota_{2q}]\in\pi_{4q-1}(S^{2q})$ has finite order. Then the elements $[\iota_{2q}^{(j)},\iota_{2q}^{(j)}]\in\pi_{4q-1}(S^{2q})$, $j=1,2$ also have finite order since $\iota_{2q}^{(j)}$ is the image of the generator ι_{2q} under the homomorphism $\pi_{2q}(S^{2q})\longrightarrow\pi_{2q}(S^{2q}\vee S^{2q})$. Then it follows that for some integer λ

$$0=\mu_*(\lambda[\iota_{2q},\iota_{2q}])=\lambda[\iota_{2q}^{(1)},\iota_{2q}^{(1)}]+\lambda[\iota_{2q}^{(2)},\iota_{2q}^{(2)}]+2\lambda w_{2q}=2\lambda w_{2q}.$$

This contradicts to Claim 10.8. Thus the element $[\iota_{2q},\iota_{2q}]\in\pi_{4q-1}(S^{2q})$ has infinite order. $\square$

We specify Claims 10.3, 10.4 to get so called "hard part" of the Suspension Theorem.

Theorem 10.11. **(1)** *The Whitehead product* $[\iota_{2q},\iota_{2q}]\in\pi_{4q-1}(S^{2q})$ *has infinite order.*

(2) *The Whitehead product* $[\iota_{2q},\iota_{2q}]\in\pi_{4q-1}(S^{2q})$ *is in the kernel of the suspension homomorphism, i.e.* $\Sigma([\iota_{2q},\iota_{2q}])=0$ *in* $\pi_{4q}(S^{2q+1})$.

Remark. Actually, $\pi_{4q-1}(S^{2q})\cong\mathbf{Z}\oplus\{\text{finite group}\}$ and these groups are the only homotopy groups of spheres (besides $\pi_n(S^n)$) which are infinite. We shall return to the Whitehead product to study the Hopf invariant.

Now we consider the product $S^n\times S^k$.

Corollary 10.12. *The suspension* $\Sigma(S^n\times S^k)$ *is homotopy equivalent to the wedge*

$$S^{n+1}\vee S^{k+1}\vee S^{n+k+1}.$$

Exercise 10.10. Prove Corollary 10.12.

11. Homotopy groups of CW-complexes

11.1. **Changing homotopy groups by attaching a cell.** Let X be a CW-complex, and $f : S^n \longrightarrow X$ be an attaching map for new cell. Let $Y = X \cup_f D^{n+1}$. We would like to understand how do the homotopy groups $\pi_* X$ and $\pi_* Y$ relate to each other.

Theorem 11.1. *Let X be a path-connected space (not necessarily a CW-complex) with a base point $x_0 \in X$, $f : S^n \longrightarrow X$ be a map such that $f(s_0) = x_0$, where s_0 is a base point of S^n. Let $Y = X \cup_f D^{n+1}$, and $i : X \longrightarrow Y$ be the inclusion. Then the induced homomorphism*

$$i_* : \pi_q(X, x_0) \longrightarrow \pi_q(Y, x_0) \tag{33}$$

- **(1)** *is an isomorphism if $q < n$,*
- **(2)** *is an epimorphism if $q = n$, and*
- **(3)** *the kernel Ker $i_* : \pi_n(X, x_0) \longrightarrow \pi_n(Y, x_0)$ is generated by $\gamma^{-1}[f]\gamma \in \pi_n(X, x_0)$ where $\gamma \in \pi_1(X, x_0)$.*

Proof. First we prove a technical result. Let E^m be either D^m or S^m. In both cases we choose $\mathbf{R}^m$ to be a subspace of E^m:

$$S^m = \mathbf{R}^m \cup \{x_0\},$$

$$D^m = \overset{o}{D^m} \cup S^{m-1}, \quad \overset{o}{D^m} \cong \mathbf{R}^m.$$

Lemma 11.2. *Let $h : E^m \longrightarrow Y$ be a map, such that $h|_{E^m \setminus \mathbf{R}^m}$ sends $E^m \setminus \mathbf{R}^m$ to X. Then there exists a map $h_1 : E^m \longrightarrow Y$ homotopic to h such that:*

- **(a)** $h_1|_{h^{-1}(X)} = h|_{h^{-1}(X)}$.
- **(b)** *If $m \leq n$ then $h_1(E^m) \subset X$.*
- **(c)** *If $m = n+1$ there exist disks $d_1, \ldots, d_r \subset E^m$ such that*
  - **(c1)** $h_1(E^m \setminus \bigcup_{s=1}^{r} d_s) \subset X$;
  - **(c2)** *the restriction $h_1|_{\overset{o}{d_s}} : \overset{o}{d_s} \longrightarrow D^m$ is a linear homeomorphism, $s = 1, \ldots, r$.*

Proof. The proof goes down the line of arguments which we used several times starting with Free Point Lemma. We give here the outline only.

- **(1)** Consider the disks (centered at the same point) $D_1^m \subset D_2^m \subset D_3^m \subset D_4^m \subset D_5^m \subset D^m$ of radii $i\rho/5$, $i = 1, \ldots, 5$.
- **(2)** The set $h^{-1}(D_5^m) \subset E^m$ is compact, furthermore, $h^{-1}(D_5^m) \subset \mathbf{R}^m \subset E^m$. Choose a simplex $\Delta^m \supset h^{-1}(D_5^m)$ and a triangulation $\{\Delta_\alpha\}$ of Δ such that if $h(\Delta_\alpha) \cap D_i^m \neq \emptyset$ then $h(\Delta_\alpha) \subset D_{i+1}^m$ for $i = 1, 2, 3, 4$, and $\operatorname{diam}(h(\Delta_\alpha)) < \rho/5$.

(3) Let $K = \bigcup_{h(\Delta_\alpha)\cap D_4^m\neq\emptyset} \Delta_\alpha$. Then we construct a map $h' : K \longrightarrow D^m$ by extending linearly h restricted on the vertices of K.

(4) We assume that the center y_0 of the disks D_i^m is not in the image of any face of the simplices Δ_α. If it happens to be in such image, we choose a homotopy which moves a point y_0 away from those images. Thus there is a small disk d_0^m centered at y_0 so that the points of d_0 are not in the image of any face of the simplices Δ_α as well.

(5) Now we use the same formula as before to construct a map $h'' : E^m \longrightarrow Y$ so that h'' coincides with h outside of K and h'' coincides with h' on $h^{-1}(D_2^m)$.

(6) We notice that for each simplex Δ_α of the above triangulation the disk d_0^m is either in the image of the interior of $h''(\Delta_\alpha)$ or $d_0^m \cap h'(\Delta_\alpha) = \emptyset$. If the latter holds for all simplices Δ_α, then the map h is homotopic to a map $h_1 : E^m \longrightarrow Y$ so that $h_1(E^m) \subset X$ since we can just blow off the map h'' out of the free point y_0.

(7) Notice that if $d_0^m \cap h''(\Delta_\alpha) \neq \emptyset$, then $(h'')^{-1}(d_0^m)$ is an ellipsoid since h'' restricted on the simplex Δ_α is linear. Thus the inverse image of the disk d_0^m is a finite number of ellipsoids $d_1^m, \ldots, d_r^m \subset \mathbf{R}^m$.

(8) Now we stretch the disk d_0^m up to the disk D^m: it gives a a map h_1' (homotopic to h'') which sends each ellipsoid d_j^m linearly to the disk D^m.

Lemma 11.2 is proved. □

Conclusion of proof of Theorem 11.1. Lemma 11.2 implies that the homomorphism (33) is epimorphism if $q < n$. (Notice that the surjectivity of i_* for $q = n$ follows directly from the cell-aproximation arguments.)

Now let $g : S^n \longrightarrow X$ be a map representing an element of Ker i_*, where

$$i_* : \pi_n(X, x_0) \longrightarrow \pi_n(Y, x_0),$$

i.e. g extends to a map $h : D^{n+1} \longrightarrow Y$. We apply Lemma 11.2 to the map h to construct a map $h_1 : D^{n+1} \longrightarrow Y$ such that $h_1|_{S^n} = g = h|_{S^n}$, and that the map h_1 restricted to the boundary of each disk d_j^{n+1} coincides with the composition

$$\partial(d_j^{n+1}) \xrightarrow{\ell_j} S^n \xrightarrow{f} X$$

where ℓ_j is a linear map. Now we can use the argument simillar to the one we used to prove Theorem 6.5. We choose a path γ_j connecting the base point s_0 with some point $s_j \in \partial(d_j)$ in the same way as we did in Theorem 6.5, see Fig 11.1.

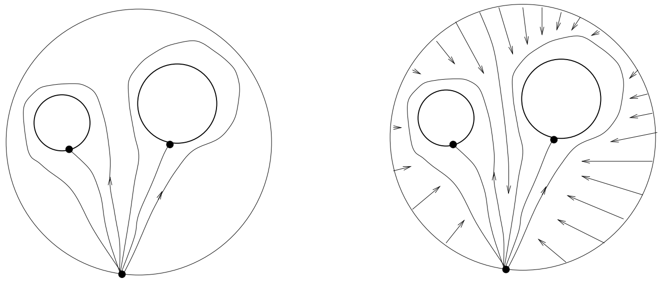


Fig. 11.1.

The rest of the proof is left to a reader. $\square$

Corollary 11.3. *Let $A \subset X$ be a CW-pair, such that $X \setminus A$ does not contain cells of dimension $\leq n$. Then the homomorphism $i_* : \pi_q(A) \longrightarrow \pi_q(X)$ is isomorphism if $q < n$ and is epimorphism if $q = n$. In particular $\pi_n(X^{(n+1)}) \cong \pi_n(X)$, where $X^{(n+1)}$ is the $(n+1)$th skeleton of X.*

11.2. **Homotopy groups of a wedge.**

Theorem 11.4. *Let X be an n-connected CW-complex, and Y be a k-connected CW-complex. Then*

- **(1)** $\pi_q(X \vee Y) \cong \pi_q(X) \oplus \pi_q(Y)$ *for* $q \leq n+k$;
- **(2)** *for each $q \geq 1$ the group $\pi_q(X \vee Y)$ contains a direct summand isomorphic to $\pi_q(X) \oplus \pi_q(Y)$.*

Proof. By Theorem 5.11 the CW-complexes X and Y are homotopy equivalent to CW-complexes without cells in dimensions in between 0 and $n+1$ (for X) and in between 0 and $k+1$ (for Y). Thus we may assume that X and Y are such complexes. Consider the product $X \times Y$ with the product cell-structure. The wedge $X \vee Y$ is a subcomplex of $X \times Y$. Furthermore the difference $X \times Y \setminus X \vee Y$ has cells of dimension at least $n+k+2$. By Corollary 11.3,

$$\pi_q(X \vee Y) \cong \pi_q(X \times Y) \cong \pi_q(X) \oplus \pi_q(Y).$$

To prove the second statement we notice that the composition

$$\pi_q(X) \oplus \pi_q(Y) \xrightarrow{i^X_* \oplus i^Y_*} \pi_q(X \vee Y) \longrightarrow \pi_q(X \times Y) \xrightarrow{\cong} \pi_q(X) \oplus \pi_q(Y),$$

(where $i^X : X \longrightarrow X \vee Y$, $i^Y : Y \longrightarrow X \vee Y$ are the caninical embeddings) is the identity homomorphism. $\square$

Corollary 11.5. *There is an isomorphism $\pi_n(S^n \vee \cdots \vee S^n) \cong \mathbf{Z} \oplus \cdots \oplus \mathbf{Z}$ with generators induced by the embeddings $S^n \longrightarrow S^n \vee \cdots \vee S^n$.*

Exercise 11.1. Let X be an n-connected CW-complex, and Y be a k-connected CW-complex. Prove the isomorphism:

$$\pi_{n+k+1}(X \vee Y) \cong \pi_{n+k+1}(X) \oplus \pi_{n+k+1}(Y) \oplus [\pi_{n+1}(X), \pi_{k+1}(Y)].$$

In particular it follows that $\pi_3(S^2\vee S^2)\cong \mathbf{Z}\oplus\mathbf{Z}\oplus\mathbf{Z}$.

11.3. **The first nontrivial homotopy group of a CW-complex.** Let X be $(n-1)$-connected CW-complex. We know very well that the homotopy groups $\pi_q(X)=0$ if $q\leq n-1$. Our goal is to describe the group $\pi_n(X)$. Again we can assume that X does not contain cells of dimension in between 0 and n. Then the n-skeleton $X^{(n)}$ is a wedge of spheres: $X^{(n)}=S^n_1\vee\cdots\vee S^n_s$. Let $g_i : S^n\longrightarrow S^n_1\vee\cdots\vee S^n_s$ be the embedding of the i-th sphere, and let $r_j : S^n\longrightarrow S^n_1\vee\cdots\vee S^n_s$ be the attaching maps of the $n+1$ cells $e^{n+1}_1,\ldots,e^{n+1}_\beta$. The maps g_i determine the generators of the group $\pi_n(X^{(n)})$, and let $\rho_j\in\pi_n(X^{(n)})$ be the elements determined by the maps r_j. The following theorem is a straightforward corollary of Theorem 11.1.

Theorem 11.6. *The homotopy group $\pi_n(X)$ is isomorphic to the factor group of the homotopy group $\pi_n(X^{(n)})\cong\mathbf{Z}\oplus\cdots\oplus\mathbf{Z}$ by the subgroup generated by ρ_j, $j=1,\ldots,\beta$.*

Remark. Theorem 11.6 is analogous to Theorem 6.5 about the fundamental group. This result gives an impression that we can calculate the first nontrivial homotopy group of any CW-complex without any problems. However, we do not offer here an efficient algorithm to do this calculation. The difficulty shows up when we start with any CW-complex X and construct new CW-complex X' homotopy equivalent to X and without cells in dimensions $\leq n-1$. The process we described in Theorem 5.11 is not really algoriphmic. Thus Theorem 11.6 should not be considered as a computational tool, but rather as a "theoretical device" which allows to prove general facts about homotopy groups.

Exercise 11.2. Let (X,A) be a CW-pair with connected subcomplex A, and such that $X\setminus A$ contains cells of dimension $\geq n$, where $n\geq 3$. Let $\pi=\pi_1(A)$ acting on $\pi_n(X,A)$ by changing a base point. This action gives $\pi_n(X,A)$ a structure of $\mathbf{Z}[\pi]$-module. Prove that the $\mathbf{Z}[\pi]$-module $\pi_n(X,A)$ is generated by the n-cells of $X\setminus A$ with relations corresponding the $(n+1)$-cells of $X\setminus A$.

Exercise 11.3. Let (X,A) be a CW-pair with simply connected subcomplex A, and such that $X\setminus A$ contains cells of dimension $\geq n\geq 2$. Prove that the natural map $j : (X,A)\longrightarrow (X/A,*)$ induces isomomorphism $j_* : \pi_n(X,A)\longrightarrow \pi_n(X/A)$.

11.4. **Weak homotopy equivalence.** Recall that spaces X and Y are weak homotopy equivalent if there is a natural bijection $\varphi_Z : [Z,X]\longrightarrow [Z,Y]$ for any CW-complex Z (natural with respect to maps $Z\longrightarrow Z'$. We have seen that the fibers of a Serre fiber bundle are weak homotopy equivalent. The definition of weak homotopy equivalence does not offer any hint how to construct the bijection φ_Z. The best possible case is when the bijection φ_Z is induced by a map $f : X\longrightarrow Y$.

A map $f : X\longrightarrow Y$ is a *weak homotopy equivalence* if for any CW-complex Z the induced map $f_* : [Z,X]\longrightarrow [Z,Y]$ is a bijection.

Remark. Clearly if $f : X \longrightarrow Y$ is a weak homotopy equivalence, then X is weak homotopy equivalent to Y. The opposite statement fails. Indeed, let $X = \mathbf{Z} \subset \mathbf{R}$, and $Y = \mathbf{Q} \subset \mathbf{R}$ with induced topology. It is easy to check that $\mathbf{Z} \overset{w}{\sim} \mathbf{Q}$, however there is no continiuos bijection $f : \mathbf{Q} \longrightarrow \mathbf{Z}$. Thus there is no bijection $[pt, \mathbf{Q}] \longrightarrow [pt, \mathbf{Z}]$ induced by any continuous map f. However, if any two (reasonably good spaces, like Hausdorff) X, Y are weak homotopy equivalent, then we will prove soon that there exist a CW-complex W and weak homotopy equivalences $f : W \longrightarrow X$, $g : W \longrightarrow Y$. Also we are about to prove that weak homotopy equivalence coincides with homotopy equivalence on the category of CW-complexes.

Theorem 11.7. *Let $f : X \longrightarrow Y$ be a continuous map. Then the following statements are equivalent.*

(1) *The map $f : X \longrightarrow Y$ is weak homotopy equivalence.*
(2) *The induced homomorphism $f_* : \pi_n(X, x_0) \longrightarrow \pi_n(Y, f(x_0))$ is isomorphism for all n and $x_0 \in X$.*
(3) *Let (W, A) be a CW-pair, and $h : A \longrightarrow X$, $g : W \longrightarrow Y$ be such maps that the following diagram commutes up to homotopy*

$$\begin{tikzcd} X \arrow[rr, "f"] & & Y \\ A \arrow[u, "h"] \arrow[rr, "i"] \arrow[urr, "f\circ h"] & & W \arrow[u, "g"'] \end{tikzcd} \tag{34}$$

i.e. $f \circ h \sim g|_A = i \circ g$. Then there exists a map $\tilde{h} : W \longrightarrow X$ such that $\tilde{h}|_A = \tilde{h} \circ i = h$ and $f \circ \tilde{h} \sim g$ in the diagram

$$\begin{tikzcd} X \arrow[rr, "f"] & & Y \\ A \arrow[u, "h"] \arrow[rr, "i"] & & W \arrow[u, "g"'] \arrow[ull, "\tilde{h}"'] \end{tikzcd} \tag{35}$$

Proof. The implication **(1)** $\Longrightarrow$ **(2)** is obvious.

(3) $\Longrightarrow$ **(1)**. Let $(W, A) = (Z, \emptyset)$. Then we have that for any map $g : Z \longrightarrow Y$ there exists a map $\tilde{h} : Z \longrightarrow X$ so that the triangle

$$\begin{tikzcd} X \arrow[rr, "f"] & & Y \\ & & Z \arrow[u, "g"'] \arrow[ull, "\tilde{h}"'] \end{tikzcd} \tag{36}$$

commutes up to homotopy. It implies that the map $f_*:[Z,X]\longrightarrow[Z,Y]$ is epimorphism. To prove that f_* is injective, consider the pair $(W,A)=(Z\times I, Z\times\{0\}\cup Z\times\{1\})$. Let $h_0:Z\longrightarrow X$, $h_1:Z\longrightarrow X$ be two maps so that the compositions $h_0\circ f:Z\longrightarrow Y$, $h_1\circ f:Z\longrightarrow Y$ are homotopic. Let $G:Z\times I\longrightarrow Y$ be a homotopy between the maps $h_0\circ f$ and $h_1\circ f$. The statement (3) implies that there exists a homotopy $\tilde{H}:Z\times I\longrightarrow X$ so that the diagram

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ Z\times\{0\}\cup Z\times\{1\} \arrow[u, "h_0\cup h_1"] \arrow[r, "i"] & Z\times I \arrow[u, "G"'] \arrow[ul, "\tilde{H}"'] \end{tikzcd} \tag{37}$$

commutes up to homotopy. In particular, it means that the maps $h_0,h_1:Z\longrightarrow X$ were homotopic in the first place.

(2) $\Longrightarrow$ (3). Let $f:X\longrightarrow Y$ satisfy (2). We assume that $W=A\cup_\alpha D^{n+1}$, where $\alpha:S^n\longrightarrow A$ is the attaching map. Let $h:A\longrightarrow X$, $g:A\cup_\alpha D^{n+1}\longrightarrow Y$ be such maps that $f\circ h\sim g|_A$. Consider the diagram:

$$\begin{tikzcd} & X \arrow[r, "f"] & Y \\ S^n \arrow[r, "\alpha"] & A \arrow[u, "h"] \arrow[r, "i"] & A\cup_\alpha D^{n+1} \arrow[u, "g"'] \end{tikzcd} \tag{38}$$

The composition $i\circ\alpha:S^n\longrightarrow A\cup_\alpha D^{n+1}$ is null-homotopic by construction, hence the map $g\circ i\circ\alpha:S^n\longrightarrow Y$ is null-homotopic as well. Thus $[g\circ i\circ\alpha]=0$ in the group $\pi_n(Y)$. Notice that the map $f\circ h\circ\alpha:S^n\longrightarrow Y$ gives the same homotopy class as the map $g\circ i\circ\alpha$ since the diagram (38) is commutative up to homotopy by conditions of the statement (3). In particular, we have that $f_*([h\circ\alpha])=[g\circ i\circ\alpha]=0$. Hence $[h\circ\alpha]=0$ in the group $\pi_n(X)$ since $f_*:\pi_n(X)\longrightarrow\pi_n(Y)$ is isomorphism. It implies that there exists a map $\beta:D^{n+1}\longrightarrow X$ extending the map $h\circ\alpha:S^n\longrightarrow X$. We have the following diagram:

$$\begin{tikzcd} D^{n+1} \arrow[r, "\beta"] & X \arrow[r, "f"] & Y \\ S^n \arrow[u] \arrow[r, "\alpha"] & A \arrow[u, "h"] \arrow[r, "i"] & A\cup_\alpha D^{n+1} \arrow[u, "g"'] \end{tikzcd} \tag{39}$$

where the left square is commutative, and the right one is commutative up to homotopy. The left square gives us a map $\tilde{h}':A\cup_\alpha D^{n+1}\longrightarrow X$, so that $f\circ\tilde{h}'|_A=f\circ h\sim g|_A$. We choose a homotopy $H:A\times I\longrightarrow Y$ so that

$$H|_{A\times\{0\}}=g|_A,\quad H|_{A\times\{1\}}=f\circ\tilde{h}'|_A=f\circ h.$$

Consider the cylinder $D^{n+1} \times I$ and its boundary $S^{n+1} = \partial(D^{n+1} \times I)$. No we construct a map $\gamma : S^{n+1} \longrightarrow Y$ as it is shown below, see Fig. 11.2.

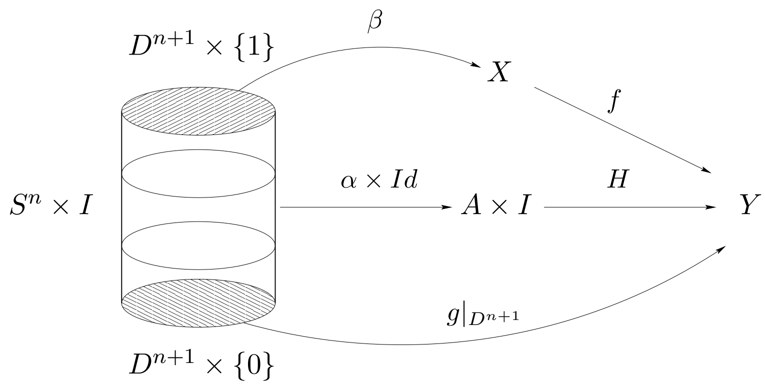


Fig. 11.2.

If the map $\gamma : S^{n+1} \longrightarrow Y$ is homotopic to zero, then we are done since we can extend γ to the interior of the cylinder $D^{n+1} \times I$, and it will give us a homotopy between $f \circ \tilde{h}'$ and g. However, there is no any reason to assume that $\gamma \sim 0$. To correct the construction we make the following observation.

Lemma 11.8. *Let $\xi \in \pi_q(Y, y_0)$ be any element, and $\beta : D^q \longrightarrow Y$, such that $\beta(s_0) = y_0$, where $x_0 \in S^{q-1} = \partial(D^q)$. Then there exists a map $\beta' : D^q \longrightarrow Y$ such that*

(a) $\beta'|_{S^{q-1}} = \beta|_{S^{q-1}}$*;*
(b) *the map $\beta \cup \beta' : S^q \longrightarrow Y$ represents the element $\xi \in \pi_q(Y, y_0)$.*

Proof. Let $\varphi : S^q \longrightarrow Y$ be any map representing the element $\xi \in \pi_q(Y, y_0)$. We consider the sphere $S^q_1 = D^q_N \cup_{S^{q-1}} D^q_S$. Let $p : S^q_1 \longrightarrow S^q$ be a map which takes the southern hemisphere D^q_S to the base point $s_0 \in S^q$. Clearly the composition

$$S^q_1 \xrightarrow{p} S^q \xrightarrow{\varphi} Y$$

represents the same element $\xi \in \pi_q(Y, y_0)$. Fig. 11.3 below is supposed to hint how to construct new map $\varphi' : S^q_1 \longrightarrow Y$ so that $\varphi' = \beta \cup \beta'$ represents the element $\xi \in \pi_q(Y, y_0)$. The details are left to you. $\square$

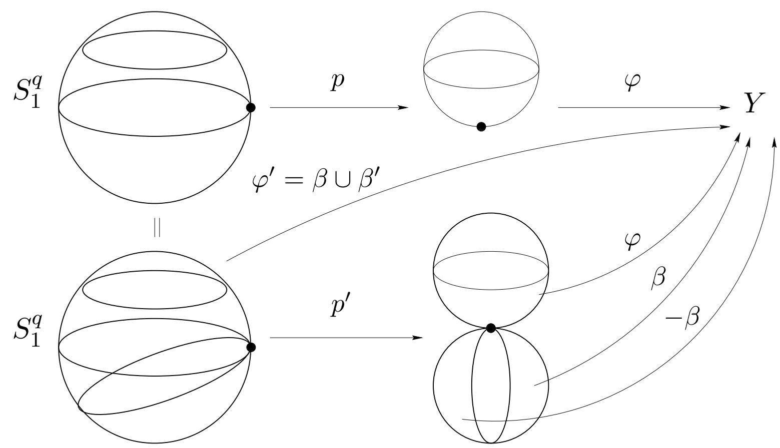


Fig. 11.3.

Now we complete a proof of the implication **(2)** $\Longrightarrow$ **(3)**. The above map $\gamma : S^{n+1} \longrightarrow Y$ gives an element $\gamma \in \pi_{n+1}(Y)$. Then we consider the element $-\gamma \in \pi_{n+1}(Y)$ and use Lemma 11.8 to find a map $\beta' : D^{n+1} \longrightarrow Y$, such that $\beta'|_{S^n} = \beta|_{S^n}$ and the map $\beta' \cup \beta$ represents the element $-\gamma$. We put together the maps we constructed to get new map $\gamma' : S^{n+1} \longrightarrow Y$ which homotopic to zero, see Fig. 11.4. Since $\gamma' \sim 0$ we are done in the case when $W = A \cup_\alpha D^{n+1}$. The general case follows then by induction: the n-th step is to do the above construction for all $(n+1)$-cells of the difference $W \setminus A$.

□

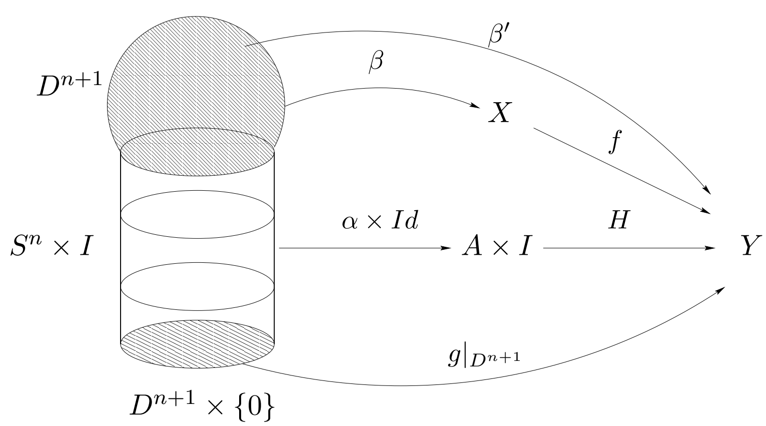


Fig. 11.4. The map $\gamma' : S^{n+1} \longrightarrow Y$.

Corollary 11.9. (Whitehead Theorem) *Let X, Y be CW-complexes. Then if a map $f : X \longrightarrow Y$ induces isomorphism*

$$f_* : \pi_n(X, x_0) \longrightarrow \pi_n(Y, f(x_0))$$

for all $n \geq 0$ and $x_0 \in X$, then f is a homotopy equivalence.

Exercise 11.4. Prove Corollary 11.9.

Exercise 11.5. Prove that the homotopy groups of the spaces $S^3 \times \mathbf{CP}^\infty$ and S^2 are isomorphic, and that they are not homotopy equivalent.

Exercise 11.6. Let $k \neq n$. Prove that the homotopy groups of the spaces $\mathbf{RP}^n \times S^k$ and $S^k \times \mathbf{RP}^k$ are isomorphic, and that they are not homotopy equivalent.

11.5. **Cellular approximation of topological spaces.** Let X be an arbitrary Hausdorff space. There is a natural question: is there a natural cellular approximation of the space X? This is the answer:

Theorem 11.10. *Let X be a Hausdorff topological space. There exists a CW-complex K and a weak homotopy equivalence $f : K \longrightarrow X$. The CW-complex K is unique up to homotopy equivalence.*

Proof. We assume that X is a path-connected space. We construct a chain of CW-complexes

$$K_0 \subset K_1 \subset K_2 \subset \cdots \subset K_{n-1} \subset K_n \subset \cdots$$

and maps $f_j : K_j \longrightarrow X$ so that

(1) $f_j|_{K_{j-1}} = f_{j-1} : K_{j-1} \longrightarrow X$,
(2) $(f_j)_* : \pi_q(K_j) \longrightarrow \pi_q(X)$ is an isomorphism for all $q \leq j$.

Let $K_0 = \{x_0\}$, and $f_0 : K_0 \longrightarrow X$ be a choice of a base point. Assume that we have constructed the maps $f_j : K_j \longrightarrow X$ for all $j \leq n-1$ satisfying the above conditions. Let $\pi = \pi_1(X, x_0)$. We consider the group $\pi_n(X, x_0)$ and choose generators g_α of $\pi_n(X, x_0)$ as a $\mathbf{Z}[\pi]$-module and representing maps $g_\alpha : S^n_\alpha \longrightarrow X$. Let

$$K_n' = K_{n-1} \vee \left(\bigvee_\alpha S^n_\alpha \right)$$

We define $f_n' : K_n' \longrightarrow X$ to be f_{n-1} on K_{n-1} and to be $\bigvee_\alpha g_\alpha$ on $\bigvee_\alpha S^n_\alpha$. The induction hypothesis and Theorem 11.1 implies that f_n' induces isomorphism

$$\pi_q(K_n') \xrightarrow{(f_n')_*} \pi_q(X)$$

for $q \leq n-1$. The homomorphism $(f_n')_* : \pi_n(K_n') \longrightarrow \pi_n(X)$ is epimorphism since all generators g_α are in the image. However it may not monomorphic. We choose generators h_β of the kernel $\mathrm{Ker}\ (f_n')_* \subset \pi_n(K_n')$ (which is also a $\mathbf{Z}[\pi]$-module) and representatives $h_\beta : S^n_\beta \longrightarrow K_n'$. Now we attach the cells e^{n+1}_β using the maps h_β as attaching maps. Let K_n be the resulting CW-complex. The map $f_n' : K_n' \longrightarrow X$ may be extended to $f_n : K_n \longrightarrow X$ since each composition

$$S^n_\beta \xrightarrow{h_\beta} K_n' \xrightarrow{f_n'} X$$

is homotopic to zero. Thus $f'_n : K'_n \longrightarrow X$ may be extended to all cells e^{n+1}_β we attached. Theorem 11.1 implies that $(f_n)_* : \pi_q(K_n) \longrightarrow \pi_q(X)$ is an isomorphism for $q \leq n-1$ and also that $\pi_n(K_n) \cong \pi_n(K'_n)/\mathrm{Ker}\ (f'_n)_* \cong \pi_n(X)$. Thus $(f_n)_* : \pi_n(K_n) \longrightarrow \pi_n(X)$ is an isomorphism as well.

Exercise 11.7. Prove that the CW-complex K we constructed is unique up to homotopy.

This conclude the proof of Theorem 11.10. □

Exercise 11.8. Let X, Y be two weak homotopy equivalent spaces. Prove that there exist a CW-complex K and maps $f : K \longrightarrow X$, $g : K \longrightarrow Y$ which both are weak homotopy equivalences.

11.6. **Eilenberg-McLane spaces.** Let n be a positive integer and π be a group (abelian) if $n \geq 2$. A space X is called an *Eilenberg-McLane space of the type* $K(\pi, n)$ if

$$\pi_q(X) = \begin{cases} \pi & \text{if } q = n \\ 0 & \text{else.} \end{cases}$$

Theorem 11.11. *Let n be a positive integer and π be a group (abelian) if $n \geq 2$. Then the Eilenberg-McLane space of the type $K(\pi, n)$ exists and unique up to weak homotopy equivalence.*

Remark. If a space X is an Eilenberg-McLane space of the type $K(\pi, n)$, we will say that X is $K(\pi, n)$.

Proof of Theorem 11.11. Let $\{g_\alpha\}$ be generators of the group π, and $\{r_\beta\}$ be relations (if $n > 1$ we mean relations in the abelian group). Let $X_n = \bigvee_\alpha S^n_\alpha$. Then $\pi_q(X_n) = 0$ if $q \leq n-1$ and $\pi_n(X_n) = \bigoplus_\alpha \mathbf{Z}$ (or free group with generators $\{g_\alpha\}$ if $n = 1$). Each relation r_β defines a unique element $r_\beta \in \pi_n(X_n)$. We choose maps $r_\beta : S^n_\beta \longrightarrow X_n$ representing the above relations and attach cells e^{n+1}_β using r_β as the attaching maps. Let X_{n+1} be the resulting space. Theorem 11.1 implies that $\pi_q(X_{n+1}) = 0$ if $q \leq n-1$ and $\pi_n(X_{n+1}) = \pi$. Then we choose generators of $\pi_{n+1}(X_{n+1})$ and attach $(n+2)$-cells using maps representing these generators as the attaching maps. Let X_{n+2} be the resulting CW-complex. Again Theorem 11.1 implies that $\pi_q(X_{n+2}) = 0$ if $q \leq n-1$ or $q = n+1$ and $\pi_n(X_{n+2}) = \pi$. Now we proceed by induction killing the homotopy group $\pi_{n+2}(X_{n+2})$ and so on.

Exercise 11.9. Prove that an Eilenberg-McLane space of the type $K(\pi, n)$ is unique up to weak homotopy equivalence, i.e. if K_1, K_2 are two Eilenberg-McLane spaces of the type $K(\pi, n)$ then there exist weak homotopy equivalences $f_1 : X \longrightarrow K_1$ and $f_2 : X \longrightarrow K_2$, where X is the space we just constructed.

This concludes the proof of Theorem 11.11 □

Remark. The above construction is not algorithmic at all: we have no idea what groups $\pi_{n+k}(X_{n+k})$ we are going to get in this process.

Examples. **(1)** $K(\mathbf{Z},1)=S^1$.

(2) $K(\mathbf{Z}/2,1)=\mathbf{RP}^\infty$.

(3) $K(\mathbf{Z},2)=\mathbf{CP}^\infty$.

(4) Let $L^{2n-1}(\mathbf{Z}/m)$ be the lens space we defined at the end of Section 7, and let

$$L^\infty(\mathbf{Z}/m)=\lim_{n\longrightarrow\infty} L^{2n-1}(\mathbf{Z}/m).$$

Then $L^\infty(\mathbf{Z}/m)=K(\mathbf{Z}/m,1)$.

Exercise 11.10. Construct the space $K(\pi,1)$, where π is a finitely generated abelian group.

Exercise 11.11. Let $X=K(\pi,n)$. Prove that $\Omega X=K(\pi,n-1)$.

11.7. **Killing the homotopy groups.** There are two constructions we discuss here. The first one we used implicitly several times. Let X be a space, then for each n there is a space X_n and a map $f_n : X \longrightarrow X_n$, such that

(1) $\pi_q(X_n)=\begin{cases}\pi_q(X) & \text{if } q\leq n\\ 0 & \text{else}\end{cases}$

(2) $(f_n)_* : \pi_q(X)\longrightarrow \pi_q(X_n)$ is isomorphism if $q\leq n$.

We know how to construct X_n: start with generators $\{g_\alpha\}$ of the group $\pi_{n+1}(X)$, then attach the cells e^{n+2}_α using the maps $g_\alpha : S^{n+1}_\alpha \longrightarrow X$. Then the resulting space Y_{n+1} has the homotopy groups $\pi_{n+1}(Y_{n+1})=0$ and $\pi_q(Y_{n+1})=\pi_q(X)$ if $q\leq n$. Then one kills in the same way the homotopy group $\pi_{n+2}(Y_{n+1})$ to construct the space Y_{n+2} with $\pi_{n+1}(Y_{n+2})=0$, $\pi_{n+2}(Y_{n+2})=0$, and $\pi_q(Y_{n+2})=\pi_q(X)$ if $q\leq n$, and so on. The limiting space is X_n with the above properties. The map $f_n : X\longrightarrow X_n$ is the embedding.

Let X be $(n-1)$-connected. Then $X_n = K(\pi_n(X), n)$. This construction may be organized so that there is a commutative diagram

(40)
$$
\begin{tikzcd}
 & \vdots \arrow[d, "i_{n+k+1}"'] \\
 & X_{n+k} \arrow[d, "i_{n+k}"'] \\
 & \vdots \arrow[d, "i_{n+2}"'] \\
X \arrow[uur, "f_{n+k}"] \arrow[r, "f_{n+1}"] \arrow[dr, "f_n"] & X_{n+1} \arrow[d, "i_{n+1}"'] \\
 & K(\pi_n, n)
\end{tikzcd}
$$

where $\pi_q = \pi_q(X)$. The maps $i_q : X_q \longrightarrow X_{q-1}$ in the diagram (40) are homotopy equivalent to Serre fiber bundles, so that the diagram (40) becomes commutative up to homotopy. Let F_q be the fiber of the Serre bundle $X_q \xrightarrow{i_q} X_{q-1}$. The exact sequence in homotopy

$$\cdots \longrightarrow \pi_j(F_q) \longrightarrow \pi_j(X_q) \longrightarrow \pi_j(X_{q-1}) \longrightarrow \pi_{j-1}(F_q) \longrightarrow \cdots$$

for the Serre bundle $X_q \xrightarrow{i_q} X_{q-1}$ immediately implies that $F_q = K(\pi_q, q) = \Omega K(\pi_q, q+1)$.

Consider for a moment the Eileberg-McLane space $K(\pi, q+1)$. We have a canonical Serre fiber bundle $\pi : \mathcal{E}(K(\pi, q+1)) \to K(\pi, q+1)$. It is easy to identify the fiber $\Omega K(\pi, q+1)$ with the space $K(\pi, q)$ (up to weak homotopy equivalence).

Here there is an important fact which we state without a proof:

Claim 11.1. *Let $p : E \to B$ be a Serre fiber bundle with a fiber $F = K(\pi, q)$. Then there exists a map $k : B \to K(\pi, q+1)$, such that the following diagram commutes up to homotopy:*

$$
\begin{tikzcd}
E \arrow[r, "\hat{k}"] \arrow[d, "\Omega K(\pi,q+1)"', "p"] & E(K(\pi, q+1)) \arrow[d, "\Omega K(\pi,q+1)"', "\pi"] \\
B \arrow[r, "k"] & K(\pi, q+1)
\end{tikzcd}
$$

where we identify $\Omega K(\pi, q+1)$ with $K(\pi, q)$.

In particular, we obtain the following commutative diagram:

$$\begin{tikzcd}
X_q \arrow[r, "\hat{k}_q"] \arrow[d, "\Omega K(\pi_q, q+1)"', "i_q"] & E(K(\pi_q, q+1)) \arrow[d, "\Omega K(\pi_q, q+1)"', "p"] \\
X_{q-1} \arrow[r, "k_q"] & K(\pi_q, q+1)
\end{tikzcd} \tag{41}$$

Here the maps $k_q : X_{q-1} \longrightarrow K(\pi_{q+1}, q+2)$ are known as the *Postnikov invariants* of the space X. In fact, the maps $k_q : X_{q-1} \longrightarrow K(\pi_{q+1}, q+2)$ are defined up to homotopy and determine the elements in cohomology

$$k_q \in H^{q+2}(X; \pi_{q+1}), \quad q \geq n.$$

The diagram

$$\begin{tikzcd}
 & \vdots \arrow[d, "i_{n+k+1}"'] & \\
 & X_{n+k} \arrow[r, "k_{n+k+1}"] \arrow[d, "i_{n+k}"', "K(\pi_{n+k}, n+k)"] & K(\pi_{n+k}, n+k+1) \\
 & \vdots \arrow[d, "i_{n+2}"', "K(\pi_{n+2}, n+2)"] & \\
X \arrow[uuur, "f_{n+k}"] \arrow[r, "f_{n+1}"] \arrow[dr, "f_n"'] & X_{n+1} \arrow[r, "k_{n+2}"] \arrow[d, "i_{n+1}"', "K(\pi_{n+1}, n+1)"] & K(\pi_{n+2}, n+3) \\
 & K(\pi_n, n) \arrow[r, "k_{n+1}"] & K(\pi_{n+1}, n+2)
\end{tikzcd} \tag{42}$$

is called the *Postnikov tower* of the space X. The Postnikov tower exists and unique up to homotopy under some restrictions on X. For instance, it exists when X is a simply-connected CW-complex. The existence of diagram (42) shows that the Eilenberg-McLane spaces are the "elmentary building blocks" for any simply connected space X. The Postnikov tower also shows that there are many spaces with the same homotopy groups, while these spaces are not homotopy equivalent. Again, this construction does not provide an algorithm to compute the homotopy groups, however it leads to some computational procedure called the Adams spectral sequence. We are not ready even to discuss this, and we shall return to the above constructions later on.

There is the second way to kill homotopy groups. Let X be $(n-1)$-connected as above. The map $f_n : X \longrightarrow X_n = K(\pi_n, n)$ may be turned into Serre fiber bundle. Let $X|_n$ be its fiber, and $j_n : X|_n \longrightarrow X$ be the inclusion map. The exact sequence in homotopy for the fiber bundle $X \xrightarrow{X|_n} K(\pi_n, n)$ implies that the map $j_n : X|_n \longrightarrow X$ induces isomorphism $\pi_q(X|_n) \cong \pi_q(X)$ if $q \geq n+1$, and also that $\pi_q(X|_n) = 0$ if $q \leq n$. One can iterate this construction to build the space $X|_{n+k}$ and the map $j_{n+k} : X|_{n+k} \longrightarrow X$ so that the induced homomorphism $\pi_q(X|_{n+k}) \longrightarrow \pi_q(X)$ is isomorphism if $q \geq n+k$ and $\pi_q(X|_{n+k}) = 0$ if $q \leq n+k-1$.

Exercise 11.13. Let $X = S^2$. Prove that $X|_3 = S^3$.

Exercise 11.14. Let $X = \mathbf{CP}^n$. Prove that $X|_3 = X|_{2n+1} = S^{2n+1}$.

12. Homology groups: basic constructions

The homotopy groups $\pi_q(X)$ are very important invariants. They are defined in the most natural way, and capture an important information about topological spaces. However it is very difficult to compute the homotopy groups, as we have seen. There are just few finite CW-complexes for which all homotopy groups are known. Even for the sphere S^n the problem to compute the homotopy groups is far from to be solved. Here we define different invariants: homology groups $H_n(X)$ and cohomology groups $H^n(X)$. These groups are much easier to compute: we will be able to compute the homology groups for all basic examples. However, their definition requires more work.

12.1. **Singular homology.** We alredy defined the *standard q-simplex*:

$$\Delta^q = \left\{ (t_0, \ldots, t_q) \mid t_0 \geq 0, \ldots, t_q \geq 0, \ \sum_{i=0}^{q} t_i = 1 \right\} \subset \mathbf{R}^{q+1}.$$

Remark. Note that the standard simplex Δ^q has vertices $A_0 = (1, 0, \ldots, 0)$, $A_1 = (0, 1, 0, \ldots, 0), \ldots, A_q = (0, 0, \ldots, 0, 1)$ in the space $\mathbf{R}^{q+1}$. In particular it defines the *orientation* of Δ^q. The simplex Δ^q has the i-th face $(i = 0, \ldots, q)$

$$\Delta^{q-1}(i) = \{ (t_0, \ldots, t_q) \mid t_i = 0 \ \}$$

which is a standard $(q-1)$-simplex in the space

$$\mathbf{R}^q(i) = \{ (t_0, \ldots, t_q) \mid t_i = 0 \ \} \subset \mathbf{R}^{q+1}$$

with the *induced orientation.*

A *singular q-simplex of the space X* is a continuous map $f : \Delta^q \longrightarrow X$. A *singular q-chain* is a finite linear combination $\sum k_i f_i$, where each $f_i : \Delta^q \longrightarrow X$ is a singular q-simplex, $k_i \in \mathbf{Z}$. The group q-chains $C_q(X)$ is a free abelian group generated by all singular q-simplices of the space X.

Now we define the "boundary homomorphism" $\partial_q : C_q(X) \longrightarrow C_{q-1}(X)$ as follows. Let $f : \Delta^q \longrightarrow X$ be a singular simplex, then we denote $\Gamma_i(f) = f|_{\Delta^{q-1}(i)}$ its restriction on the i-th face $\Delta^{q-1}(i)$. We define:

$$\partial_q f = \sum_{i=0}^{q} (-1)^i \Gamma_i(f).$$

Lemma 12.1. *The composition*

$$C_{q+1}(X) \xrightarrow{\partial_{q+1}} C_q(X) \xrightarrow{\partial_q} C_{q-1}(X)$$

is trivial, i.e. Im $\partial_{q+1} \subset$ Ker ∂_q.

Proof. It follows from the definition and the identity:

$$\Gamma_i(\Gamma_j(f)) = \begin{cases} \Gamma_{j-1}(\Gamma_i(f)) & \text{for } j > i, \\ \Gamma_j(\Gamma_{i+1}(f)) & \text{for } j \leq i. \end{cases} \tag{43}$$

Exercise 12.1. Check the identity (43) and complete the proof of Lemma 12.1.

Main Definition: The group $H_q(X) = \text{Ker } \partial_q / \text{Im } \partial_{q+1}$ is a *q-th homology group* of the space X. (The group $H_0(X) = C_0(x)/\text{Im } \partial_1$, and $H_q(X) = 0$ for $q < 0$.)

The group $Z_q(X) = \text{Ker } \partial_q$ is called the *group of cycles*, and the group $B_q(X) = \text{Im } \partial_{q+1}$ the *group of boundaries*. Thus $H_q(X) = Z_q(X)/B_q(X)$. If $c_1, c_2 \in C_q(X)$ are such elements that $c_1 - c_2 = \partial_{q+1}(d)$, then we say that the chain c_1 is *homologic to* c_2. We call a class $[c] \in H_q(X)$ a *homological class of a cycle c*.

The group $H_q(X)$ is an abelian group; if it is finitely-generated, then $H_q(X) \equiv \mathbf{Z} \oplus \ldots \oplus \mathbf{Z} \oplus \mathbf{Z}_{k_1} \oplus \ldots \oplus \mathbf{Z}_{k_m}$; the rank of this group (i.e. the number of $\mathbf{Z}$'s in this decomposition) is the *Betti* number of the space X.

12.2. **Chain complexes, chain maps and chain homotopy.** A *chain complex* $\mathcal{C}$ is a sequence of abelian groups and homomorphisms

$$\ldots \longrightarrow C_{q+1} \xrightarrow{\partial_{q+1}} C_q \xrightarrow{\partial_q} C_{q-1} \longrightarrow \ldots \longrightarrow C_1 \xrightarrow{\partial_1} C_0 \longrightarrow 0 \,, \tag{44}$$

such that $\partial_q \circ \partial_{q+1} = 0$ for all $q \geq 1$. For a given chain complex $\mathcal{C}$ the group $H_q(\mathcal{C}) = \text{Ker } \partial_q / \text{Im } \partial_{q+1}$ is the q-th homology group of $\mathcal{C}$. The chain complex

$$\ldots \longrightarrow C_{q+1}(X) \xrightarrow{\partial_{q+1}} C_q(X) \xrightarrow{\partial_q} C_{q-1}(X) \longrightarrow \ldots \longrightarrow C_1(X) \xrightarrow{\partial_1} C_0(X) \longrightarrow 0 \,, \tag{45}$$

will be denoted as $\mathcal{C}(X)$. Thus $H_q(X) = H_q(\mathcal{C}(X))$.

Let $\mathcal{C}'$, $\mathcal{C}''$ be two chain complexes. A *chain map* $\varphi : \mathcal{C}' \longrightarrow \mathcal{C}''$ is a collection of homomorphisms $\varphi_q : C'_q \longrightarrow C''_q$ such that the diagram

$$\begin{tikzcd}
\cdots \arrow[r, "\partial'_{q+1}"] & C'_q \arrow[r, "\partial'_q"] \arrow[d, "\varphi_q"] & C'_{q-1} \arrow[r, "\partial'_{q-1}"] \arrow[d, "\varphi_{q-1}"] & \cdots \arrow[r] & C'_1 \arrow[r, "\partial'_1"] \arrow[d, "\varphi_1"] & C'_0 \arrow[r] \arrow[d, "\varphi_0"] & 0 \\
\cdots \arrow[r, "\partial''_{q+1}"] & C''_q \arrow[r, "\partial''_q"] & C''_{q-1} \arrow[r, "\partial''_{q-1}"] & \cdots \arrow[r] & C''_1 \arrow[r, "\partial''_1"] & C''_0 \arrow[r] & 0
\end{tikzcd} \tag{46}$$

commutes. It is clear that a chain map $\varphi : \mathcal{C}' \longrightarrow \mathcal{C}''$ induces the homomorphisms $\varphi_* : H_q(\mathcal{C}') \longrightarrow H_q(\mathcal{C}'')$. In particular, a map $g : X \longrightarrow Y$ induces the homomorphism $g_\# : C_q(X) \longrightarrow C_q(Y)$ (which maps a singular simplex $f : \Delta^q \longrightarrow X$ to a singular simplex $g \circ f : \Delta^q \longrightarrow Y$). It defines a chain map $g_\# : \mathcal{C}(X) \longrightarrow \mathcal{C}(Y)$ and homomorphisms $g_* : H_q(X) \longrightarrow H_q(Y)$.

Exercise 12.2. Prove the following statements:
1. Let $g : X \longrightarrow Y$, $h : Y \longrightarrow Z$ be two maps. Then $(h \circ g)_\# = h_\# \circ g_\#$, and $(h \circ g)_* = h_* \circ g_*$.
2. Let $i : X \longrightarrow X$ be the identity map. Then $i_* = Id$.

Let $\varphi, \psi : \mathcal{C}' \longrightarrow \mathcal{C}''$ be two chain maps. We say the φ, ψ are *chain homotopic* if there are homomorphisms $D_q : C'_q \longrightarrow C''_{q-1}$ such that for each q

$$D_{q-1} \circ \partial'_q + \partial''_{q+1} \circ D_q = \varphi_q - \psi_q,$$

(here $D_{-1} = 0$). In that case we will write down $\varphi \sim \psi$.

Theorem 12.2. *Let $\varphi, \psi : \mathcal{C}' \longrightarrow \mathcal{C}''$ be two chain maps, and $\varphi \sim \psi$. Then*

$$\varphi_* = \psi_* : H_q(\mathcal{C}') \longrightarrow H_q(\mathcal{C}'').$$

Exercise 12.3. Prove Theorem 12.2.

Theorem 12.3. *Let $g, h : X \longrightarrow Y$ be homotopic maps, then $g_* = h_* : H_q(X) \longrightarrow H_q(Y)$. In other words, homotopic maps induce the same homomorphism in homology groups.*

Proof. By definition we have a homotopy $H : X \times I \longrightarrow Y$, such that $H|_{X\times\{0\}} = g$, $H|_{X\times\{1\}} = h$. Then for any singular simplex $f : \Delta^q \longrightarrow X$ we have a map $H \circ (f \times I) : \Delta^q \times I \longrightarrow Y$. The cylinder $\Delta^q \times I$ has a canonical simplicial structure: we subdivide $\Delta^q \times I$ into $(q+1)$-simplices $\overline{\Delta}^{q+1}(i)$, $i = 0, \ldots, q$, as follows:

$$\overline{\Delta}^{q+1}(i) = \{(t_0, \ldots, t_q, \tau) \in \Delta^q \times I \mid t_0 + \ldots + t_{i-1} \le \tau \le t_0 + \ldots + t_i \},$$

see Fig. 12.1. for $q = 1, 2$:

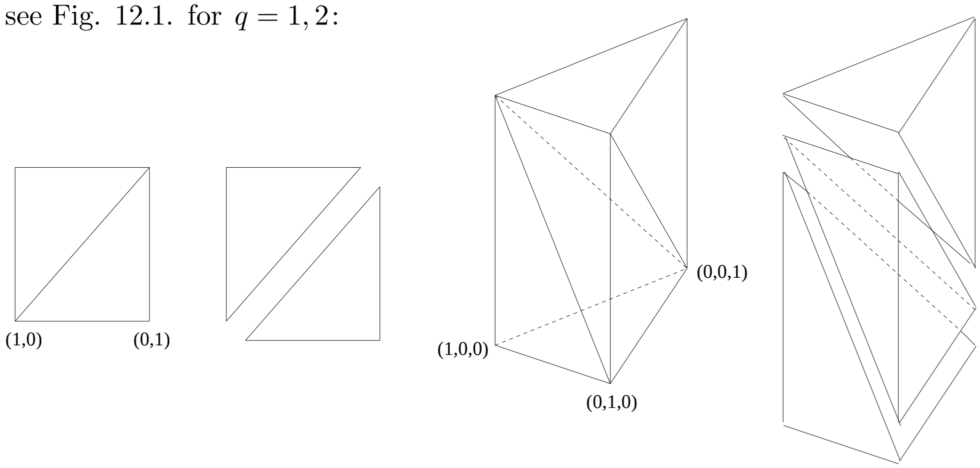


Fig. 12.1.

The map $G = H \circ (f \times I) : \Delta^q \times I \longrightarrow Y$ defines $(q+1)$ singular simplices of dimension $(q+1)$. We define

$$D(f) = \sum_{i=0}^{q} (-1)^i G|_{\overline{\Delta}^{q+1}(i)}.$$

It is easy to check that the homomorphisms

$$D_q : C_q(X) \longrightarrow C_{q+1}(Y), \quad D_q\left(\sum k_i f_i\right) = \sum k_i D_q(f_i)$$

define a chain homotopy $D : \mathcal{C}(X) \longrightarrow \mathcal{C}(Y)$. $\square$

Corollary 12.4. *Let X and Y be homotopy equivalent spaces. Then $H_q(X) \cong H_q(Y)$ for all q.*

Remark. There is a natural question: what happens if X and Y are weak homotopy equivalent? We will find the answer on this question in the next section.

12.3. **First computations.** By definition, the groups $C_q(X)$ are really huge, and it is difficult to compute homology directly. We will learn how to do this in a while, however even now we can prove several important facts.

Let $*$ be a space consisting of a single point. Clearly there is a unique map $f_q : \Delta^q \longrightarrow *$ for any q. We have that $C_q(*) = \mathbf{Z}$ for all $q \geq 0$. By definition, $\partial_q(f_q) = \sum(-1)^i \Gamma_i(f_q) = \sum(-1)^i f_{q-1}$. It follows then that

$$\partial_q(f_q) = \begin{cases} 0, & \text{for odd } q, \\ 1, & \text{for even } q. \end{cases}$$

The complex $\mathcal{C}(*)$ is the following:

$$\ldots \mathbf{Z} \xrightarrow{Id} \mathbf{Z} \xrightarrow{0} \mathbf{Z} \xrightarrow{Id} \mathbf{Z} \xrightarrow{0} \mathbf{Z} \longrightarrow 0.$$

The argument above proves the following statement.

Claim 12.1. $H_q(*) = \begin{cases} \mathbf{Z}, & \textit{if } q = 0 \\ 0, & \textit{else.} \end{cases}$

A space X with the same homology groups as of the point is called an *acyclic space*.

Corollary 12.5. *Let X be a contractible space, then it is acyclic.*

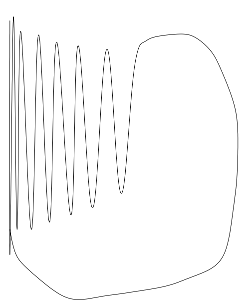


Fig. 12.2.

Remark. The opposite statement does not hold. The simplest example may be constructed out of the function $\sin \frac{1}{x}$, see Fig. 12.2.

Exercise 12.4. Prove that $H_0(X) \cong \mathbf{Z}$ if X is a path-connected space.

Exercise 12.5. Prove that $H_0(X) \cong \mathbf{Z} \oplus \ldots \oplus \mathbf{Z}$, where the number of $\mathbf{Z}$'s is the same as the number of path-connected components of X.

Exercise 12.6. Prove that if $f : X \longrightarrow Y$ is a map of path-connected spaces, then $f_* : H_0(X) \longrightarrow H_0(Y)$ is an isomorphism.

12.4. **Relative homology groups.** Let A be a subspace of X. Then $C_q(A) \subset C_q(X)$, and $\partial_q(C_q(A)) \subset C_{q-1}(A)$ by definition. Notice that each generator of the group $C_q(A)$ maps to a generator of the group $C_q(X)$. The group $C_q(X, A) = C_q(X)/C_q(A)$ is a group of *relative q-chains* of the space X modulo subspace A. Note that $C_q(X,A)$ is a free abelian group. Alternatively the group $C_q(X,A)$ may be defined as a free abelian group with generators

$$f:\Delta^q \longrightarrow X, \quad f(\Delta^q)\cap (X\setminus A)\neq \emptyset.$$

The boundary operator $\partial_q : C_q(X) \longrightarrow C_{q-1}(X)$ induces the operator $\partial_q : C_q(X,A) \longrightarrow C_{q-1}(X,A)$, and we obtain the complex $\mathcal{C}(X,A)$:

$$\cdots \longrightarrow C_q(X,A) \xrightarrow{\partial_q} C_{q-1}(X,A) \xrightarrow{\partial_{q-1}} \cdots \xrightarrow{\partial_2} C_1(X,A) \xrightarrow{\partial_1} C_0(X,A) \longrightarrow 0. \tag{47}$$

It is easy to check that we have a *short exact sequence of complexes*:

$$0 \longrightarrow \mathcal{C}(A) \xrightarrow{i_\#} \mathcal{C}(X) \xrightarrow{j_\#} \mathcal{C}(X,A) \longrightarrow 0. \tag{48}$$

It is very common situation in the homological algebra to work with a short exact sequence of complexes. The nature of the complexes is not important for the following statement: the complexes below may be over any abelian category.

Lemma 12.6. (LES-Lemma) *Let $0 \longrightarrow \mathcal{C}' \xrightarrow{i} \mathcal{C} \longrightarrow \mathcal{C}'' \longrightarrow 0$ be a short exact sequence of complexes. Then there is a long exact sequence of homology groups*

$$\cdots \longrightarrow H_q(\mathcal{C}') \xrightarrow{i_*} H_q(\mathcal{C}) \xrightarrow{j_*} H_q(\mathcal{C}'') \xrightarrow{\partial} H_{q-1}(\mathcal{C}') \xrightarrow{i_*} \cdots \tag{49}$$

where the homomorphisms i_* and j_* are induced by i and j respectively, and ∂ is the boundary homorphism to be defined.

Proof. First we define the boundary homomorphism $\partial : H_q(\mathcal{C}'') \longrightarrow H_{q-1}(\mathcal{C}')$. We have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & C'_{q+1} \arrow[r, "i_{q+1}"] \arrow[d, "\partial'_{q+1}"'] & C_{q+1} \arrow[r, "j_{q+1}"] \arrow[d, "\partial_{q+1}"'] & C''_{q+1} \arrow[r] \arrow[d, "\partial''_{q+1}"'] & 0 \\
0 \arrow[r] & C'_q \arrow[r, "i_q"] \arrow[d, "\partial'_q"'] & C_q \arrow[r, "j_q"] \arrow[d, "\partial_q"'] & C''_q \arrow[r] \arrow[d, "\partial''_q"'] & 0 \\
0 \arrow[r] & C'_{q-1} \arrow[r, "i_{q-1}"] & C_{q-1} \arrow[r, "j_{q-1}"] & C''_{q-1} \arrow[r] & 0
\end{tikzcd} \tag{50}$$

Let $\alpha \in H_q(\mathcal{C}'')$, and $c'' \in \operatorname{Ker} \partial''_q$ such that $\alpha = [c'']$. Choose an element $\widetilde{c} \in C_q$ such that $j_q(\widetilde{c}) = c''$, then the element $c = \partial_q(\widetilde{c}) \in C_{q-1}$ is such that $j_{q-1}(c) = 0$ by commutativity of (50). The exactness of the bottom row gives that there exists an element $c' \in C'_{q-1}$ such that $i_q(c') = c$.

Now we notice that $c' \in \text{Ker } \partial'_{q-1}$: it follows from the commutative diagram

(51)
$$
\begin{tikzcd}
0 \arrow[r] & C'_{q-1} \arrow[r, "i_{q-1}"] \arrow[d, "\partial'_{q-1}"'] & C_{q-1} \arrow[r, "j_{q-1}"] \arrow[d, "\partial_{q-1}"'] & C''_{q-1} \arrow[r] \arrow[d, "\partial''_{q-1}"'] & 0 \\
0 \arrow[r] & C'_{q-2} \arrow[r, "i_{q-2}"] & C_{q-2} \arrow[r, "j_{q-2}"] & C''_{q} \arrow[r] & 0
\end{tikzcd}
$$

since i_{q-1} is monomorphism, $c = \partial_q(\widetilde{c})$, and

$$i_{q-1} \circ \partial'_{q-1}(c') = \partial_{q-1} \circ i_q(c) = \partial_{q-1} \circ \partial_q(\widetilde{c}) = 0.$$

Thus $c' \in \text{Ker } \partial_{q-1}$, and we define $\partial(\alpha) = [c'] \in H_{q-1}(\mathcal{C}')$.

Exercise 12.7. Prove that the homomorphism $\partial : H_q(\mathcal{C}'') \longrightarrow H_{q-1}(\mathcal{C}')$ is well-defined.

The proof that the sequence (49) is exact is rather routine exercise. We will prove only the exactness at the term $H_q(\mathcal{C}'')$, i.e. that $\text{Im } j_* = \text{Ker } \partial$.

The inclusion $\text{Im } j_* \subset \text{Ker } \partial$ follows immediately from the definition. Now we prove that $\text{Ker } \partial \subset \text{Im } j_*$. Let $\alpha \in \text{Ker } \partial$. As above (in the definition of ∂) we consider a cycle $c'' \in C''_q$, an element $\widetilde{c}$, such that $j_q(\widetilde{c}) = c''$, then the element $c = \partial_q(\widetilde{c})$, and, finally, the element c' such that $i_{q-1}(c') = c$. We know that $[c'] = 0$, i.e. $c' \in \text{Im } \partial_q$. Let $b' \in C'_q$ be such that $\partial_q(b') = c'$. Let $\widetilde{c}_1 = i_q(b')$. By commutativity of (50) $\partial_q(\widetilde{c} - \widetilde{c}_1) = 0$, and by exatness of the second row of (50) $j_q(\widetilde{c} - \widetilde{c}_1) = c''$. Thus the element $d = \widetilde{c} - \widetilde{c}_1 \in C_q$ is a cycle, and $j_q(d) = c''$, and $j_*([d]) = \alpha$.

Exercise 12.8. Prove the exactness of (49) at the term $H_q(\mathcal{C}')$.

The exactness of (49) at the term $H_q(\mathcal{C})$ is an easy exercise. □

Now we specify Lemma 12.6 in the case of the exact sequence of complexes

$$0 \longrightarrow \mathcal{C}(A) \xrightarrow{i_\#} \mathcal{C}(X) \xrightarrow{j_\#} \mathcal{C}(X,A) \longrightarrow 0.$$

The boundary operator $\partial_q : C_q(X,A) \longrightarrow C_{q-1}(X,A)$ is induced by the boundary operator $\partial_q : C_q(X) \longrightarrow C_{q-1}(X)$, and clearly $\partial_q(c) \in C_{q-1}(A)$ if $c \in C_{q-1}(X,A)$ is a cycle.

Corollary 12.7. *Let (X,A) be a pair of spaces. Then there is an exact sequence of homology groups:*

(52)
$$\cdots \xrightarrow{\partial} H_q(A) \xrightarrow{i_*} H_q(X) \xrightarrow{j_*} H_q(X,A) \xrightarrow{\partial} H_{q-1}(A) \xrightarrow{i_*} \cdots .$$

Let $B \subset A \subset X$ be a triple of spaces. We have the following maps of pairs:

(53)
$$(A,B) \xrightarrow{i} (X,B) \xrightarrow{j} (X,A)$$

which induce the homomorphisms $\mathcal{C}(A,B) \xrightarrow{i_\#} \mathcal{C}(X,B) \xrightarrow{j_\#} \mathcal{C}(X,A)$.

Exercise 12.9. Prove that the sequence of complexes

(54)
$$0 \longrightarrow \mathcal{C}(A,B) \xrightarrow{i_\#} \mathcal{C}(X,B) \xrightarrow{j_\#} \mathcal{C}(X,A) \longrightarrow 0$$

is exact.

Exercise 12.9 and the LES-Lemma imply the following result.

Corollary 12.8. *Let $B \subset A \subset X$ be a triple of spaces. Then there is a long exact sequence in homology:*

$$\cdots \longrightarrow H_q(A,B) \xrightarrow{i_*} H_q(X,B) \xrightarrow{j_*} H_q(X,A) \xrightarrow{\partial} H_{q-1}(A,B) \xrightarrow{i_*} \cdots . \tag{55}$$

The relative homology groups are natural invariant.

Exercise 12.10. Let $B \subset A \subset X$ and $B' \subset A' \subset X'$ be two triples of spaces, and $f : X \longrightarrow X'$ be such a map that $f(B) \subset B'$, and $f(A) \subset A'$. Prove that the following diagram commutes:

$$\begin{tikzcd}
\cdots \arrow[r] & H_q(A,B) \arrow[r, "i_*"] \arrow[d, "f_*"'] & H_q(X,B) \arrow[r, "j_*"] \arrow[d, "f_*"'] & H_q(X,A) \arrow[r, "\partial"] \arrow[d, "f_*"'] & H_{q-1}(A,B) \arrow[r] \arrow[d, "f_*"'] & \cdots \\
\cdots \arrow[r] & H_q(A',B') \arrow[r, "i_*"] & H_q(X',B') \arrow[r, "j_*"] & H_q(X',A') \arrow[r, "\partial"] & H_{q-1}(A',B') \arrow[r] & \cdots
\end{tikzcd}$$

Exercise 12.11. Let $f : (X,A) \longrightarrow (X',A')$ be such map of pairs that the induced maps $f : X \longrightarrow X'$ and $f|_A : A \longrightarrow A'$ are homotopy equivalences. Prove that $f_* : H_q(X,A) \longrightarrow H_q(X',A')$ is an isomorphism for each q.

Remark. One may expect that there is a long exact sequence in homology groups for a Serre fiber bundle $E \longrightarrow B$. However there is no such exact sequence in general case: here there is a *spectral sequence* which relates the homology groups of the base, the total space and the fiber. Again, we are not ready even to discuss this yet.

12.5. **Relative homology groups and regular homology groups.** Let (X,A) be a pair of spaces. The space X/A has a base point a (the image of A under the projection $X \longrightarrow A$. There is a map of pairs $p : (X,A) \longrightarrow (A,a)$ induced by the projection $X \longrightarrow X/A$. Besides, the is the inclusion map $i : X \longrightarrow X \cup C(A)$, and thus the map of pairs $i : (X,A) \longrightarrow (X \cup C(A), C(A))$. Let v be the vertex of the cone $C(A)$.

Theorem 12.9. *Let (X,A) be a pair of spaces. Then the inclusion*

$$i : (X,A) \longrightarrow (X \cup C(A), C(A))$$

induces the isomorphism $H_q(X,A) \cong H_q(X \cup C(A), C(A)) = H_q(X \cup C(A), v)$.

We have to get ready to prove Theorem 12.9. Recall that for each simplex Δ^q there is the barycentric subdivision of Δ^q. First we examine the barycentric subdivision one more time. Let Δ^q be given by the vertices $A_0, \ldots, A_q$. Let $f : \Delta^q \longrightarrow X$ be a singular simplex. We would like to give a natural description of all q-simplices of the barycentric subdivision of Δ^q in terms of the symmetric group Σ_{q+1} acting on the vertices $(A_0, \dots, A_q)$ of Δ^q.

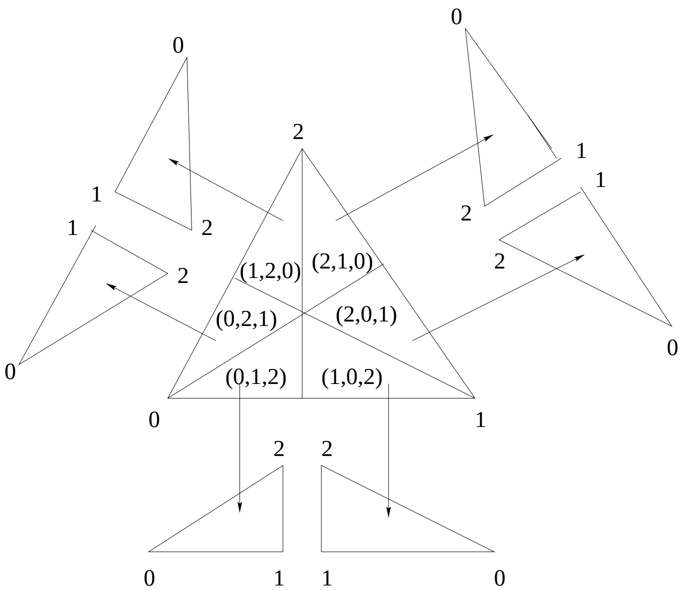


Fig. 12.3.

First, let $q = 1$, then Δ^1 is given by vertices (A_0, A_1). Let B_0 is the barycenter of Δ^1. Then we let $\overline{\Delta}^1(0,1) := (A_0, B_0)$ and $\overline{\Delta}^1(0,1) := (A_1, B_0)$. Here the simplex $\overline{\Delta}^1(0,1)$ is obtained from $\overline{\Delta}^1(0,1)$ by permutation $(0,1)$ which acts on the vertices (A_0, A_1). By induction, let $\overline{\Delta}^q(0,\dots,q)$ be the simplex which has the same first q vertices as the simplex $\overline{\Delta}^{q-1}(0,\dots,q-1)$ and the last one being the barycenter of the simplex Δ^q. The symmetric group Σ_{q+1} acts on the vertices $(A_0,\dots,A_q)$ of Δ^q, and each permutation $\sigma \in \Sigma_{q+1}$ gives a linear map $\sigma : \Delta^q \to \Delta^q$ leaving the barycenter B_0 of Δ^q fixed.

Then the simplex $\overline{\Delta}^q(\sigma)$ is defined as the image $\sigma(\overline{\Delta}^q(0,\dots,q))$. Thus we can list all simplices $\overline{\Delta}^q(\sigma)$ of the barycentric subdivision of Δ^q by the elements $\sigma \in \Sigma_{q+1}$. Let $(-1)^\sigma$ be the sign of the permutation $\sigma \in \Sigma_{q+1}$, see Fig. 12.4.

Now we define a chain map $\beta : \mathcal{C}(X) \longrightarrow \mathcal{C}(X)$ as follows. Let $f : \Delta^q \longrightarrow X$ be a generator, and $f_\sigma = f|_{\overline{\Delta}^q(\sigma)}$. Then

$$\beta(f : \Delta^q \longrightarrow X) = \sum_{\sigma \in \Sigma_{q+1}} (-1)^\sigma (f_\sigma : \overline{\Delta}^q(\sigma) \longrightarrow X).$$

Then we define $\beta(\sum_i \lambda_i f_i) = \sum_i \lambda_i \beta(f_i)$. It is easy to check that $\beta\partial_q = \partial_q\beta$. (Here the choice of the above sign $(-1)^\sigma$ is important.)

Lemma 12.10. *The chain map $\beta : \mathcal{C}(X) \longrightarrow \mathcal{C}(X)$ induces the identity homomorphism in homology:*

$$Id = \beta_* : H_q(\mathcal{C}(X)) \longrightarrow H_q(\mathcal{C}(X)) \quad \textit{for each } q \geq 0.$$

Proof. It is enough to construct a chain homotopy $D_q : C_q(X) \longrightarrow C_{q+1}(X)$ so that $\beta - Id = D_{q-1} \circ \partial'_q + \partial''_{q+1} \circ D_q$. We construct the triangulation of $\Delta^q \times I$ as follows.

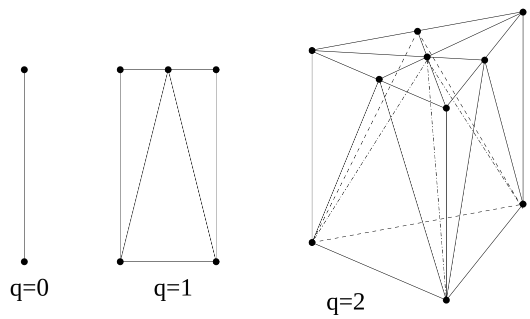


Fig. 12.4.

The cases $q = 0, 1, 2$ are shown at Fig. 12.4. Now the bottom simplex $\Delta^q \times \{0\}$ is given the standard triangulation (just one simplex), and the top simplex $\Delta^q \times \{1\}$ is given the barycentric subdivision. The side $\partial\Delta^q \times I$ is given the subdivision by induction. Now consider the center v of the simplex $\Delta^q \times \{1\}$, and consider the cones with the vertex v over each q-simplex $\overline{\Delta}^q$, where

$$\overline{\Delta}^q \subset \Delta^q \times \{0\} \cup \partial\Delta^q \times I \cup \Delta^q \times \{1\}.$$

This triangulation gives the chain $D_q(f)$, where $f : \Delta^q \longrightarrow X$ is a singular simplex. We notice that $D_q(f)$ is defined as via the map

$$G : \Delta^q \times I \xrightarrow{\text{projection}} \Delta^q \times \{0\} \xrightarrow{f} X$$

by restricting G on the corresponding simplices. Lemma 12.9 follows. $\square$

Let $\mathfrak{U} = \{U_i\}$ be a finite open covering of a space X. We define the group

$$C_q^{\mathfrak{U}}(X) = \{\text{free abelian group}\}\,(f : \Delta^q \longrightarrow X \mid f(\Delta^q) \subset U_i \text{ for some } U_i \in \mathfrak{U}).$$

Clearly $C_q^{\mathfrak{U}}(X) \subset C_q(X)$ and the restriction of the boundary operator $\partial_q : C_q(X) \longrightarrow C_{q-1}(X)$ defines the operator $\partial_q : C_q^{\mathfrak{U}}(X) \longrightarrow C_{q-1}^{\mathfrak{U}}(X)$. Thus we have the complex $\mathcal{C}^{\mathfrak{U}}(X)$.

Lemma 12.11. *The chain map (inclusion) $i : \mathcal{C}^{\mathfrak{U}}(X) \longrightarrow \mathcal{C}(X)$ induces isomorphism in the homology groups*

$$i_* : H_q(\mathcal{C}^{\mathfrak{U}}(X)) \xrightarrow{\cong} H_q(\mathcal{C}(X)). \tag{56}$$

Proof. Let $\alpha \in H_q(\mathcal{C}(X)) = H_q(X)$, and $\alpha = [c]$, where $c \in Z_q(X)$ is a cycle. To prove that i_* is epimorphism, it is enough to prove that

(i) *there is $c' \in Z_q^{\mathfrak{U}}(X)$ and $d \in C_{q+1}(X)$ so that $\partial_{q+1}(d) = c - c'$.*

Let $\alpha' \in H_q(\mathcal{C}^{\mathfrak{U}}(X))$, $\alpha' = [c']$, where $c' \in Z_q^{\mathfrak{U}}(X)$. Assume that $i_*(\alpha') = 0$, i.e. $c' = \partial_{q+1}d$ where $d \in C_{q+1}(X)$. To prove that i_* is monomorphism, it is enough to show that

(ii) *there is $d' \in C_{q+1}^{\mathfrak{U}}(X)$ such that $\partial_{q+1}(d') = c'$.*

The above statements follow from the following three observations:

(1) For any $c \in C_q(X)$ there is $n \geq 1$ so that $\beta^n c \in C_q^{\mathfrak{U}}(X)$.

(2) For any $c \in C_q(X)$ and $n \geq 1$ there is $d \in C_q(X)$ such that $\partial_{q+1}(d) = c - \beta^n c$. (Lemma 12.10.)

(3) Let $c' \in Z_q^{\mathfrak{U}}(X)$, then for any $n \geq 1$ there is $d' \in C_{q+1}^{\mathfrak{U}}(X)$ such that $\partial_{q+1} d' = c' - \beta^n c'$.

Exercise 12.12. Prove the properties (1) and (3).

Exercise 12.13. Show that the above statements (i), (ii) follow from (1), (2), (3).

This concludes the proof. □

Remark. Let $\mathfrak{V} = \{V_j\}$ be a finite covering of X, such that $X = \bigcup_j \overset{o}{V}_j$, where $\overset{o}{V}$ is the interior of V. Then the chain map $\mathcal{C}^{\mathfrak{V}}(X) \longrightarrow \mathcal{C}(X)$ also induces isomorphism in the homology groups.

Remark. Let (X, A) be a pair of spaces. Then a covering $\mathfrak{U} = \{U_j\}$ induces a covering $\{U_j \cap A\}$. We denote a corresponding chain complex by $\mathcal{C}^{\mathfrak{U}}(A)$. Then for each q we have a short exact sequence

$$0 \to C_q^{\mathfrak{U}}(A) \longrightarrow C_q^{\mathfrak{U}}(X) \longrightarrow C_q^{\mathfrak{U}}(X, A) \to 0$$

which determines the relative chain complex $\mathcal{C}^{\mathfrak{U}}(X, A)$. It easy to modify the proof of Lemma 12.11 (and use five-lemma) to show that the natural chain map $\mathcal{C}^{\mathfrak{U}}(X, A) \longrightarrow \mathcal{C}(X, A)$ induces an isomorphism in the homology groups

$$H_q(\mathcal{C}^{\mathfrak{U}}(X, A)) \xrightarrow{\cong} H_q(\mathcal{C}(X, A)) = H_q(X, A).$$

Proof of Theorem 12.9. Consider the following covering of the space $X \cup C(A)$. Let $U_1 = (X \cup C(A)) \setminus X$ and $U_2 = X \cup \overline{C}(A)$, where $\overline{C}(A)$ is the half-cone over A, i.e. $\overline{C}(A) = \{(a, t) \in C(A) \mid 0 \leq t < 1/2\}$. The relative version of Lemma 12.11 (see the above remark) implies that the embedding

$$\mathcal{C}^{\mathfrak{U}}(X \cup C(A), C(A)) \longrightarrow \mathcal{C}(X \cup C(A), C(A))$$

induces an isomorphism in the homology groups. By definition of a relative chain complex, we have the isomorphism:

$$C_q^{\mathfrak{U}}(X \cup C(A), C(A)) \cong C_q^{\mathfrak{U}}(X \cup C(A))/C_q^{\mathfrak{U}}(C(A)).$$

Then we observe that there is an isomorphism

$$C_q^{\mathfrak{U}}(X \cup C(A))/C_q^{\mathfrak{U}}(C(A)) \cong C_q(X \cup \overline{C}(A))/C_q(\overline{C}(A)) = C_q(X \cup \overline{C}(A), \overline{C}(A)).$$

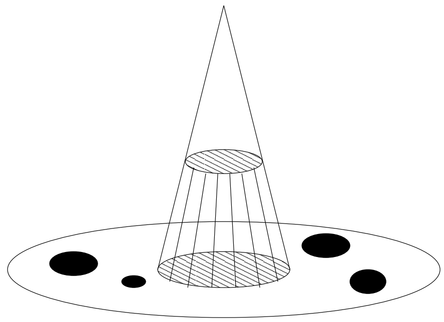


Fig. 12.5.

Indeed, let $f : \Delta^q \to X \cup C(A)$ be a generator in $C_q^{\mathfrak{U}}(X \cup C(A), C(A))$, i.e.

$$f(\Delta^q) \cap (X \cup C(A) \setminus C(A)) \neq \emptyset$$

and $f(\Delta^q) \subset U_1$ or $f(\Delta^q) \subset U_2$. Since $U_1 = (X \cup C(A)) \setminus X$, $f(\Delta^q) \subset U_2 = X \cup \overline{C}(A)$ and

$$f(\Delta^q) \cap ((X \cup \overline{C}(A)) \setminus \overline{C}(A)) \neq \emptyset,$$

the map $f : \Delta^q \to X \cup C(A)$ is a generator of the free abelian group

$$C_q(X \cup \overline{C}(A), \overline{C}(A)) = C_q(X \cup \overline{C}(A))/C_q(\overline{C}(A)).$$

It is also easy to check that any generator in $C_q(X \cup \overline{C}(A))/C_q(\overline{C}(A))$ gives a generator in the group $C_q^{\mathfrak{U}}(X \cup C(A), C(A))$.

Since $X \cup \overline{C}(A)$ is homotopy equivalent to X, and $\overline{C}(A) \sim A$, we obtain the isomorphisms

$$H_q(X \cup C(A), C(A)) \cong H_q(X \cup \overline{C}(A), \overline{C}(A)) \cong H_q(X, A).$$

This concludes the proof of Theorem 12.9. $\square$

Corollary 12.12. *Let (X, A) be a Borsuk pair. Then the projection $p : (X, A) \longrightarrow (A, a)$ induces the isomorphism $p_* : H_q(X, A) \longrightarrow H_q(X/A, a)$ for each q.*

Exercise 12.14. Prove Corollary 12.12.

12.6. **Excision Theorem.** Let (X, A) be a pair of spaces, and $B \subset A$. The map of pairs $e : (X \setminus B, A \setminus B) \longrightarrow (A, B)$ induces the *excision homomorphism*:

$$e_* : H_q(X \setminus B, A \setminus B) \longrightarrow H_q(X, A) \tag{57}$$

The following result is known as the *Excision Theorem.*

Theorem 12.13. *Let (X, A) be a pair of Hausdorff spaces, and $B \subset A$ so that $\overline{B} \subset \overset{o}{A}$. Then the homomorphism $e_* : H_q(X \setminus B, A \setminus B) \longrightarrow H_q(X, A)$ is an isomorphism.*

Proof. We use the condition $\overline{B} \subset \overset{o}{A}$ to notice that

$$(X \overset{o}{\setminus} B) \supset X \setminus \overline{B} \supset X \setminus \overset{o}{A}.$$

Thus $\overset{o}{A} \cup (X \overset{o}{\setminus} B) = X$. We consider the covering $\mathfrak{V} = \{V_1, V_2\}$ of X, where $V_1 = A$, $V_2 = X \setminus B$. The chain complex $\mathcal{C}^{\mathfrak{V}}(X)$ (see the remark following Exercise 12.13) gives the chain map $i : \mathcal{C}^{\mathfrak{V}}(X) \longrightarrow \mathcal{C}(X)$. Note that for each q

$$C_q^{\mathfrak{V}}(X) \cong C_q(A) + C_q(X \setminus B) \subset C_q(X).$$

Consider the relative chain complex $\mathcal{C}^{\mathfrak{V}}(X, A)$. To prove the excision property we consider the following commutative diagram of chain complexes:

$$\begin{tikzcd}
\mathcal{C}(X \setminus B, A \setminus B) \arrow[r, "j_1"] \arrow[dr, "j_2"'] & \mathcal{C}(X, A) \\
& \mathcal{C}^{\mathfrak{V}}(X, A) \arrow[u, "j_3"']
\end{tikzcd} \tag{58}$$

Here the chain maps j_1 and j_3 are induced by natural inclusions.

Now we construct the chain map j_2. By definition

$$C_q(X \setminus B, A \setminus B) = C_q(X \setminus B)/C_q(A \setminus B) \cong C_q(X \setminus B)/(C_q(X \setminus B) \cap C_q(A))$$

since $C_q(A \setminus B) = C_q(X \setminus B) \cap C_q(A)$. Similarly,

$$C_q^{\mathfrak{V}}(X, A) = C_q^{\mathfrak{V}}(X)/C_q^{\mathfrak{V}}(A) = (C_q(X \setminus B) + C_q(A))/C_q(A).$$

Now recall the following standard fact from the group theory.

Claim 12.2. *Let $G_1, G_2 \subset G$ be subgroups of an abelian group G. Then*

$$G_1/(G_1 \cap G_2) \cong (G_1 + G_2)/G_2.$$

If we let $G_1 := C_q(X \setminus B)$, $G_2 := C_q(A)$, then the isomorphism

$$j_2 : C_q(X \setminus B, A \setminus B) \longrightarrow C_q^{\mathfrak{V}}(X, A)$$

is given by Claim 12.2. We obtain the induced commutative diagram in homology groups

$$\begin{tikzcd}
H_q(X \setminus B, A \setminus B) \arrow[r, "(j_1)_*"] \arrow[dr, "(j_2)_*"'] & H_q(X, A) \\
& H_q(\mathcal{C}^{\mathfrak{V}}(X, A)) \arrow[u, "(j_3)_*"']
\end{tikzcd} \tag{59}$$

where $(j_1)_*$, $(j_2)_*$ and $(j_3)_*$ are isomorphisms. Thus $H_q(X \setminus B, A \setminus B) \cong H_q(X, A)$. $\square$

12.7. **Mayer-Vietoris Theorem.** Let $X = X_1 \cup X_2$. We notice that $\mathcal{C}(X_1 \cap X_2) = \mathcal{C}(X_1) \cap \mathcal{C}(X_2)$, and that $\mathcal{C}(X_1)$, $\mathcal{C}(X_2)$ are subcomplexes of $\mathcal{C}(X_1 \cup X_2)$. In particular, the complex $\mathcal{C}(X_1)+\mathcal{C}(X_2) \subset \mathcal{C}(X_1 \cup X_2)$ is well-defined. Let

$$j^{(1)} : \mathcal{C}(X_1 \cap X_2) \longrightarrow \mathcal{C}(X_1), \quad j^{(2)} : \mathcal{C}(X_1 \cap X_2) \longrightarrow \mathcal{C}(X_2),$$

$$i^{(1)} : \mathcal{C}(X_1) \longrightarrow \mathcal{C}(X_1 \cup X_2), \quad i^{(2)} : \mathcal{C}(X_2) \longrightarrow \mathcal{C}(X_1 \cup X_2)$$

be the inclusions. Consider the following sequence of complexes:

$$0 \longrightarrow \mathcal{C}(X_1 \cap X_2) \xrightarrow{\alpha} \mathcal{C}(X_1) \oplus \mathcal{C}(X_2) \xrightarrow{\beta} \mathcal{C}(X_1) + \mathcal{C}(X_2) \longrightarrow 0. \tag{60}$$

where $\alpha(c) = j^{(1)}(c) \oplus j^{(2)}(c)$, and $\beta(c_1 \oplus c_2) = c_1 - c_2 \in \mathcal{C}(X_1) + \mathcal{C}(X_2)$.

Claim 12.3. *The sequence (60) is a short exact sequence of chain complexes.*

Exercise 12.15. Prove Claim 12.3.

Lemma 12.14. *Let $X_1, X_2 \subset X$, and $X_1 \cup X_2 = X$, $\overset{o}{X}_1 \cup \overset{o}{X}_2 = X$. Then the chain map $\mathcal{C}(X_1) + \mathcal{C}(X_2) \longrightarrow \mathcal{C}(X_1 \cup X_2)$ induces isomorphism in the homology groups.*

Proof. Consider the covering $\mathfrak{V} = \{X_1, X_2\}$. Then by definition $\mathcal{C}^{\mathfrak{V}}(X_1 \cup X_2) = \mathcal{C}(X_1) + \mathcal{C}(X_2)$. Lemma 12.11 and the remark following Lemma 12.11 completes the proof. □

Theorem 12.15. (Mayer-Vietoris Theorem) *Let X be a space, and $X = X_1 \cup X_2$, and $X = \overset{o}{X}_1 \cup \overset{o}{X}_2$. Then there is a long exact sequence*

$$\cdots \to H_q(X_1 \cap X_2) \xrightarrow{\alpha_*} H_q(X_1) \oplus H_q(X_2) \xrightarrow{\beta_*} H_q(X_1 \cup X_2) \xrightarrow{\partial} H_{q-1}(X_1 \cap X_2) \to \cdots \tag{61}$$

This is the Mayer-Vietoris long exact sequence.

Proof. The short exact sequence of chain complexes (60) induces the long exact sequnce

$$\cdots \to H_q(\mathcal{C}(X_1 \cap X_2)) \xrightarrow{\alpha_*} H_q(\mathcal{C}(X_1)) \oplus H_q(\mathcal{C}(X_2)) \xrightarrow{\beta_*} H_q(\mathcal{C}(X_1) + \mathcal{C}(X_2))$$

$$\xrightarrow{\partial} H_{q-1}(\mathcal{C}(X_1 \cap X_2)) \xrightarrow{\alpha_*} H_{q-1}(\mathcal{C}(X_1)) \oplus H_{q-1}(\mathcal{C}(X_2)) \to \cdots$$

To complete the proof we replace the groups $H_q(\mathcal{C}(X_1) + \mathcal{C}(X_2))$ by $H_q(X_1 \cup X_2)$ using Lemma 12.14. □

13. Homology groups of CW-complexes

The main goal of this section is to develop a technique to compute homology groups of CW-complexes. The singular chain complex $\mathcal{C}(X)$ is far too big to peform computations. We will construct here a *cellular chain complex* $\mathcal{E}(X)$ which is much smaller than $\mathcal{C}(X)$. We start with computations of homology groups of spheres and wedges of spheres.

13.1. **Homology groups of spheres.**

Theorem 13.1.

$$\widetilde{H}_q(S^n) \cong \begin{cases} \mathbf{Z} & \text{if } q = n, \\ 0 & \text{else.} \end{cases}$$

Remark. We use here reduced homology groups to unify the formula for $n = 0$ and $n \geq 1$. We already know that $H_0(S^0) = \mathbf{Z} \oplus \mathbf{Z}$, hence $\widetilde{H}_0(S^0) = \mathbf{Z}$.

Proof. Consider a long exact sequence for the pair (D^n, S^{n-1}):

$$\widetilde{H}_q(S^{n-1}) \longrightarrow \widetilde{H}_q(D^n) \longrightarrow H_q(D^n, S^{n-1}) \longrightarrow \widetilde{H}_{q-1}(S^{n-1}) \longrightarrow \widetilde{H}_{q-1}(D^n) \ .$$

We have $\widetilde{H}_q(D^n) = 0$, $\widetilde{H}_{q-1}(D^n) = 0$. Thus $H_q(D^n, S^{n-1}) \cong \widetilde{H}_{q-1}(S^{n-1})$. Induction on n concludes the proof. □

Theorem 13.2. *Let X be a space. Then $\widetilde{H}_{q+1}(\Sigma X) \cong \widetilde{H}_q(X)$ for each q.*

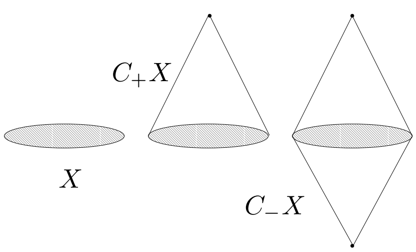


Fig. 13.1.

Proof. We notice that $\Sigma X = C_+X \cup C_-X$, see Fig. 13.1. Consider a long exact sequence in homology for the pair (C_+X, X):

$$\cdots \to \widetilde{H}_q(C_+X) \longrightarrow H_q(C_+X, X) \longrightarrow \widetilde{H}_{q-1}(X)$$

$$\longrightarrow \widetilde{H}_{q-1}(C_+X) \to \cdots$$

Clearly we have that $\widetilde{H}_*(C_+X) = 0$ since the cone C_+X is contractible.

Thus $H_q(C_+X, X) \cong \widetilde{H}_{q-1}(X)$. Notice that the pair (C_+X, X) is always a Borsuk pair, thus

$$H_q(C_+X, X) \cong \widetilde{H}_q(C_+X/X) \cong \widetilde{H}_q(C_+X \cup C_-X) = \widetilde{H}_q(\Sigma X).$$

Theorem 13.2 is proved. □

Remark. The homeomorphism $\Delta^q \xrightarrow{\cong} D^q$ gives a particular repesentative for a generator $\bar{\iota}_q \in H_q(D^q, S^{q-1}) \cong \mathbf{Z}$. The composition $\Delta^q \xrightarrow{\cong} D^q \xrightarrow{pr} D^q/S^{q-1}$ gives a particular repesentative for the generator $\iota_q \in H_q(S^q) \cong \mathbf{Z}$. Clearly $\bar{\iota}_q$ maps to ι_{q-1} under the boundary homomorphism $H_q(D^q, S^{q-1}) \to H_{q-1}(S^{q-1})$.

Theorem 13.2 leads to the following construction. Let $f : \Delta^q \longrightarrow X$ be a singular simplex. Consider the composition

$$\Sigma f : \Delta^{q+1} = C\Delta^q \xrightarrow{Cf} CX \xrightarrow{\text{projection}} CX/X \cong \Sigma X.$$

Thus we have the chain map $\Sigma : C_q(X) \longrightarrow C_{q+1}(\Sigma X)$.

Excercise 13.1. Show that the map $\Sigma : C_q(X) \longrightarrow C_{q+1}(\Sigma X)$ commutes with the boundary operator and induces the isomorphism $\Sigma : \widetilde{H}_q(X) \longrightarrow \widetilde{H}_{q+1}(\Sigma X)$.

13.2. **Homology groups of a wedge.**

Theorem 13.3. *Let A be a set of indices, and S^n_α be a copy of the n-th sphere, $\alpha \in A$. Then*

$$\widetilde{H}_q\left(\bigvee_{\alpha\in A} S^n_\alpha\right) = \begin{cases} \bigoplus_{\alpha\in A} \mathbf{Z}(\alpha), & \text{if } q = n, \\ 0, & \text{else.} \end{cases}$$

Here $\bigoplus_{\alpha\in A} \mathbf{Z}(\alpha)$ is a free abelian group with generators $\alpha \in A$.

This result follows from Theorem 13.2 because of the homotopy equivalence

$$\Sigma\left(\bigvee_{\alpha\in A} S^n_\alpha\right) \sim \bigvee_{\alpha\in A} \Sigma S^n_\alpha = \bigvee_{\alpha\in A} S^{n+1}_\alpha.$$

On the other hand Theorem 13.3 is a particular case in the following result.

Theorem 13.4. *Let (X_α, x_α) be based spaces, $\alpha \in A$. Assume that the pair (X_α, x_α) is Borsuk pair for each $\alpha \in A$. Then*

$$\widetilde{H}_q\left(\bigvee_{\alpha\in A} X_\alpha\right) = \bigoplus_{\alpha\in A} \widetilde{H}_q(X_\alpha).$$

Excercise 13.2. Prove Theorem 13.4. **Hint:** The wedge $\bigvee_{\alpha\in A} X_\alpha$ is a factor-space of the disjoint union $\bigsqcup_{\alpha\in A} X_\alpha$ by the union of the base points.

13.3. **Maps** $g : \bigvee_{\alpha\in A} S^n_\alpha \to \bigvee_{\beta\in B} S^n_\beta$**.**

Let $f : S^n \longrightarrow S^n$ be a map. Then the homotopy class $[f] = d\iota_n$, where $d \in \mathbf{Z}$, and $\iota_n \in \pi_n(S^n)$ is a generator represented by the identity map $S^n \longrightarrow S^n$. Recall that $d = \deg f$.

Claim 13.1. *Let $f : S^n \longrightarrow S^n$ be a map of degree $d = \deg f$. Then the induced homomorphism $f_* : H_n(S^n) \longrightarrow H_n(S^n)$ is multiplication by d.*

Proof. We constructed earlier a map $g(d) : S^n \longrightarrow S^n$ of the degree d:

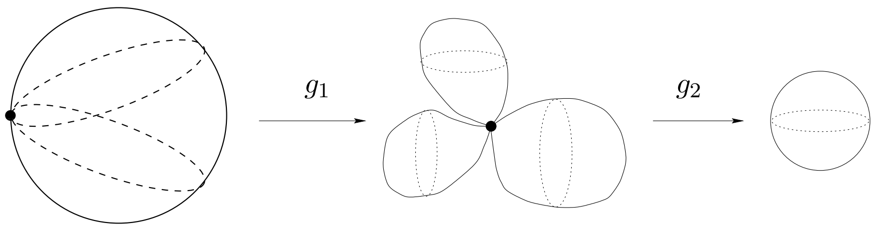


Fig. 13.2.

where the map $g_1 : S^n \longrightarrow \bigvee_j^d S_j^n$ collapses $(d-1)$ spheres S^{n-1} as it is shown at Fig. 13.2, and $g_2 : \bigvee_j^d S_j^n \longrightarrow S^n$ is a folding map. The map $g(d) = g_2 \circ g_1$ is of degree d since this is a representative of $d\iota_n$. Thus $g(d) \sim f$. The composition of the map g_1 with a projection $p_j : \bigvee_j^d S_j^n \longrightarrow S_j^n$ is homotopic to the identinty. Thus $(g_1)_* : H_n(S^n) \longrightarrow H_n(\bigvee_j^d S_j^n)$ gives $(g_1)_*(1) = 1 \oplus \cdots \oplus 1$. We notice that the map $S_j^n \xrightarrow{i_j} \bigvee_j^d S_j^n \xrightarrow{g_2} S^n$ is homotopic to the identity map. Thus $(g_2)_*(1 \oplus \cdots \oplus 1) = 1 + \cdots + 1 = d$. This implies that the homomorphism $g(d)_* = (g_2)_* \circ (g_1)_* : H_n(S^n) \longrightarrow H_n(S^n)$ is the multiplication by d. This completes the proof since the map f is homotopic to the map $g(d)$. $\square$

Now we consider a map $g : \bigvee_{\alpha \in A} S_\alpha^n \xrightarrow{g} \bigvee_{\beta \in B} S_\beta^n$. Let $i_\alpha : S_\alpha^n \longrightarrow \bigvee_{\alpha \in A} S_\alpha^n$ be the canonical inclusion, and $p_\beta : \bigvee_{\beta \in B} S_\beta^n \longrightarrow S_\beta^n$ be the projection on the β-th summand. We have the commutative diagram:

$$\begin{tikzcd} \bigvee_{\alpha\in A} S_\alpha^n \arrow[r, "g"] & \bigvee_{\beta\in B} S_\beta^n \arrow[d, "p_\beta"] \\ S_\alpha^n \arrow[u, "i_\alpha"] \arrow[r, "g_{\alpha\beta}"] & S_\beta^n \end{tikzcd} \tag{62}$$

Let the map $g_{\alpha\beta} : S_\alpha^n \longrightarrow S_\beta^n$ have degree $d_{\alpha\beta}$, and let $\{d_{\alpha\beta}\}_{\alpha\in A, \beta\in B}$ be the matrix of those degrees.

Theorem 13.5. *Let $g : \bigvee_{\alpha \in A} S_\alpha^n \xrightarrow{g} \bigvee_{\beta \in B} S_\beta^n$ be a map. Then the homomorphism*

$$\bigoplus_{\alpha\in A} \mathbf{Z}(\alpha) = H_n\left(\bigvee_{\alpha\in A} S_\alpha^n\right) \xrightarrow{g_*} H_n\left(\bigvee_{\beta\in B} S_\beta^n\right) = \bigoplus_{\beta\in B} \mathbf{Z}(\beta)$$

is given by multiplication with the matrix $\{d_{\alpha\beta}\}_{\alpha\in A,\beta\in B}$, where $d_{\alpha\beta} = \deg g_{\alpha\beta}$.

Excercise 13.3. Use Claim 13.1 to prove Theorem 13.5.

13.4. **Cellular chain complex.** Let X be a CW-complex, and $X^{(q)}$ be its q-th skeleton. The factor-space $X^{(q)}/X^{(q-1)}$ is homeomorphic to the wedge $\bigvee_{i\in E_q} S^q_i$, where E_q is the set of q-cells of X. It implies that

$$H_j(X^{(q)},X^{(q-1)}) = \begin{cases} \bigoplus_{i\in E_n} \mathbf{Z}(i) & \text{if } j=q, \\ 0 & \text{else.} \end{cases} \tag{63}$$

We define the cellular chain complex $\mathcal{E}(X)$ as follows. Let $\mathcal{E}_q(X) = H_q(X^{(q)},X^{(q-1)}) = \bigoplus_{i\in E_n} \mathbf{Z}(i)$. The boundary operator $\tilde{\partial}_q : \mathcal{E}_q(X) \longrightarrow \mathcal{E}_{q-1}(X)$ is the boundary homomorphism in the long exact sequnce for the triple $(X^{(q)},X^{(q-1)},X^{(q-2)})$:

$$\begin{tikzcd}
\cdots \to H_q(X^{(q)},X^{(q-2)}) \arrow[r] & H_q(X^{(q)},X^{(q-1)}) \arrow[r, "\partial_*"] & H_q(X^{(q-1)},X^{(q-2)}) \to \cdots \\
& \mathcal{E}_q(X) \arrow[u, "\cong"] \arrow[r, "\tilde{\partial}_q"] & \mathcal{E}_{q-1}(X) \arrow[u, "\cong"]
\end{tikzcd}$$

The following result implies that $\mathcal{E}(X)$ is a chain complex.

Claim 13.2. *The composition* $\mathcal{E}_{q+1}(X) \xrightarrow{\tilde{\partial}_{q+1}} \mathcal{E}_q(X) \xrightarrow{\tilde{\partial}_q} \mathcal{E}_{q-1}(X)$ *is zero.*

Proof. We have the following commutative diagram of pairs:

$$\begin{tikzcd}
(X^{(q)},X^{(q-2)}) \arrow[r] \arrow[d, "\alpha"] & (X^{(q+1)},X^{(q-2)}) \arrow[r] \arrow[d, "\gamma"] & (X^{(q+1)},X^{(q)}) \arrow[d, "Id"] \\
(X^{(q)},X^{(q-1)}) \arrow[r] & (X^{(q+1)},X^{(q-1)}) \arrow[r] & (X^{(q+1)},X^{(q)})
\end{tikzcd}$$

This diagram gives the following commutative diagram in homology groups:

$$
\begin{tikzcd}
 & & \vdots \arrow[d] & \\
H_{q+1}(X^{(q+1)}, X^{(q-2)}) \arrow[r] \arrow[d, "\gamma_*"] & H_{q+1}(X^{(q+1)}, X^{(q)}) \arrow[r, "\partial_*"] \arrow[d, "Id"] \arrow[dr, "\eth_{q+1}"] & H_q(X^{(q)}, X^{(q-2)}) \arrow[r] \arrow[d, "\alpha_*"'] & \cdots \\
H_{q+1}(X^{(q+1)}, X^{(q-1)}) \arrow[r] & H_{q+1}(X^{(q+1)}, X^{(q)}) \arrow[r, "\partial_*"] & H_q(X^{(q)}, X^{(q-1)}) \arrow[r] \arrow[d, "\eth_q"'] & \cdots \\
 & & H_{q-1}(X^{(q-1)}, X^{(q-2)}) \arrow[d] & \\
 & & \vdots &
\end{tikzcd}
$$

Here the right column is the long exact sequence for the triple $(X^{(q)}, X^{(q-1)}, X^{(q-2)})$. The boundary operator $\eth_{q+1} = \alpha_* \circ \partial_*$ by commutativity. Thus $\eth_q \circ \eth_{q+1} = \eth_q \circ (\alpha_* \circ \partial_*) = (\eth_q \circ \alpha_*) \circ \partial_* = 0$ since $\eth_q \circ \alpha_* = 0$ by the exactness of the column. ☐

The chain complex $\mathcal{E}(X)$

$$\cdots \longrightarrow \mathcal{E}_q(X) \xrightarrow{\eth_q} \mathcal{E}_{q-1}(X) \longrightarrow \cdots \longrightarrow \mathcal{E}_1(X) \xrightarrow{\eth_1} \mathcal{E}_0(X) \longrightarrow 0$$

is the *cellular chain complex of* X.

Theorem 13.6. *There is an isomorphism* $H_q(\mathcal{E}(X)) \cong H_q(X)$ *for each* q *and any CW-complex* X.

Proof. We prove the following three isomorphisms:

- **(a)** $H_q(\mathcal{E}(X)) \cong H_q(X^{(q+1)}, X^{(q-2)})$,
- **(b)** $H_q(X^{(q+1)}, X^{(q-2)}) \cong \widetilde{H}_q(X^{(q+1)})$,
- **(c)** $H_q(X^{(q+1)}) \cong H_q(X)$.

(a) Consider the following commutative diagram

$$\begin{tikzcd}
 & H_q(X^{(q-1)}, X^{(q-2)}) = 0 \arrow[d] & & \\
H_{q+1}(X^{(q+1)}, X^{(q)}) \arrow[r, "\partial_*"] \arrow[dr, "\overset{\wr}{\partial}_{q+1}"'] & H_q(X^{(q)}, X^{(q-2)}) \arrow[r, "\alpha"] \arrow[d, "\beta"] & H_q(X^{(q+1)}, X^{(q-2)}) \arrow[r] & H_q(X^{(q+1)}, X^{(q)}) = 0 \\
 & H_q(X^{(q)}, X^{(q-1)}) = \mathcal{E}_q(X) \arrow[d, "\overset{\wr}{\partial}_q"] & & \\
 & H_{q-1}(X^{(q-1)}, X^{(q-2)}) = \mathcal{E}_{q-1}(X) & &
\end{tikzcd}$$

The exactness at the term $H_q(X^{(q)}, X^{(q-2)})$ implies

$$H_q(X^{(q+1)}, X^{(q-2)}) \cong H_q(X^{(q)}, X^{(q-2)})/\mathrm{Ker}\ \alpha \cong H_q(X^{(q)}, X^{(q-2)})/\mathrm{Im}\ \partial_*.$$

The homomorphism β is monomorphism since $H_q(X^{(q-1)}, X^{(q-2)}) = 0$. Thus

$$H_q(X^{(q)}, X^{(q-2)})/\mathrm{Im}\ \partial_* \cong \beta(H_q(X^{(q)}, X^{(q-2)}))/\beta(\mathrm{Im}\ \partial_*) \cong$$

$$\mathrm{Im}\ \beta/\mathrm{Im}\ (\beta \circ \partial_*) \cong \mathrm{Ker}\ \overset{\wr}{\partial}_q/\mathrm{Im}\ \overset{\wr}{\partial}_{q+1} = H_q(\mathcal{E}(X)).$$

(b) Consider long exact sequence for the triple $(X^{(q+1)}, X^{(i)}, X^{(i-1)})$ where $i = q-2,\ q-3, \ldots, 1, 0$:

$$0 = H_q(X^{(i)}, X^{(i-1)}) \longrightarrow H_q(X^{(q+1)}, X^{(i-1)}) \xrightarrow{\cong} H_q(X^{(q+1)}, X^{(i)}) \longrightarrow H_{q-1}(X^{(i)}, X^{(i-1)}) = 0$$

Thus we obtain the isomorphisms:

$$H_q(X^{(q+1)}, X^{(q-2)}) \cong H_q(X^{(q+1)}, X^{(q-3)}) \cong \cdots \cong H_q(X^{(q+1)}, X^{(0)}) \cong \widetilde{H}_q(X^{(q+1)}).$$

(c) Consider long exact sequence in homology for the pair $(X^{(j)}, X^{(q+1)})$ for $j = q+2,\ q+3, \ldots$:

$$0 = H_{q+1}(X^{(j)}, X^{(q+1)}) \longrightarrow H_q(X^{(q+1)}) \xrightarrow{\cong} H_q(X^{(j)}) \longrightarrow H_q(X^{(j)}, X^{(q+1)}) = 0.$$

Thus $H_q(X^{(q+1)}) \cong H_q(X^{(q+2)}) \cong \cdots \cong H_q(X)$. $\square$

13.5. **Geometric meaning of the boundary homomorphism $\overset{\wr}{\partial}_q$.** Consider closely the groups $\mathcal{E}_q(X)$ and the boundary operator $\overset{\wr}{\partial}_q : \mathcal{E}_q(X) \longrightarrow \mathcal{E}_{q-1}(X)$. First we recall that

$$\mathcal{E}_q(X) = \left\{ \textstyle\sum_{i \in E_q} \lambda_i e_i^q \right\}$$

where e_i^q are the q-th cells of the CW-complex X. The isomorphism of the group $\mathcal{E}_q(X)$ with free abelian group is not unique: it depends on the choice of the homeomorphism $X^{(q)}/X^{(q-1)} \cong \bigvee S^q$. The choice of this homeomorphism is detemined by the characteristic maps

$$\begin{tikzcd} D^q \arrow[r, "\Phi_i"] & X^{(q)} \\ S^{q-1} \arrow[u] \arrow[r, "\varphi_i"] & X^{(q-1)} \arrow[u] \end{tikzcd}$$

The map of pairs $(\Phi_i, \varphi_i) : (D^q, S^{q-1}) \to (X^{(q)}, X^{(q-1)})$ induces the homeomorphism

$$\bar{\Phi}_i : S^q = D^q/D^{q-1} \longrightarrow \bar{e}^q/\partial e^q \subset X^{(q)}/X^{(q-1)}.$$

Definition 13.7. We say that two characteristic maps $\Phi_i, \Phi_i' : (D^q, S^{q-1}) \longrightarrow (X^{(q)}, X^{(q-1)})$ are *of the same orientation* if the composition (which is a homeomorphism)

$$S^q \xrightarrow{\bar{\Phi}_i} \bar{e}^q/\partial e^q \xrightarrow{(\bar{\Phi}_i')^{-1}} S^q$$

has degree one. It means that the map $\bar{\Phi}_i \circ (\bar{\Phi}_i')^{-1}$ is homotopic to the identity map. If the degree of the map $\bar{\Phi}_i \circ (\bar{\Phi}_i')^{-1}$ is -1, the characteristic maps Φ_i, Φ_i' have the *opposite orientation.*

Thus the group

$$\mathcal{E}_q(X) = \left\{ \textstyle\sum_{i \in E_q} \lambda_i e_i^q \right\}$$

should be thought as a free abelian group with *oriented q-cells* as generators.

Let e_i^q be a q-cell of X, and σ_j^{q-1} be a $(q-1)$-cell. The attaching map $\varphi_i : S^{q-1} \longrightarrow X^{(q-1)}$ defines the map

$$\psi : S^{q-1} \xrightarrow{\varphi_i} X^{(q-1)} \xrightarrow{\text{projection}} X^{(q-1)}/(X^{(q-2)} \cup \left\{ \text{all } (q-1)\text{-cells except } \sigma_j^{q-1} \right\}) =$$

$$\bar{\sigma}_j^{q-1}/\partial \sigma_j^{q-1} \xrightarrow{(\bar{\Phi}_j)^{-1}} S^{q-1}.$$

Let $\deg \psi = [e_i^q : \sigma_j^{q-1}]$.

Remark. The number $[e_i^q : \sigma_j^{q-1}]$ depends on the choice of characteristic maps for the cells e_i^q, σ_j^{q-1} only through the orientation. It is easy to see that the number will change the sign if the orientation of either cell e_i^q or σ_j^{q-1} would be different. It is important to notice that $[e_i^q : \sigma_j^{q-1}] = 0$ if the cells e_i^q, σ_j^{q-1} do not intersect. Thus the number $[e_i^q : \sigma_j^{q-1}] \neq 0$ only for finite number of cells σ_j^{q-1}.

Theorem 13.8. *The boundary operator $\tilde{\partial}_q : \mathcal{E}_q(X) \longrightarrow \mathcal{E}_{q-1}(X)$ is given by the formula:*

(64) $$\tilde{\partial}_q(e^q) = \sum_{j \in E_{q-1}} [e^q : \sigma_j^{q-1}] \sigma_j^{q-1}.$$

Proof. Let $\bar{\Phi} : (D^q, S^{q-1}, \emptyset) \longrightarrow (X^{(q)}, X^{(q-1)}, X^{(q-2)})$ be the map determined by the characteristic map of the cell e^q:

$$\begin{tikzcd}
D^q \arrow[r, "\Phi"] & X^{(q)} \\
S^{q-1} \arrow[u] \arrow[r, "\varphi"] & X^{(q-1)} \arrow[u]
\end{tikzcd}$$

We have the following commutative diagram in homology groups:

$$\begin{tikzcd}
 & \mathbf{Z} \arrow[r, "\cong"] & H_q(D^q, S^{q-1}) \arrow[r, "\partial_*"] \arrow[d, "\alpha"] & H_{q-1}(S^{q-1}) \arrow[d, "\beta"] & \mathbf{Z} \arrow[l, "\cong"'] \\
\widetilde{H}_q(X^{(q)}/X^{(q-1)}) \cong & & H_q(X^{(q)}, X^{(q-1)}) \arrow[r, "\partial"] \arrow[d, "\cong"] & H_{q-1}(X^{(q-1)}, X^{(q-2)}) \arrow[d, "\cong"] & \cong \widetilde{H}_{q-1}(X^{(q-1)}/X^{(q-2)}) \\
 & & \mathcal{E}_q(X) \arrow[r, "\tilde{\partial}_q"] & \mathcal{E}_{q-1}(X) &
\end{tikzcd}$$

Here α and β are induced by the map $\bar{\Phi}$ and $\alpha(1)=e^q$, $\beta(1)=\overset{\flat}{\partial}\, e^q$. Consider the composition

$$\begin{aligned}\gamma : H_{q-1}(S^{q-1}) &\xrightarrow{\beta} H_{q-1}(X^{(q-1)},X^{(q-2)}) = \widetilde{H}_{q-1}(X^{(q-1)}/X^{(q-2)}) = H_{q-1}(\bigvee_j S_j^{q-1}) \\ &= \bigoplus_j H_{q-1}(S_j^{q-1}) \xrightarrow{\text{projection}} H_{q-1}(S^{q-1}_{\sigma_j^{q-1}}).\end{aligned}$$

By definition the degree of the homomorphism γ the coefficient with σ_j^{q-1} in $\overset{\flat}{\partial}\, e^q$ is equal to $[e^q : \sigma_j^{q-1}]$. Thus

$$\overset{\flat}{\partial}_q\ e^q = \sum_j [e^q:\sigma_j^{q-1}]\sigma_j^{q-1}. \qquad \square$$

13.6. **Some computations.** First we compute again the homology groups S^n. We choose the standard cell decomposition: $S^n = e^n \cup e^0$.

$n=1$. Here we have $\mathcal{E}_0 = \mathbf{Z}$ with the generator e^0, $\mathcal{E}_1 = \mathbf{Z}$ with the generator e^1, $\mathcal{E}_q = 0$ if $q\neq 0,1$. Clearly $\partial e^1 = e^0 - e^0 = 0$.

(1) $n>1$. Here we have $\mathcal{E}_0 = \mathbf{Z}$ with the generator e^0, $\mathcal{E}_n = \mathbf{Z}$ with the generator e^n, $\mathcal{E}_q = 0$ if $q\neq 0,n$. Clearly $\overset{\flat}{\partial}_q e^n = 0$.

Thus in both cases we have that $\widetilde{H}_n(S^n) = \mathbf{Z}$, and $\widetilde{H}_q(S^n)=0$ if $q\neq n$.

13.7. **Homology groups of $\mathbf{RP}^n$.** Here we have to work a bit harder. We need the following geometric fact. Let the sphere $S^n \subset \mathbf{R}^{n+1}$ is given by the equation $x_1^2+\cdots+x_{n+1}^2=1$.

Lemma 13.9. *Let $A : S^n \longrightarrow S^n$ be the antipodal map, $A : x\mapsto -x$, and $\iota_n \in \pi_n(S^n)$ be the generator represented by the identity map $S^n \longrightarrow S^n$. Then the homotopy class $[A]\in\pi_n(S^n)$ is equal to*

$$[A] = \begin{cases} \iota_n, & \text{if } n \text{ is odd,} \\ -\iota_n, & \text{if } n \text{ is even.}\end{cases}$$

Excercise 13.4. Prove Lemma 13.9.

Let $e^0,\dots,e^n$ be the cells in the standard cell decomposition of $\mathbf{RP}^n$. Recall that $(\mathbf{RP}^n)^{(q)} = \mathbf{RP}^q$, and $e^q = \mathbf{RP}^q/\mathbf{RP}^{q-1}$, and that the Hopf map $S^{q-1}\longrightarrow \mathbf{RP}^{q-1}$ is the attaching map of the cell e^q.

Lemma 13.10. *Let $e^0,\dots,e^n$ be the cells in the standard cell decomposition of $\mathbf{RP}^n$. Then*

$$[e^q : e^{q-1}] = \begin{cases} 2 & \text{if } q \text{ is odd,} \\ 0, & \text{if } q \text{ is even.}\end{cases}$$

Proof. Let $h : S^{q-1} \longrightarrow \mathbf{RP}^{q-1}$ be the Hopf map. We identify $\mathbf{RP}^{q-1}$ with the sphere S^{q-1} where the points $x, -x \in S^{q-1}$ are identified. The projective space $\mathbf{RP}^{q-2} \subset \mathbf{RP}^{q-1}$ is then the equator sphere $S^{q-2} \subset S^{q-1}$ with the intipodal points identified as well. Now the composition

$$S^{q-1} \xrightarrow{h} \mathbf{RP}^{q-1} \longrightarrow \mathbf{RP}^{q-1}/\mathbf{RP}^{q-2} = S^{q-1}$$

represents the element $[A] + \iota_{q-1} \in \pi_{q-1}(S^{q-1})$, see Fig. 13.3. Lemma 13.9 implies the desired formula. $\square$

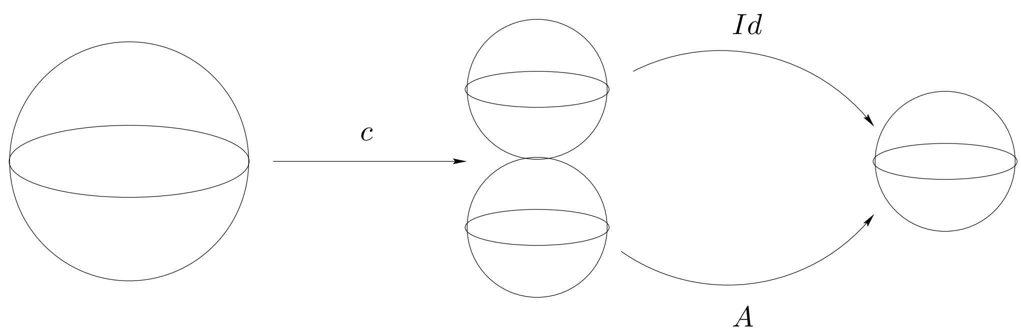


Fig. 13.3.

Thus the chain complex $\mathcal{E}(\mathbf{RP}^{2k+1})$ is the following one

$$0 \longrightarrow \overset{2k+1}{\mathbf{Z}} \xrightarrow{0} \overset{2k}{\mathbf{Z}} \xrightarrow{\cdot 2} \overset{2k-1}{\mathbf{Z}} \xrightarrow{0} \cdots \xrightarrow{0} \overset{2}{\mathbf{Z}} \xrightarrow{\cdot 2} \overset{1}{\mathbf{Z}} \xrightarrow{0} \overset{0}{\mathbf{Z}}$$

Hence we have

$$\widetilde{H}_q(\mathbf{RP}^{2k+1}) = \begin{cases} \mathbf{Z}/2, & q = 1, 3, \ldots, 2k-1, \\ \mathbf{Z}, & q = 2k+1, \\ 0, & \text{else.} \end{cases}$$

The chain complex $\mathcal{E}(\mathbf{RP}^{2k})$ is the following one

$$0 \longrightarrow \overset{2k}{\mathbf{Z}} \xrightarrow{\cdot 2} \overset{2k-1}{\mathbf{Z}} \xrightarrow{0} \overset{2k-1}{\mathbf{Z}} \xrightarrow{0} \cdots \xrightarrow{0} \overset{2}{\mathbf{Z}} \xrightarrow{\cdot 2} \overset{1}{\mathbf{Z}} \xrightarrow{0} \overset{0}{\mathbf{Z}}$$

Thus we have

$$\widetilde{H}_q(\mathbf{RP}^{2k}) = \begin{cases} \mathbf{Z}/2, & q = 1, 3, \ldots, 2k-1, \\ 0, & \text{else.} \end{cases}$$

13.8. **Homology groups of $\mathbf{CP}^n$, $\mathbf{HP}^n$.** These groups are very easy to compute since $\mathcal{E}_{2q}(\mathbf{CP}^n) = \mathbf{Z}$, $q = 0, 2, \ldots, 2n$, and $\mathcal{E}_{2q+1}(\mathbf{CP}^n) = 0$. Similarly $\mathcal{E}_{4q}(\mathbf{HP}^n) = \mathbf{Z}$, $q = 0, 4, \ldots, 4n$, and $\mathcal{E}_q(\mathbf{HP}^n) = 0$ for all other q. Thus

$$H_q(\mathbf{CP}^n) = \begin{cases} \mathbf{Z}, & q = 0, 2, \ldots, 2n, \\ 0, & \text{else.} \end{cases}, \quad H_q(\mathbf{HP}^n) = \begin{cases} \mathbf{Z}, & q = 0, 4, \ldots, 4n, \\ 0, & \text{else.} \end{cases}$$

Exercise 13.5. Prove that there is no map $f : D^n \longrightarrow S^{n-1}$ so that the restriction

$$f|_{S^{n-1}} : S^{n-1} \longrightarrow S^{n-1}$$

has nonzero degree.

Theorem 13.11. (Brouwer Fixed Point Theorem) *Let $g : D^n \longrightarrow D^n$ be a continious map. Then there exists a fixed point of g, i.e. such $x \in D^n$ that $g(x) = x$.*

Exercise 13.6. Use Exercise 13.5 to prove Theorem 13.11.

Exercise 13.7. Let $M_g = T^2\# \cdots \#T^2$ (g times). Compute the following homology groups:

(a) $H_q(M_g)$,
(b) $H_q(M_g\#\mathbf{RP}^2)$,
(c) $H_q(M_g\#Kl^2)$.

Exercise 13.8. Let $G(n,k)$ be the real Grassmannian manifold. Use the CW-decomposition of $G(n,k)$ given in Section 4 to compute the homology groups:

(a) $H_q(G(4,2))$,
(b) $H_q(G(5,3))$.

Exercise 13.9. Compute the homology groups:

(a) $H_q(\mathbf{RP}^2 \times \mathbf{RP}^3)$,
(b) $H_q(\mathbf{RP}^5 \times \mathbf{RP}^3)$,
(c) $H_q(\mathbf{RP}^2 \times \mathbf{RP}^4)$.

Exercise 13.10. Let $f(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a complex polynomial with $a_n \neq 0$. Show that a polynomial $f(z)$, viewed as a map $f : \mathbf{C} \to \mathbf{C}$, can be always extended to a continious map $\hat{f} : S^2 \to S^2$. Prove that the degree of the map $\hat{f}$ equals to n.

Exercise 13.11. Construct a map $f : S^{2n-1} \to S^{2n-1}$ without fixed points.

Exercise 13.12. Let $f, g : S^n \to S^n$ be two maps. Assume that $f(x) \neq -g(x)$ for all $x \in S^n$. Prove that $f \sim g$.

Exercise 13.13. Let $f : S^n \to S^n$ be a map with $\deg f \neq (-1)^{n+1}$. Then there exists $x \in S^n$ with $f(x) = x$.

Exercise 13.14. Let $f : S^{2n} \to S^{2n}$ be a map. Prove that there exists a point $x \in S^{2n}$ such that either $f(x) = x$ or $f(x) = -x$.

Exercise 13.15. Let $f : S^n \to S^n$ be a map of degree zero. Prove that there exist two points $x, y \in S^n$ with $f(x) = x$ and $f(y) = -y$.

Exercise 13.16. Construct a surjective map $f : S^n \to S^n$ of degree zero.

14. Homology and homotopy groups

14.1. **Homology groups and weak homotopy equivalence.** Our goal here is to prove the following fact.

Theorem 14.1. *Let $f: X \longrightarrow Y$ be a weak homotopy equivalence. Then the induced homomorphism $f_*: H_q(X) \longrightarrow H_q(Y)$ is an isomorphism for all $q \geq 0$.*

We start with a preliminary lemma.

Lemma 14.2. *Let X be a topological space, $\alpha \in H_q(X)$. Then there exist a CW-complex K, a map $f: K \longrightarrow X$, an element $\beta \in H_q(K)$ such that $f_*(\beta) = \alpha$.*

Proof. Let $c = \sum_i \lambda_i f_i$, $f_i : \Delta^q_i \longrightarrow X$, be a chain representing $\alpha \in H_q(X)$. Consider the space

$$K' = \bigsqcup_i \Delta^q_i.$$

Recall that the simplex $\Delta^q \subset \mathbf{R}^{q+1}$ is given by the vertices $\Delta^q = (v_0, \ldots, v_q)$, where $v_0 = (1, 0, \ldots, 0)$, $\ldots$, $v_q = (0, \ldots, 0, 1)$. We can describe all subsimplices of Δ^q as follows. Let $0 \leq t_1 < \cdots < t_{q-r} \leq q$. Then

$$\Gamma^r_{t_1,\ldots,t_{q-r}}(\Delta^q) = (v_0, \ldots, \widehat{v}_{t_1}, \ldots, \widehat{v}_{t_{q-r}}, \ldots, v_q)$$

is an r-dimensional simplex with the vertices $(v_0, \ldots, \widehat{v}_{t_1}, \ldots, \widehat{v}_{t_{q-r}}, \ldots, v_q)$. We introduce the following equivalence relation in K:

$$\Gamma^r_{t_1,\ldots,t_{q-r}}(\Delta^q_i) \equiv \Gamma^r_{s_1,\ldots,s_{q-r}}(\Delta^q_j) \quad \text{iff} \quad f_i|_{\Gamma^r_{t_1,\ldots,t_{q-r}}(\Delta^q_i)} = f_j|_{\Gamma^r_{s_1,\ldots,s_{q-r}}(\Delta^q_j)}.$$

Let $K = K'/\sim$. The maps $f_i : \Delta^q_i \longrightarrow X$ determine a map $f: K \longrightarrow X$. Furthermore, let

$$g_j : \Delta^q_j \xrightarrow{\text{inclusion}} K' \xrightarrow{\text{projection}} K.$$

Then the chain $\bar{c} = \sum_j \lambda_j g_j \in C_q(K)$ maps to the chain c by construction. One has to notice that $\bar{c}$ is a cycle since c is a cycle. Then $\beta = [\bar{c}]$ maps to α under the induced homomorphism f_*. $\square$

There is the relative version of Lemma 14.2 which may be proved by slight modification of the above proof.

Lemma 14.3. *Let (X, A) be a pair of topological spaces, $\alpha \in H_q(X, A)$. The there exist a CW-pair (K, L), a map $f: (K, L) \longrightarrow (X, A)$, and element $\beta \in H_q(K, L)$, such that $f_*(\beta) = \alpha$.*

Exercise 14.1. Prove Lemma 14.3.

Proof of Theorem 14.1. Recall that $f: X \longrightarrow Y$ is a weak homotopy equivalence if for any CW-complex Z, the induced map $f_\# : [Z, X] \longrightarrow [Z, Y]$ is a bijection.

(1): f_* is an epimorphism. Let $\alpha \in H_q(Y)$. Then by Lemma 14.2 there exists a CW-complex K, a map $g : K \longrightarrow Y$ such that $g_*(\beta) = \alpha$. Consider a map $h \in f_{\#}^{-1}([g]) \in [K, X]$. We have the following diagram

$$\begin{tikzcd}
X \arrow[rr, "f"] & & Y \\
& & K \arrow[u, "g"'] \arrow[ull, "h"]
\end{tikzcd}$$

which commutes up to homotopy. Now we obtain a commutative diagram:

$$\begin{tikzcd}
H_q(X) \arrow[rr, "f_*"] & & H_q(Y) \\
& & H_q(K) \arrow[u, "g"'] \arrow[ull, "h_*"]
\end{tikzcd}$$

Thus $f_*(h_*(\beta)) = \alpha$.

(2): f_* is a monomorphism. First we can change the map $f : X \longrightarrow Y$ to a homotopy equivalent map $f' : X' \longrightarrow Y'$, so that f' is an embedding. Thus we assume that $X \subset Y$ and $f = i : X \subset Y$ is an embedding. Let $\alpha \in H_q(X)$, and $i_*(\alpha) = 0$. Consider the long exact sequence in homology groups:

$$\cdots \to H_{q+1}(Y, X) \xrightarrow{\partial_*} H_q(X) \xrightarrow{i_*} H_q(Y) \to \cdots$$

The exactness implies that there is $\gamma \in H_{q+1}(Y, X)$ such that $\partial_*(\gamma) = \alpha$. By Lemma 14.3 there exist a pair (K, L), a map $g : (K, L) \longrightarrow (Y, X)$, and $\beta \in H_{q+1}(K, L)$ such that $g_*(\beta) = \gamma$. Then since $f : X \longrightarrow Y$ is a weak homotopy equivalence, there exists a map $h : K \longrightarrow X$ making the diagram

$$\begin{tikzcd}
X \arrow[r, "f"] & Y \\
L \arrow[u, "g|_L"] \arrow[r, "i"] & K \arrow[u, "g"'] \arrow[ul, "h"']
\end{tikzcd}$$

is commutative up to homotopy. Furthermore, by Theorem 11.7 the map $h : K \longrightarrow Y$ may be chosen so that $(i \circ h)|_L = g|_L$. We have the commutative diagram

$$\begin{tikzcd}
H_{q+1}(Y, X) \arrow[r, "\partial_*"] & H_q(X) \arrow[r, "i_*"] & H_q(Y) \arrow[r] & \cdots \\
H_{q+1}(K, L) \arrow[u, "g_*"] \arrow[r, "\partial_*"] & H_q(L) \arrow[u, "(g|_L)_* = ((i \circ h)|_L)_*"'] \arrow[r, "i_*"] & H_q(K) \arrow[u, "g_*"'] \arrow[ul, "h_*"'] \arrow[r] & \cdots
\end{tikzcd}$$

Thus we have that $\alpha = \partial_*(\gamma) = \partial_*(g_*(\beta)) = (g|_L)_*(\partial_*(\gamma) = h_*(i_*(\partial_*(\beta))) = 0$ because of the exactness. Theorem 14.1 is proved. $\square$

Recall that we proved (Theorem 11.7) that a map $f : X \longrightarrow Y$ is weak homotopy equivalence if the induced homomorphism $f_* : \pi_q(X, x_0) \longrightarrow \pi_q(Y, f(x_0))$ is an isomorphism for all $q \geq 0$ and $x_0 \in X$. We reformulate Theorem 14.1:

Theorem 14.4. *Let a map $f : X \longrightarrow Y$ induce an isomorphism in homotopy groups $f_* : \pi_q(X, x_0) \longrightarrow \pi_q(Y, f(x_0))$ for all $q \geq 0$ and $x_0 \in X$. Then f induces isomorphism in the homology groups $f_* : H_q(X) \longrightarrow H_q(Y)$ for all $q \geq 0$.*

The following exercises show that some naive generalizations of Theorem 14.4 fail.

Exercise 14.2. Show that the spaces $\mathbf{CP}^\infty \times S^3$ and S^2 have isomorphic homotopy groups and different homology groups. Thus these spaces are not homotopy equivalent.

Exercise 14.3. Show that the spaces $\mathbf{RP}^n \times S^m$ and $S^n \times \mathbf{RP}^m$ $(n \neq m,\ m, n \geq 2)$ have isomorphic homotopy groups and different homology groups.

Exercise 14.4. Show that the spaces $S^1 \vee S^1 \vee S^2$ and $S^1 \times S^1$ have the same homology groups and different homotopy groups.

Exercise 14.5. Show that the Hopf map $h : S^3 \longrightarrow S^2$ induces trivial homomorphism in reduced homology groups, and nontrivial homomorphism in homotopy groups.

Exercise 14.6. Show that the projection

$$S^1 \times S^1 \xrightarrow{\text{projection}} (S^1 \times S^1)/(S^1 \vee S^1) = S^2$$

induces trivial homomorphism in homotopy groups, and nontrivial homomorphism in homology groups.

14.2. **Hurewicz homomorphism.** Let X be a topological space with a base point $x_0 \in X$. Let s_n be a canonical generator of $H_n(S^n)$, $n = 1, 2, \ldots$, given by the homeomorphism $\partial\Delta^{n+1} \xrightarrow{\cong} S^n$. For any element $\alpha \in \pi_n(X, x_0)$ consider a representative $f : S^n \longrightarrow X$, $[f] = \alpha$. We have the induced homomorphism $f_* : H_n(S^n) \longrightarrow H_n(X)$. Let

$$\mathsf{h}(\alpha) = f_*(s_n) \in H_n(X).$$

Clearly the element $h(\alpha) \in H_n(X)$ does not depend on the choice of the representing map f. Furthermore, the correspondence $\alpha \mapsto \mathsf{h}(\alpha)$ determines the homomorphism

$$\mathsf{h} : \pi_n(X, x_0) \longrightarrow H_n(X), \quad n = 1, 2, \ldots .$$

The homomorphism h is the *Hurewicz homomorphism*. The Hurewicz homomorphism is natural with respect to maps $(X, x_0) \longrightarrow (Y, y_0)$ of based spaces.

Exercise 14.7. Prove that $\mathsf{h} : \pi_n(X, x_0) \longrightarrow H_n(X)$ is a homomorphism.

Exercise 14.8. Let $x_0, x_1 \in X$, and $\gamma : I \longrightarrow X$ be a path connecting the points x_0, x_1: $\gamma(0) = x_0$, and $\gamma(1) = x_1$. The path γ determines the isomorphism

$$\gamma_{\#} : \pi_n(X, x_0) \longrightarrow \pi_n(X, x_1).$$

Prove that the following diagram commutes:

$$\begin{tikzcd}
\pi_n(X, x_0) \arrow[rr, "\gamma_{\#}"] \arrow[dr, "\mathsf{h}"'] & & \pi_n(X, x_1) \arrow[dl, "\mathsf{h}"] \\
& H_n(X) &
\end{tikzcd}$$

Theorem 14.5. (Hurewicz) *Let (X, x_0) be a based space, such that*

$$\pi_0(X, x_0) = 0, \ \pi_1(X, x_0) = 0, \cdots, \pi_{n-1}(X, x_0) = 0, \tag{65}$$

where $n \geq 2$. Then

$$H_1(X) = 0, \quad H_2(X) = 0, \cdots, H_{n-1}(X) = 0,$$

and the Hurewicz homomorphism $\mathsf{h} : \pi_n(X, x_0) \longrightarrow H_n(X)$ is an isomorphism.

Proof. By Theorem 11.10 there exist a CW-complex K and a weak homotopy equivalence $f : K \longrightarrow X$. Theorem 14.1 guarantees that f induces an isomorphism in homology groups. Thus it is enough to prove the statement in the case when X is a CW-complex. Then the condition (65) means that X is $(n-1)$-connected CW-complex. Theorem 5.9 implies that up to homotopy equivalence X may be chosen so that it has a single zero-cell, and it does not have any cells of dimensions $1, 2, \ldots, n-1$. In particular, this implies that $H_1(X) = 0$, $H_2(X) = 0$, $\cdots$, $H_{n-1}(X) = 0$. Now the n-th skeleton of X is a wedge of spheres: $X^{(n)} = \bigvee_i S^n_i$. Let $g_i : S^n_i \longrightarrow \bigvee_i S^n_i$ be the embedding of the i-th sphere, and let $r_j : S^n \longrightarrow \bigvee_i S^n_i$ be the attaching maps of the $(n+1)$-cells e^{n+1}_j. The maps g_i determine the generators of the group $\pi_n(X^{(n)})$, and let $\rho_j \in \pi_n(X^{(n)})$ be the elements determined by the maps r_j.

Theorem 11.6 describes the first nontrivial homotopy group $\pi_n(X, x_0)$ as a factor-group of the homotopy group $\pi_n(X^{(n)}) \cong \mathbf{Z} \oplus \cdots \oplus \mathbf{Z}$ by the subgroup generated by ρ_j. Notice that the cellular chain group

$$\mathcal{E}_n(X) = H_n(X^{(n)}) = H_n\left(\bigvee_i S^n_i\right),$$

and $H_n(X) = \mathcal{E}_n(X)/\mathrm{Im}\ \overset{b}{\partial}_{n+1}$. Finally we notice that the Hurewicz homomorphism $h : \pi_n(S^n) \longrightarrow H_n(S^n)$ is an isomorphism. Thus we have the commutative diagram

$$\begin{tikzcd}
\pi_n\left(\bigvee S_i^n\right) \arrow[r, "h"] & H_n\left(\bigvee S_i^n\right) \\
\pi_n(S_j^n) \arrow[u, "(\rho_j)_*"] \arrow[r, "h"] & H_n(S_j^n) \arrow[u, "(\rho_j)_*"']
\end{tikzcd}$$

where the horizontal homomorphisms are isomorphisms. Hence h induces an isomorphism $\pi_n(X, x_0) \longrightarrow H_n(X)$. □

Corollary 14.6. *Let X be a simply connected space, and $H_1(X) = 0$, $H_2(X) = 0$ $\cdots$ $H_{n-1}(X) = 0$. Then $\pi_1(X) = 0$, $\pi_2(X) = 0$ $\cdots$ $\pi_{n-1}(X) = 0$ and the Hurewicz homomorphism $h : \pi_n(X, x_0) \longrightarrow H_n(X)$ is an isomorphism.*

Corollary 14.7. *Let X be a simply-connected CW-complex with $\widetilde{H}_n(X) = 0$ for all n. Then X is contractible.*

Exercise 14.9. Prove Corollary 14.6.

Exercise 14.10. Prove Corollary 14.7.

Remark. Let X be a CW-complex. The above results imply that if $\pi_q(X, x_0) = 0$ for all $q \geq 0$ or $H_q(X) = 0$ for all $q \geq 0$, then X is homotopy equivalent to a point. However for a given map $f : X \longrightarrow Y$ the fact that f induces trivial homomorphism in homotopy or homology groups does not imply that f is homotopic to a constant map. The following exercises show that even if f induces trivial homomorphism in both homotopy and homology groups, it does not imply that f is homotopic to a constant map.

Exercise 14.11. Consider the torus $X = S^1 \times S^1 \times S^1$. We give X an obvious product CW-structure. In particular, $X^{(1)} = S^1 \vee S^1 \vee S^1$. Consider the map

$$f : S^1 \times S^1 \times S^1 \xrightarrow{\text{projection}} (S^1 \times S^1 \times S^1)/(S^1 \times S^1 \times S^1)^{(2)} = S^3 \xrightarrow{\text{Hopf}} S^2.$$

Prove that f induces trivial homomorphism in homology and homotopy groups, however f is not homotopic to a constant map.

Exercise 14.12. Consider the map

$$g : S^{2n-2} \times S^3 \xrightarrow{\text{projection}} (S^{2n-2} \times S^3)/(S^{2n-2} \vee S^3) = S^{2n+1} \xrightarrow{\text{Hopf}} \mathbf{CP}^n.$$

Prove that g induces trivial homomorphism in homology and homotopy groups, however g is not homotopic to a constant map.

14.3. **Hurewicz homomorphism in the case** $n = 1$**.**

Theorem 14.8. (Poincarè) *Let* X *be a connected space. Then the Hurewicz homomorphism* $h : \pi_1(X, x_0) \longrightarrow H_1(X)$ *is epimorphism, and the kernel of* h *is the commutator* $[\pi_1(X, x_0), \pi_1(X, x_0)] \subset \pi_1(X, x_0)$*. Thus* $H_1(X) \cong \pi_1(X, x_0)/[\pi_1(X, x_0), \pi_1(X, x_0)]$.

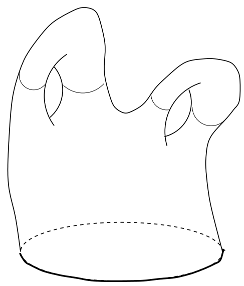


Fig. 14.1.

Exercise 14.13. Prove Theorem 14.8.

Exercise 14.14. We say that a map $f : S^1 \longrightarrow X$ is cobordant to zero if there is an oriented surface M with boundary $\partial M = S^1$ and a map $F : M \longrightarrow X$ such that $F|_{\partial M} = f$, see Fig. 14.1. Let $f : S^1 \longrightarrow X$ be a map representing an element $\alpha \in \pi_1(X, x_0)$. Prove that $\alpha \in \operatorname{Ker} h$ if and only if the map $f : S^1 \longrightarrow X$ is cobordant to zero.

Exercise 14.15. Let M^2_g be an oriented surface of genus g. As we know, $H_2(M^2_g) \cong \mathbf{Z}$, and a generator $s \in H_2(M^2_g)$ may be represented by the identity map $M^2_g \longrightarrow M^2_g$. Let X be a simply connected space. Prove that for any class $\alpha \in H_2(X)$ there exist a surface M^2_g, and a map $f : M^2_g \longrightarrow X$ such that $f_*(s) = \alpha$.

14.4. **Relative version of the Hurewicz Theorem.** One defines the relative Hurewicz homomorphism $h : \pi_n(X, A, x_0) \longrightarrow H_n(X, A)$ similarly to the regular Hurewicz homomorphism. Let $s \in H_n(D^n, S^{n-1})$ be a canonical generator given by the homeomorphism $(\Delta^n, \partial\Delta^n) \xrightarrow{\cong} (D^n, S^{n-1})$. Let $f : (D^n, S^{n-1}) \longrightarrow (X, A)$ be a map representing an element $\alpha \in \pi_n(X.A)$. Then $h(\alpha) = f_*(s_n) \in H_n(X, A)$. There is a relative version of the above Hurewics Theorem:

Theorem 14.9. *Let* (X, A) *be a pair of simply connected spaces,* $x_0 \in A$*, such that*

$$\pi_0(X, A, x_0) = 0, \ \pi_1(X, A, x_0) = 0, \cdots, \pi_{n-1}(X, A, x_0) = 0, \tag{66}$$

where $n \geq 2$*. Then*

$$H_1(X, A) = 0, \quad H_2(X, A) = 0, \cdots, H_{n-1}(X, A) = 0,$$

and the Hurewicz homomorphism $h : \pi_n(X, A, x_0) \longrightarrow H_n(X, A)$ *is an isomorphism.*

We do not give a proof of Theorem 14.9, however it is very similar to the proof of the above Hurewicz Theorem.

Exercise 14.16. Prove Theorem 14.9.

Theorem 14.10. (Whitehead Theorem-II) *Let* X, Y *be simply connected spaces and* $f : X \longrightarrow Y$ *be a map.*

(a) *If the induced homomorphism in homotopy groups* $f_*:\pi_q(X,x_0)\longrightarrow\pi_q(Y,f(x_0))$ *is isomorphism for* $q=2,3,\dots,n-1$, *and epimorphism if* $q=n$, *then the homomorphism in homology groups* $f_*:H_q(X)\longrightarrow H_q(Y)$ *is isomorphism for* $q=2,3,\dots,n-1$, *and epimorphism if* $q=n$.

(b) *If the induced homomorphism in homology groups* $f_*:H_q(X)\longrightarrow H_q(Y)$ *is isomorphism for* $q=2,3,\dots,n-1$, *and epimorphism if* $q=n$, *then the homomorphism in homotopy groups* $f_*:\pi_q(X,x_0)\longrightarrow\pi_q(Y,f(x_0))$ *is isomorphism for* $q=2,3,\dots,n-1$, *and epimorphism if* $q=n$.

Proof. **(a)** We can assume that $f:X\longrightarrow Y$ is an embedding, see Claim 9.1. Then the long exact sequence in homotopy

$$\cdots\to\pi_q(X,x_0)\xrightarrow{f_*}\pi_q(Y,x_0)\longrightarrow\pi_q(Y,X,x_0)\xrightarrow{\partial}\pi_{q-1}(X,x_0)\to\cdots \tag{67}$$

gives that if $f_*:\pi_q(X,x_0)\longrightarrow\pi_q(Y,f(x_0))$ is isomorphism for $q=1,2,3,\dots,n-1$, and epimorphism if $q=n$, then $\pi_q(Y,X,x_0)=0$ if $q\leq n$. Then Theorem 14.9 implies that $H_q(Y,X)=0$ for $q\leq n$. Then the long exact sequence in homology

$$\cdots\to H_q(X)\xrightarrow{f_*}H_q(Y)\longrightarrow H_q(Y,X)\longrightarrow H_{q-1}(X)\to\cdots \tag{68}$$

implies that $f_*:H_q(X)\to H_q(Y)$ is isomorphism for $q=1,2,3,\dots,n-1$, and epimorphism if $q=n$.

(b) Analogously, let $f_*:H_q(X)\to H_q(Y)$ be an isomorphism for $q=1,2,3,\dots,n-1$, and epimorphism if $q=n$. Then the long exact sequence in homology (68) implies that $H_q(Y,X,x_0)=0$ for $q\leq n$. Then again, Theorem 14.9 implies that $\pi_q(Y,X,x_0)=0$ for $q\leq n$ and the exact sequence (67) gives that $f_*:\pi_q(X,x_0)\longrightarrow\pi_q(Y,f(x_0))$ is isomorphism for $q=1,2,3,\dots,n-1$, and epimorphism if $q=n$. Thus in both cases the relative Hurewicz Theorem 14.9 implies the desired result. □

Corollary 14.11. *Let* X, Y *be simply connected spaces and* $f:X\longrightarrow Y$ *be a map which induces isomorphism* $f_*:H_q(X)\longrightarrow H_q(Y)$ *for all* $q\geq 0$. *Then* f *is weak homotopy equivalence. (In particular, if* X, Y *are* CW*-complexes, then* f *is homotopy equivalence.)*

Exercise 14.17. Let X be a connected, simply connected CW-complex with $\widetilde{H}_n(X)=\mathbf{Z}$, $n\geq 2$, and $\widetilde{H}_q(X)=0$ if $q\neq n$. Prove that X is homotopy equivalent to S^n.

15. Homology with coefficients and cohomology groups

Here we define homology and cohomology groups with coefficients in arbitrary abelian group G. We should be aware that these constructions could be done rather formally by means of basic homological algebra. We choose to avoid a total “algebraization” of those constructions: there are great classical books (say, by S. MacLane, Homology, Springer, 1967 or by S. Eilenberg & N. Steenrod, Foundations of Algebraic Topology) where you can find the most abstract algebraic approach concerning the homology and cohomology theories. We will describe only those algebraic constructions which are really necessary.

15.1. **Definitions.** Let G be an abelian group. A singular q-chain of a space X with coefficients in G is a linear combination

$$\sum_i \lambda_i f_i, \quad \text{where } \lambda_i \in G, \text{ and } f_i : \Delta^q \longrightarrow X \text{ is a singular simplex.}$$

We denote a group of q-chains $C_q(X;G)$. Clearly, $C_q(X;G) = C_q(X)\otimes G$. The boundary operator $\partial_q : C_q(X;G) \longrightarrow C_{q-1}(X;G)$ is induced by the regular boundary operator $\partial_q : C_q(X) \longrightarrow C_q(X)$. In more detail, recall that a simplex Δ^q is defined by the vertices $(v_0,\dots,v_q)$, and $\Gamma_j\Delta^q$ is the face of Δ^q given by the vertices $(v_0,\dots,\hat{v}_j,\dots,v_q)$. Then

$$\partial_q(f:\Delta^q \longrightarrow X) = \sum_{j=0}^{q}(-1)^j(f|_{\Gamma_j\Delta^q} : \Gamma_j\Delta^q \longrightarrow X).$$

Let $\mathcal{C}_*(X)$ be the singular chain complex:

$$\cdots \xrightarrow{\partial_{q+1}} C_q(X) \xrightarrow{\partial_q} C_{q-1}(X) \xrightarrow{\partial_{q-1}} \cdots \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \longrightarrow 0. \tag{69}$$

Then we have the chain complex $\mathcal{C}_*(X)\otimes G$:

$$\cdots \xrightarrow{\partial_{q+1}} C_q(X)\otimes G \xrightarrow{\partial_q} C_{q-1}(X)\otimes G \xrightarrow{\partial_{q-1}} \cdots \xrightarrow{\partial_2} C_1(X)\otimes G \xrightarrow{\partial_1} C_0(X)\otimes G \longrightarrow 0.$$

We define the homology groups with coefficients in G:

$$H_q(X;G) = H_q(\mathcal{C}_*(X)\otimes G) = \frac{\mathrm{Ker}(\partial_q : C_q(X)\otimes G \longrightarrow C_{q-1}(X)\otimes G)}{\mathrm{Im}(\partial_{q+1} : C_{q+1}(X)\otimes G \longrightarrow C_q(X)\otimes G)}.$$

Now we consider the *cochain complex* $\mathcal{C}^*(X;G) = \mathrm{Hom}(\mathcal{C}_*(X),G)$:

$$\cdots \xleftarrow{\delta^q} \mathrm{Hom}(C_q(X),G) \xleftarrow{\delta^{q-1}} \mathrm{Hom}(C_{q-1}(X),G) \xleftarrow{\delta^{q-2}} \cdots \xleftarrow{\delta^0} \mathrm{Hom}(C_0(X),G) \longleftarrow 0.$$

In other words, a cochain $\xi \in \mathrm{Hom}(C_q(X),G) = C^q(X;G)$ is a linear function on $C_q(X)$ with values in the group G, $\xi : C_q(X) \to G$.

It is convenient to denote $\xi(c) = \langle \xi, c\rangle \in G$. Notice that by definition, $\langle \delta^q\xi, a\rangle = \langle \xi, \partial_{q+1}a\rangle$, where $\xi \in C^q(X;G)$, and $a \in C_{q+1}(X)$. Clearly $\delta^{q+1}\delta^q = 0$ since

$$\langle \delta^{q+1}\delta^q\xi, a\rangle = \langle \delta^q\xi, \partial_{q+2}a\rangle = \langle \xi, \partial_{q+1}\partial_{q+2}a\rangle = 0.$$

We define the cohomology groups

$$H^q(X;G) = H^q(\mathcal{C}^*(X;G)) = \frac{\mathrm{Ker}(\delta^q : C^q(X;G) \longrightarrow C^{q+1}(X;G))}{\mathrm{Im}(\delta^{q-1} : C^{q-1}(X;G) \longrightarrow C_q(X;G))}.$$

Recall that there is a canonical homomorphism $\epsilon : C_0(X) \longrightarrow \mathbf{Z}$ sending $c = \sum_j \lambda_j f_j$ to the sum $\sum_j \lambda_j \in \mathbf{Z}$. The homomorphism ϵ induces the homomorphisms

$$\epsilon_* : C_0(X;G) \longrightarrow G \quad \text{and} \quad \epsilon^* : G \longrightarrow C^0(X;G).$$

Clearly $\epsilon_* : \sum_j \lambda_j f_j \mapsto \sum_j \lambda_j \in G$, and $\epsilon^* : \lambda \mapsto \xi_\lambda$, where $\langle \xi_\lambda, f_j \rangle = \lambda$ for any generator $f_j \in C_0(X)$. Then we define the complexes $\widetilde{\mathcal{C}}_*(X) \otimes G$ and $\widetilde{\mathcal{C}}^*(X;G)$ as

$$\cdots \xrightarrow{\partial_{q+1}} C_q(X) \otimes G \xrightarrow{\partial_q} C_{q-1}(X) \otimes G \xrightarrow{\partial_{q-1}} \cdots \xrightarrow{\partial_1} C_0(X) \otimes G \xrightarrow{\epsilon_*} G \longrightarrow 0,$$

$$\cdots \xleftarrow{\delta^q} C^q(X;G) \xleftarrow{\delta^{q-1}} C^{q-1}(X;G) \xleftarrow{\delta^{q-2}} \cdots \xleftarrow{\delta^0} C^0(X;G) \xleftarrow{\epsilon^*} G \longleftarrow 0.$$

Thus $\widetilde{H}_0(X;G) = \mathrm{Ker}\ \epsilon_*/\mathrm{Im}\ \partial_1$, and $\widetilde{H}^0(X;G) = \mathrm{Ker}\ \delta^0/\mathrm{Im}\ \epsilon^*$. It is convenient to call the elements of $C^q(X;G)$ a *cochain*, the elements of $Z^q(X;G) = \mathrm{Ker}\ \delta^q \subset C^q(X;G)$ *cocycles*, and the elements of $B^q(X;G) = \mathrm{Im}\ \delta^{q-1} \subset Z^q(X;G)$ *coboundaries*.

15.2. **Basic propertries of $H_*(-;G)$ and $H^*(-;G)$.** Here we list those properties of homology and cohomology groups which are parallel to the above features of the integral homology groups.

(1) (Naturality) The homology groups $H_q(X;G)$ and cohomology groups $H^q(X;G)$ are natural, i.e. if $f : X \longrightarrow Y$ is a map, then it induces the homomorphisms

$$f_* : H_q(X;G) \longrightarrow H_q(Y;G), \quad \text{and} \quad f^* : H^q(Y;G) \longrightarrow H^q(X;G).$$

In other words, the homology $H_*(-;G)$ is a *covariant functor* on the category of spaces, and the cohomology $H^*(-;G)$ is a *contravariant functor*.

(2) (Homotopy invariance) Let $f \sim g : X \longrightarrow Y$. then

$$f_* = g_* : H_q(X;G) \longrightarrow H_q(Y;G), \quad \text{and} \quad f^* = g^* : H^q(Y;G) \longrightarrow H^q(X;G).$$

(3) (Additivity) Let $X = \bigsqcup_j X_j$ be a disjoint union. Then

$$H_*\left(\bigsqcup_j X_j; G\right) \cong \bigoplus_j H_*(X_j;G), \quad \text{and} \quad H^*\left(\bigsqcup_j X_j; G\right) \cong \prod_j H^*(X_j;G).$$

(4) (Homology of the point) $H_0(pt;G) = G$, $H^q(pt;G) = 0$, and $H_q(pt;G) = 0$, $H^q(pt;G) = 0$ for $q \geq 1$.

(5) (Long exact sequences) For any pair (X, A) there are the following long exact sequences:

$$\cdots \to H_q(A;G) \longrightarrow H_q(X;G) \longrightarrow H_q(X,A;G) \xrightarrow{\partial} H_{q-1}(A;G) \to \cdots,$$

$$\cdots \to H^q(X,A;G) \longrightarrow H^q(X;G) \longrightarrow H^q(A;G) \xrightarrow{\delta} H^{q+1}(X,A;G) \to \cdots$$

(6) (Excision) If (X,A) is a Borsuk pair, then

$$H_q(X,A;G) \cong \widetilde{H}_q(X/A;G), \qquad H^q(X,A;G) \cong \widetilde{H}^q(X/A;G).$$

In general case, there are the excision isomorphisms:

$$H_q(X\setminus B, A\setminus B) \cong H_q(X,B), \quad H^q(X\setminus B, A\setminus B) \cong H^q(X,B),$$

under the same assumptions as before (i.e. that $B\subset A$ and $\overline{B}\subset \overset{o}{A}$).

(7) (Mayer-Vietoris long exact sequnces) Let $X_1, X_2 \subset X$, and $X_1\cup X_2 = X$, $\overset{o}{X}_1 \cup \overset{o}{X}_2 = X$. Then there are the Mayer-Vietoris long exact sequnces:

$$\cdots \to H_q(X_1\cap X_2;G) \xrightarrow{\alpha_*} H_q(X_1;G)\oplus H_q(X_2;G) \xrightarrow{\beta_*} H_q(X_1\cup X_2;G) \xrightarrow{\partial} \cdots$$

$$\cdots \to H^q(X_1\cup X_2;G) \xrightarrow{\alpha^*} H^q(X_1;G)\oplus H^q(X_2;G) \xrightarrow{\beta^*} H^q(X_1\cap X_2;G) \xrightarrow{\delta} \cdots$$

Exercise 15.1. Compute the groups $H_q(S^n;G)$ and $H^q(S^n;G)$.

Exercise 15.2. Compute the groups $H_q(\mathbf{CP}^n;G)$ and $H^q(\mathbf{CP}^n;G)$.

Exercise 15.3. Compute the groups $H_q(\mathbf{RP}^n;\mathbf{Z}/p)$, $H^q(\mathbf{RP}^n;\mathbf{Z}/p)$ for any prime p.

Exercise 15.4. Let M_g be an oriented surface of genus g. Compute the groups $H_q(M_g;\mathbf{Z}/p)$ and $H^q(M_g;\mathbf{Z}/p)$ for any prime p.

Exercise 15.5. Compute the homology and cohomology groups $H_q(M_g\#\mathbf{RP}^2;\mathbf{Z}/p)$ and $H^q(M_g\#\mathbf{RP}^2;\mathbf{Z}/p)$ for any prime p.

Exercise 15.6. Compute the homology and cohomology groups $H_q(M_g\#Kl^2;\mathbf{Z}/p)$ and $H^q(M_g\#Kl^2;\mathbf{Z}/p)$ for any prime p.

15.3. **Coefficient sequences.** We have to figure out the relationship between homology and cohomology groups with different coefficients. Let $\varphi : G \longrightarrow H$ be a homomorphism of abelian groups. Then clearly φ induces the chain (cochain) maps of complexes:

$$\varphi_{\#} : \mathcal{C}_*(X;G) \longrightarrow \mathcal{C}_*(X;H), \quad \text{and} \quad \varphi^{\#} : \mathcal{C}^*(X;G) \longrightarrow \mathcal{C}^*(X;H).$$

(Notice that the homomorphisms $\varphi_{\#}$ and $\varphi^{\#}$ are going in the same direction.) Thus φ induces the homomorphisms:

$$\varphi_* : H_*(X;G) \longrightarrow H_*(X;H) \quad \text{and} \quad \varphi^* : H^*(X;G) \longrightarrow H^*(X;H).$$

Now let $0 \longrightarrow G' \xrightarrow{\alpha} G \xrightarrow{\beta} G'' \longrightarrow 0$ be a short exact sequence of abelian groups. It is easy to notice that this short exact sequnce induces the short exact sequnces of complexes:

$$0 \longrightarrow \mathcal{C}_*(X;G') \xrightarrow{\alpha_\#} \mathcal{C}_*(X;G) \xrightarrow{\beta_\#} \mathcal{C}_*(X;G'') \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{C}^*(X;G') \xrightarrow{\alpha^\#} \mathcal{C}^*(X;G) \xrightarrow{\beta^\#} \mathcal{C}^*(X;G'') \longrightarrow 0.$$

These short exact sequences immediately imply the coefficient exact sequences:

$$\cdots \to H_q(X;G') \xrightarrow{\alpha_*} H_q(X;G) \xrightarrow{\beta_*} H_q(X;G'') \xrightarrow{\partial} H_{q-1}(X;G') \to \cdots,$$

$$\cdots \to H^q(X;G') \xrightarrow{\alpha^*} H^q(X;G) \xrightarrow{\beta^*} H^q(X;G'') \xrightarrow{\delta} H^{q+1}(X;G') \to \cdots,$$

Example. Consider the short exact sequence $0 \longrightarrow \mathbf{Z} \xrightarrow{\cdot m} \mathbf{Z} \longrightarrow \mathbf{Z}/m \longrightarrow 0$. Then we have the connecting homomorphisms

$$\partial = \beta^m : H^q(X;\mathbf{Z}/m) \longrightarrow H^{q+1}(X;\mathbf{Z}), \quad \text{and} \quad \delta = \beta_m : H_q(X;\mathbf{Z}/m) \longrightarrow H_{q+1}(X;\mathbf{Z}).$$

These homomorphisms are known as the Bockstein homomorphisms. Let $\alpha \in H_q(X;\mathbf{Z}/m)$, and $a \in C_q(X;\mathbf{Z}/m)$ a cycle representing α. Then $\partial_q(a) = 0$ in $C_{q-1}(X;\mathbf{Z}/m)$, however in general, $\partial_q(a) \neq 0$ in $C_{q-1}(X;\mathbf{Z})$. It is easy to check that $\partial_q(a) = m \cdot b$, where $b \in C_{q-1}(X;\mathbf{Z})$ is a cycle. Thus $\beta_m(\alpha) = \left[\frac{\partial_q(a)}{m}\right] \in H_{q-1}(X;\mathbf{Z})$.

15.4. **The universal coefficient Theorem for homology groups.** We recall few basic constructions from elementary group theory. Let G be an abelian group. Then there is a *free resolution of* G:

$$0 \longrightarrow R \xrightarrow{\beta} F \xrightarrow{\alpha} G \longrightarrow 0,$$

i.e. the above sequence is exact and the groups F, R are free abelian. Roughly a choice of free resolution corresponds to a choice of generators and relations for the abelian group G. This choice is not unique, however, if $0 \longrightarrow R_1 \xrightarrow{\beta_1} F_1 \xrightarrow{\alpha_1} G \longrightarrow 0$ is another free resolution, there exist homomorphisms $\psi : F \longrightarrow F_1$, $\theta : R \longrightarrow R_1$ which make the following diagram commute:

$$\begin{tikzcd}
0 \arrow[r] & R \arrow[r, "\beta"] \arrow[d, "\theta"'] & F \arrow[r, "\alpha"] \arrow[d, "\psi"'] & G \arrow[r] \arrow[d, "Id"'] & 0 \\
0 \arrow[r] & R_1 \arrow[r, "\beta_1"] & F_1 \arrow[r, "\alpha_1"] & G \arrow[r] & 0
\end{tikzcd}$$

Now let H be an abelian group.

Claim 15.1. *There is the exact sequence*

$$0 \longrightarrow \mathrm{Ker}(\beta \otimes 1) \longrightarrow R \otimes H \xrightarrow{\beta\otimes 1} F \otimes H \xrightarrow{\alpha\otimes 1} G \otimes H \longrightarrow 0.$$

Exercise 15.7. Prove Claim 15.1.

We define $\mathrm{Tor}(G,H) = \mathrm{Ker}(\beta \otimes 1)$.

Exercise 15.8. Prove that the group $\mathrm{Tor}(G,H)$ is well-defined, i.e. it does not depend on the choice of resolution.

Let $0 \longrightarrow R \xrightarrow{\beta} F \xrightarrow{\alpha} G \longrightarrow 0$ be a resolution of G. We denote $\mathcal{R}(G)$ the chain complex

$$0 \longrightarrow R \xrightarrow{\beta} F \longrightarrow 0.$$

Clearly $H_0(\mathcal{R}(G)) = G$, and $H_j(\mathcal{R}(G)) = 0$ if $j > 0$. Consider the complex $\mathcal{R}(G) \otimes H$:

$$0 \longrightarrow R \otimes H \xrightarrow{\beta} F \otimes H \longrightarrow 0.$$

By definition we have that

$$H_j(\mathcal{R}(G) \otimes H) = \begin{cases} G \otimes H & \text{if } j = 0, \\ \mathrm{Tor}(G,H) & \text{if } j = 1, \\ 0 & \text{else.} \end{cases}$$

Now consider an exact sequence

$$0 \longrightarrow G' \longrightarrow G \longrightarrow G'' \longrightarrow 0. \tag{70}$$

The sequence (70) induces a short exact sequence of the complexes:

$$0 \longrightarrow \mathcal{R}(G') \longrightarrow \mathcal{R}(G) \longrightarrow \mathcal{R}(G'') \longrightarrow 0,$$

and a short exact sequence of the complexes:

$$0 \longrightarrow \mathcal{R}(G') \otimes H \longrightarrow \mathcal{R}(G) \otimes H \longrightarrow \mathcal{R}(G'') \otimes H \longrightarrow 0.$$

Thus we have the long exact sequence in homology groups:

$$0 \to \mathrm{Tor}(G',H) \to \mathrm{Tor}(G,H) \to \mathrm{Tor}(G'',H) \to G' \otimes H \to G \otimes H \to G'' \otimes H \to 0 \tag{71}$$

Exercise 15.9. Let G, H be abelian groups. Prove that there is a canonical isomorphism $\mathrm{Tor}(G,H) \cong \mathrm{Tor}(H,G)$.

Exercise 15.10. Let G, G', G'' be abelian groups. Prove the isomorphism

$$\mathrm{Tor}(\mathrm{Tor}(G,G'),G'') \cong \mathrm{Tor}(G, \mathrm{Tor}(G',G'')).$$

Exercise 15.11. Let F be a free abelian group. Show that $\mathrm{Tor}(F,G) = 0$ for any abelian group G.

Exercise 15.12. Let G be an abelian group. Denote $T(G)$ a maximal torsion subgroup of G. Show that $\mathrm{Tor}(G,H) \cong T(G) \otimes T(H)$ for finite generated abelian groups G, H. Give an example of abelian groups G, H, so that $\mathrm{Tor}(G,H) \neq T(G) \otimes T(H)$.

Theorem 15.1. *Let X be a space, G be an abelian group. Then there is a split short exact sequence*

$$0 \to H_q(X) \otimes G \longrightarrow H_q(X;G) \longrightarrow \mathrm{Tor}(H_{q-1}(X), G) \to 0 \tag{72}$$

Remark. The splitting of the sequence (15.1) is not natural. In the course of the proof we shall see that that this splitting depends on a splitting of the chain complex $\mathcal{C}_*(X)$.

Proof. Let $0 \longrightarrow R \xrightarrow{\beta} F \xrightarrow{\alpha} G \longrightarrow 0$ be a free resolution of G. We have the five-term exact sequence

$$H_q(X;R) \xrightarrow{\beta_*} H_q(X;F) \xrightarrow{\alpha_*} H_q(X;G) \xrightarrow{\partial} H_{q-1}(X;R) \xrightarrow{\beta_*} H_{q-1}(X;F).$$

We notice that $H_q(X;R) \cong H_q(X) \otimes R$, and $H_q(X;F) \cong H_q(X) \otimes F$. Thus we have the exact sequence

$$H_q(X) \otimes R \xrightarrow{\beta_*} H_q(X) \otimes F \xrightarrow{\alpha_*} H_q(X;G) \xrightarrow{\partial} H_{q-1}(X) \otimes R \xrightarrow{\beta_*} H_{q-1}(X) \otimes F \tag{73}$$

Consider carefully the sequence (73). First, we notice that it gives a short exact sequence

$$0 \longrightarrow \mathrm{Coker}\ \beta_* \longrightarrow H_q(X;G) \longrightarrow \mathrm{Ker}\ \beta_* \longrightarrow 0,$$

where

$$\mathrm{Coker}\ \beta_* = (H_q(X) \otimes F)/\mathrm{Im}(\beta_* : H_q(X) \otimes R \xrightarrow{\beta_*} H_q(X) \otimes F) \quad \text{and}$$

$$\mathrm{Ker}\ \beta_* = \mathrm{Ker}(\beta_* : H_{q-1}(X) \otimes R \xrightarrow{\beta_*} H_{q-1}(X) \otimes F).$$

On the other hand, $\beta_* = 1 \otimes \beta = \beta \otimes 1$. Hence $\mathrm{Coker}\ \beta_* = H_q(X) \otimes G$, and $\mathrm{Ker}\ \beta_* = \mathrm{Tor}(H_{q-1}(X), G)$.

Now we have to show the splitting of the short exact sequence (72). Let $C_q = C_q(X)$. Recall that $Z_q = \mathrm{Ker}\ \partial_q$, and $B_{q-1} = \mathrm{Im}\ \partial_q \subset C_{q-1}$. We have a short exact sequence of free abelian groups:

$$0 \longrightarrow Z_q \longrightarrow C_q \longrightarrow B_{q-1} \longrightarrow 0.$$

Since the above groups are free abelian, there is a splitting $C_q = Z_q \oplus B_{q-1}$. Now we analyze the chain complex $\mathcal{C}_*$ using the above splitting for each $q \geq 0$. We have the commutative diagram:

$$\begin{tikzcd}
C_{q+1} = \arrow[d, "\partial_{q+1}"'] & Z_{q+1} \oplus B_q \arrow[d, "\text{inclusion}"] \\
C_q = \arrow[d, "\partial_q"'] & Z_q \oplus B_{q-1} \\
C_{q-1} = & Z_{q-1} \oplus B_{q-2}
\end{tikzcd}$$

This shows that the chain complex $\mathcal{C}_*$ splits into a direct sum of short chain complexes $\mathcal{C}_*(q)$:

$$\cdots \longrightarrow 0 \longrightarrow \overset{q}{B}_{q-1} \xrightarrow{\text{inclusion}} \overset{q-1}{Z}_{q-1} \longrightarrow 0 \longrightarrow \cdots$$

Clearly $H_{q-1}(\mathcal{C}_*(q)) = H_{q-1}(X)$ since we have the short exact sequence

$$0 \longrightarrow B_{q-1} \longrightarrow Z_{q-1} \longrightarrow H_{q-1}(X) \longrightarrow 0. \tag{74}$$

by definition of the homology group. Also we consider (74) as free resolution of the group $H_{q-1}(X)$. We have the isomorphism of chain complexes:

$$\mathcal{C}_* = \bigoplus_{q\geq 0} \mathcal{C}_*(q), \quad \text{and} \quad \mathcal{C}_* \otimes G = \bigoplus_{q\geq 0} (\mathcal{C}_*(q) \otimes G).$$

We notice that

$$H_j(\mathcal{C}_*(q) \otimes G) = \begin{cases} H_{q-1}(X) \otimes G & \text{if } j = q-1 \\ \mathrm{Tor}(H_{q-1}(X), G) & \text{if } j = q \\ 0 & \text{else.} \end{cases}$$

Thus

$$H_q(X; G) = H_q(\mathcal{C}_* \otimes G) = (H_q(X) \otimes G) \oplus \mathrm{Tor}(H_{q-1}(X), G).$$

This proves Theorem 15.1. □

15.5. **The universal coefficient Theorem for cohomology groups.** First we have to define the group $\mathrm{Ext}(G, H)$. I assume here that we all know basic things about the group $\mathrm{Hom}(G, H)$. Consider the short exact sequence

$$0 \longrightarrow \mathbf{Z} \xrightarrow{\cdot 2} \mathbf{Z} \longrightarrow \mathbf{Z}/2 \longrightarrow 0.$$

We apply the functor $\mathrm{Hom}(-, \mathbf{Z}/2)$ to this exact sequence:

$$0 \longleftarrow \overset{\mathbf{Z}/2}{\mathrm{Hom}}\,(\mathbf{Z}, \mathbf{Z}/2) \xleftarrow{\cdot 2} \overset{\mathbf{Z}/2}{\mathrm{Hom}}\,(\mathbf{Z}, \mathbf{Z}/2) \longleftarrow \overset{\mathbf{Z}/2}{\mathrm{Hom}}\,(\mathbf{Z}/2, \mathbf{Z}/2) \longleftarrow 0.$$

Clearly this sequence is not exact.

Let G be an abelian group and $0 \longrightarrow R \xrightarrow{\beta} F \xrightarrow{\alpha} G \longrightarrow 0$ be a free resolution.

Claim 15.2. *Let H be an abelian group. The following sequence is exact:*

$$0 \longleftarrow \mathrm{Coker}\ \beta^{\#} \longleftarrow \mathrm{Hom}(R, H) \xleftarrow{\beta^{\#}} \mathrm{Hom}(F, H) \xleftarrow{\alpha^{\#}} \mathrm{Hom}(G, H) \longleftarrow 0. \tag{75}$$

Exercise 15.13. Prove Claim 15.2.

We define $\mathrm{Ext}(G, H) = \mathrm{Coker}\ \beta^{\#}$. Consider the cochain complex $\mathrm{Hom}(\mathcal{R}(G), H)$:

$$0 \longleftarrow \mathrm{Hom}(R, H) \xleftarrow{\beta^{\#}} \mathrm{Hom}(F, H) \longleftarrow 0.$$

Then Claim 15.2 implies that

$$H^j(\mathrm{Hom}(\mathcal{R}(G),H)) = \begin{cases} \mathrm{Hom}(G,H) & \text{if } j=0, \\ \mathrm{Ext}(G,H) & \text{if } j=1, \\ 0 & \text{else.} \end{cases}$$

Exercise 15.14. Prove that the group $\mathrm{Ext}(G,H)$ is well defined, i.e. it does not depend on the choice of free resolution of G.

Exercise 15.15. Let $0 \longrightarrow G' \longrightarrow G \longrightarrow G'' \longrightarrow 0$ be a short exact sequence of abelian groups. Prove that it induces the following exact sequence:

$$0 \to \mathrm{Hom}(G'',H) \to \mathrm{Hom}(G,H) \to \mathrm{Hom}(G',H) \to$$

$$\mathrm{Ext}(G'',H) \to \mathrm{Ext}(G,H) \to \mathrm{Ext}(G',H) \to 0$$

Exercise 15.16. Prove that $\mathrm{Ext}(\mathbf{Z},H)=0$ for any group H.

Exercise 15.17. Prove the isomorphisms: $\mathrm{Ext}(\mathbf{Z}/m,\mathbf{Z}/n) \cong \mathbf{Z}/m \otimes \mathbf{Z}/n$, $\mathrm{Ext}(\mathbf{Z}/m,\mathbf{Z}) \cong \mathbf{Z}/m$.

Exercise 15.18. Let G or H be $\mathbf{Q}$, $\mathbf{R}$ or $\mathbf{C}$. Then $\mathrm{Ext}(G,H)=0$.

Theorem 15.2. *Let X be a space, G an abelian group. Then there is a split exact sequence*

$$0 \longrightarrow \mathrm{Ext}(H_{q-1}(X),G) \longrightarrow H^q(X;G) \longrightarrow \mathrm{Hom}(H_q(X),G) \longrightarrow 0 \tag{76}$$

for each $q \geq 0$. Again, the splitting of this sequence is not natural.

Proof. First, consider the splitting of the groups $C_q = C_q(X) = B_{q-1} \oplus Z_q$. We have the commutative diagram:

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & Z_q \arrow[r, "i_q"] \arrow[d, "0"] & C_q \arrow[r, "j_q"] \arrow[d, "\partial_q"] & B_{q-1} \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & Z_{q-1} \arrow[r, "i_{q-1}"] \arrow[d, "0"] & C_{q-1} \arrow[r, "j_{q-1}"] \arrow[d] & B_{q-2} \arrow[r] \arrow[d, "0"] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

This diagram may be thought as a short exact sequence of chain complexes:

$$0 \longrightarrow \mathcal{Z}_* \xrightarrow{i_\#} \mathcal{C}_* \xrightarrow{j_\#} \mathcal{B}_* \longrightarrow 0. \tag{77}$$

Remark. It is interesting to notice that the long exact sequence

$$\cdots \longrightarrow H_q(\mathcal{Z}_*) \longrightarrow H_q(\mathcal{C}_*) \longrightarrow H_q(\mathcal{B}_*) \longrightarrow H_{q-1}(\mathcal{Z}_*) \longrightarrow \cdots$$

corresponding to the short exact sequence (77) splits into the short exact sequences

$$0 \longrightarrow B_q \xrightarrow{\alpha_q} Z_q \longrightarrow H_q(X) \longrightarrow 0.$$

Exercise 15.19. Prove the above splitting.

Now we have a short exact sequence of cochain complexes:

$$0 \longleftarrow \mathrm{Hom}(\mathcal{Z}_*, G) \xleftarrow{i^\#} \mathrm{Hom}(\mathcal{C}_*, G) \xleftarrow{j^\#} \mathrm{Hom}(\mathcal{B}_*, G) \longleftarrow 0 \tag{78}$$

Notice that the cochain complexes $\mathrm{Hom}(\mathcal{Z}_*, G)$ and $\mathrm{Hom}(\mathcal{B}_*, G)$ have zero differentials, hence

$$H^q(\mathrm{Hom}(\mathcal{Z}_*, G)) = \mathrm{Hom}(Z_q, G), \quad \text{and} \quad H^q(\mathrm{Hom}(\mathcal{B}_*, G)) = \mathrm{Hom}(B_{q-1}, G).$$

The sequence (78) induces the long exact sequence in cohomology groups:

$$\mathrm{Hom}(B_q, G) \xleftarrow{\delta^q} \mathrm{Hom}(Z_q, G) \xleftarrow{i^*} H^q(\mathrm{Hom}(\mathcal{C}_*, G)) \xleftarrow{j^*} \mathrm{Hom}(B_{q-1}, G) \xleftarrow{\delta^{q-1}} \mathrm{Hom}(Z_{q-1}, G)$$

It is easy to notice that the coboundary homomorphism $\delta^q : \mathrm{Hom}(Z_q, G) \longrightarrow \mathrm{Hom}(B_q, G)$ coincides with the homomorphism $\alpha_q^\# = \mathrm{Hom}(\alpha_q, 1)$. We have the following exact sequence:

$$0 \longleftarrow \mathrm{Ker}\alpha_q^\# \longleftarrow H^q(\mathrm{Hom}(\mathcal{C}_*, G)) \longleftarrow \mathrm{Coker}\ \alpha_{q-1}^\# \longleftarrow 0.$$

Now we identify $\mathrm{Ker}\alpha_q^\# = \mathrm{Hom}(H_q(X), G)$ and $\mathrm{Coker}\ \alpha_{q-1}^\# = \mathrm{Ext}(H_{q-1}(X), G)$ to get the desired exact sequence.

Recall that we have splitting $\mathcal{C}_* = \bigoplus_{q\geq 0} \mathcal{C}_*(q)$, and hence

$$\mathrm{Hom}(\mathcal{C}_*, G) = \bigoplus_{q\geq 0} \mathrm{Hom}(\mathcal{C}_*(q), G).$$

Consider the cochain complex $\mathrm{Hom}(\mathcal{C}_*(q), G)$:

$$0 \longleftarrow \overset{q}{\mathrm{Hom}}\,(B_{q-1}, G) \longleftarrow \overset{q-1}{\mathrm{Hom}}\,(Z_{q-1}, G) \longleftarrow 0.$$

We notice that the sequence $0 \longrightarrow B_{q-1} \longrightarrow Z_{q-1} \longrightarrow H_{q-1}(X) \longrightarrow 0$ may be considered as free resolution of the group $H_{q-1}(X)$. Thus we have:

$$H^j(\mathrm{Hom}(\mathcal{C}_*(q), G)) = \begin{cases} \mathrm{Hom}(H_{q-1}(X), G) & \text{if } j = q-1, \\ \mathrm{Ext}(H_{q-1}(X), G) & \text{if } j = q, \\ 0 & \text{else.} \end{cases}$$

Thus we use the above splitting of $\mathrm{Hom}(\mathcal{C}_*, G)$ to get the isomorphism:

$$H^q(X; G) = H^q(\mathrm{Hom}(\mathcal{C}_*, G)) = \mathrm{Hom}(H_q(X), G) \oplus \mathrm{Ext}(H_{q-1}(X), G).$$

This completes the proof of Theorem 15.2. $\square$

Theorem 15.3. *Let X be a space, and G an abelian group. Then there is a split exact sequence*

$$0 \longrightarrow H^q(X;\mathbf{Z}) \otimes G \longrightarrow H^q(X;G) \longrightarrow \mathrm{Tor}(H^{q+1}(X;\mathbf{Z}),G) \longrightarrow 0$$

for any $q \geq 0$. Again the splitting is not natural.

Exercise 15.20. Prove Theorem 15.3.

Let G be a finitely generated abelian group. It is convenient to denote $F(G)$ the maximum free abelian subgroup of G, and $T(G)$ the maximum torsion subgroup, so that $G = F(G) \oplus T(G)$. Perhaps such decomposition makes only for finitely generated groups.

Exercise 15.21. Let X be a space so that the groups $H_q(X)$ are finitely generated. Prove that $H^q(X;\mathbf{Z})$ are also finitely generated and $H^q(X;\mathbf{Z}) \cong F(H_q(X;\mathbf{Z})) \oplus T(H_{q-1}(X;\mathbf{Z}))$.

Exercise 15.22. Let F be $\mathbf{Q}$, $\mathbf{R}$ or $\mathbf{C}$. Prove that

$$H_q(X;F) = H_q(X) \otimes F, \quad H^q(X;F) = \mathrm{Hom}(H_q(X),F).$$

Exercise 15.23. Let F be a free abelian group. Show that $\mathrm{Ext}(G,F) = 0$ for any abelian group G.

Exercise 15.23. Let X be a finite CW-complex, and $\mathbf{F}$ be a field. Prove that the number

$$\chi(X)_{\mathbf{F}} = \sum_{q\geq 0} (-1)^q \dim\ H_q(X;\mathbf{F})$$

does not depend on the field $\mathbf{F}$ and is equal to the Euler characteristic

$$\chi(X) = \sum_{q\geq 0} (-1)^q \{\#\text{ of } q\text{-cells of } X\}.$$

15.6. **The Künneth formula.** Let $\mathcal{C}_*$ and $\mathcal{C}'_*$ be two chain complexes:

$$\cdots \longrightarrow C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\epsilon} \mathbf{Z},$$

$$\cdots \longrightarrow C'_3 \xrightarrow{\partial'_3} C'_2 \xrightarrow{\partial'_2} C'_1 \xrightarrow{\partial'_1} C'_0 \xrightarrow{\epsilon'} \mathbf{Z}.$$

A tensor product $\mathcal{C}_* \otimes \mathcal{C}'_*$ is the complex

$$\cdots \longrightarrow \bar{C}_3 \xrightarrow{\bar{\partial}_3} \bar{C}_2 \xrightarrow{\bar{\partial}_2} \bar{C}_1 \xrightarrow{\bar{\partial}_1} \bar{C}_0 \xrightarrow{\bar{\epsilon}} \mathbf{Z}$$

where

$$\bar{C}_q = \bigoplus_{r+s=q} C_r \otimes C'_s,$$

and the boundary operator $\bar{\partial}_q : \bar{C}_q \longrightarrow \bar{C}_{q-1}$,

$$\bar{\partial}_q : \bigoplus_{r+s=q} C_r \otimes C'_s \longrightarrow \bigoplus_{r+s=q-1} C_r \otimes C'_s$$

is given by the formula (where $c \in C_r$, $c' \in C'_s$):

$$\bar{\partial}_q(c \otimes c') = (\partial_r c) \otimes c' + (-1)^r c \otimes \partial'_s c' \in (C_{r-1} \otimes C_s) \oplus (C_r \otimes C_{s-1}) \subset \bigoplus_{r+s=q-1} C_r \otimes C'_s.$$

We emphasize that the sign in the above formula is very important.

Exercise 15.24. Prove that $\bar{\partial}_{q+1}\bar{\partial}_q = 0$.

The Künneth formula describes homology groups of the produt $X \times X'$ in terms of homology groups of X and X'. It is tempted to use the same singular chain complexes we used to prove the universal coefficient formulas. However there is a serious problem here. Indeed, the singular chain complex $\mathcal{C}_*(X \times X')$ is not isomorphic to the tensor product $\mathcal{C}_*(X) \otimes \mathcal{C}_*(X')$. There is a general result showing that the chain complexes $\mathcal{C}_*(X \times X')$ and $\mathcal{C}_*(X) \otimes \mathcal{C}_*(X')$ are chain homotopy equivalent (this is the Eilenberg-Zilber Theorem). We already have some technique to avoid this general result: we can always replace the spaces X, X' to weak homotopy equivalent CW-complexes and use the cellular chain complexes. Thus the following is the key property of the cellular chain complex:

Claim 15.3. *Let X, X' be CW-complexes. We give the product $X \times X'$ the product CW-structure. Then $\mathcal{E}_*(X \times X') \cong \mathcal{E}_*(X) \otimes \mathcal{E}_*(X')$.*

Now this is the Künneth formula.

Theorem 15.4. *Let X, X' be topological spaces. Then for each $q \geq 0$ there is a split exact sequence*

$$0 \to \bigoplus_{r+s=q} H_r(X) \otimes H_s(X') \longrightarrow H_q(X \times X') \longrightarrow \bigoplus_{r+s=q-1} \mathrm{Tor}(H_r(X), H_s(X')) \to 0$$

Proof. As we mentioned, it is enough to prove the above formular in the case when X and X' are CW-complexes. Let $\mathcal{E}_* = \mathcal{E}_*(X)$, $\mathcal{E}'_* = \mathcal{E}_*(X')$. We denote $\mathcal{Z}_q = \mathrm{Ker}\ \overset{\flat}{\partial}_q$, and $\mathcal{B}_q = \mathrm{Im}\ \overset{\flat}{\partial}_{q+1}$. Again, we have the short exact sequnce

$$0 \longrightarrow \mathcal{Z}_q \longrightarrow \mathcal{E}_q \longrightarrow \mathcal{B}_{q-1} \longrightarrow 0.$$

Since all groups here are free abelian, we have a splitting

$$\mathcal{E}_q = \mathcal{Z}_q \oplus \mathcal{B}_{q-1}.$$

Similarly as we did before, this decomposition allows us to split the complex $\mathcal{E}_*$ into the direct sum of short chain complexes $\mathcal{E}_*(q)$:

$$\cdots \longrightarrow 0 \longrightarrow \overset{q}{\mathcal{B}_{q-1}} \xrightarrow{i_q} \overset{q-1}{\mathcal{Z}}_{q-1} \longrightarrow 0 \longrightarrow \cdots$$

As before, we have that $H_{q-1}(\mathcal{E}_*(q)) = H_{q-1}(\mathcal{E}_*)$ and $H_j(\mathcal{E}_*(q)) = 0$ if $j \neq q-1$.

We define such complexes $\mathcal{E}_*(q)$, $\mathcal{E}'_*(q)$, thus we have the decompositions:

$$\mathcal{E}_* = \bigoplus_{r \geq 0} \mathcal{E}_*(r), \quad \text{and} \quad \mathcal{E}'_* = \bigoplus_{s \geq 0} \mathcal{E}'_*(s).$$

Thus the tensor product $\mathcal{E}_* \otimes \mathcal{E}'_*$ is decomposed as follows:

$$\mathcal{E}_* \otimes \mathcal{E}'_* = \bigoplus_{r,s\geq 0} \mathcal{E}_*(r) \otimes \mathcal{E}'_*(s).$$

We examine the tensor product $\mathcal{E}_*(r) \otimes \mathcal{E}'_*(s)$:

$$\cdots \to 0 \to \overset{s+r}{\mathcal{B}_{r-1} \otimes \mathcal{B}'_{s-1}} \xrightarrow{\bar{\partial}_{s+r}} \overset{s+r-1}{\left(\mathcal{Z}_{r-1} \otimes \mathcal{B}'_{s-1}\right) \oplus \left(\mathcal{B}_{r-1} \otimes \mathcal{Z}'_{s-1}\right)} \xrightarrow{\bar{\partial}_{s+r-1}} \overset{s+r-2}{\mathcal{Z}_{r-1} \otimes \mathcal{Z}'_{s-1}} \to 0 \to \cdots$$

Now we have to compute the homology groups of this chain complex. First we put together all short exact sequences we need. We have the complexes $\mathcal{E}_*(r)$ and $\mathcal{E}'_*(s)$:

$$\cdots \longrightarrow 0 \longrightarrow \mathcal{B}_{r-1} \xrightarrow{i_r} \mathcal{Z}_{r-1} \longrightarrow 0 \longrightarrow \cdots$$

$$\cdots \longrightarrow 0 \longrightarrow \mathcal{B}'_{s-1} \xrightarrow{i'_s} \mathcal{Z}'_{s-1} \longrightarrow 0 \longrightarrow \cdots$$

Also we need the short exact sequences which will be considered as free resolutions of the groups $H_{r-1}(X)$, $H_{s-1}(X')$:

$$0 \longrightarrow \mathcal{B}_{r-1} \xrightarrow{i_{r-1}} \mathcal{Z}_{r-1} \xrightarrow{p_{r-1}} H_{r-1} \longrightarrow 0, \quad H_{r-1} = H_{r-1}(X),$$

$$0 \longrightarrow \mathcal{B}_{s-1} \xrightarrow{i'_{s-1}} \mathcal{Z}'_{s-1} \xrightarrow{p'_{s-1}} H'_{s-1} \longrightarrow 0, \quad H'_{s-1} = H_{s-1}(X').$$

Consider the following diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{B}_{r-1} \otimes \mathcal{B}'_{s-1} \arrow[r, "d_{r+s}"] \arrow[d, "Id"'] & \mathcal{Z}_{r-1} \otimes \mathcal{B}'_{s-1} \oplus \mathcal{B}_{r-1} \otimes \mathcal{B}'_{s-1} \arrow[r, "d_{r+s-1}"] \arrow[d, "1\otimes 1\oplus 1\otimes i'_{s-1}"'] & \mathcal{Z}_{r-1} \otimes \mathcal{B}'_{s-1} \arrow[r] \arrow[d, "1\otimes i'_{s-1}"] & 0 \\
0 \arrow[r] & \mathcal{B}_{r-1} \otimes \mathcal{B}'_{s-1} \arrow[r, "\bar{\partial}_{s+r}"] \arrow[d] & \mathcal{Z}_{r-1} \otimes \mathcal{B}'_{s-1} \oplus \mathcal{B}_{r-1} \otimes \mathcal{Z}'_{s-1} \arrow[r, "\bar{\partial}_{s+r-1}"] \arrow[d, "0\oplus 1\otimes p'_{s-1}"'] & \mathcal{Z}_{r-1} \otimes \mathcal{Z}'_{s-1} \arrow[r] \arrow[d, "1\otimes p'_{s-1}"] & 0 \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & \mathcal{B}_{r-1} \otimes H'_{s-1} \arrow[r, "i_{r-1}\otimes 1"] \arrow[d] & \mathcal{Z}_{r-1} \otimes H'_{s-1} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{79}$$

Here the homomorphisms $\bar{\partial}_{s+r}$, $\bar{\partial}_{s+r}$ are given by

$$\bar{\partial}_{s+r}(b \otimes b') = (i_{r-1}b) \otimes b' \oplus (-1)^r b \otimes (i'_{s-1}b') \in \mathcal{Z}_{r-1} \otimes \mathcal{B}'_{s-1} \oplus \mathcal{B}_{r-1} \otimes \mathcal{Z}'_{s-1},$$

$$\bar{\partial}_{s+r-1}(z \otimes b' \oplus b \otimes z') = (-1)^{r-1} z \otimes (i'_{s-1}b') + (i_{r-1}b) \otimes z' \in \mathcal{Z}_{r-1} \otimes \mathcal{Z}'_{s-1}.$$

The homomorphisms d_{r+s}, d_{r+s-1} are defined similarly:

$$d_{s+r}(b\otimes b') = (i_{r-1}b)\otimes b' \oplus (-1)^r b\otimes b' \in \mathcal{Z}_{r-1}\otimes \mathcal{B}'_{s-1}\oplus \mathcal{B}_{r-1}\otimes \mathcal{B}'_{s-1},$$

$$d_{s+r-1}(z\otimes b'_1 \oplus b\otimes b'_2) = (-1)^{r-1} z\otimes b'_1 + (i_{r-1}b)\otimes b'_2 \in \mathcal{Z}_{r-1}\otimes \mathcal{B}'_{s-1}.$$

It is easy to check that the diagram (79) commutes and the columns are exact. We consider the diagram (79) as a short exact sequence of chain complexes. We notice that the sequence

$$0\to \mathcal{B}_{r-1}\otimes \mathcal{B}'_{s-1} \xrightarrow{d_{r+s}} \mathcal{Z}_{r-1}\otimes \mathcal{B}'_{s-1}\oplus \mathcal{B}_{r-1}\otimes \mathcal{B}'_{s-1} \xrightarrow{d_{r+s-1}} \mathcal{Z}_{r-1}\otimes \mathcal{B}'_{s-1}\to 0$$

is exact. Thus the homology groups of this complex are trivial. On the other hand, the homology groups of the complex

$$0 \longrightarrow \mathcal{B}_{r-1}\otimes H'_{s-1} \xrightarrow{i_{r-1}\otimes 1} \mathcal{Z}_{r-1}\otimes H'_{s-1}\to 0$$

are equal to $H_{r-1}\otimes H'_{s-1}$ (in degree $r+s-2$), and $\mathrm{Tor}(H_{r-1},H'_{s-1})$ (in degree $r+s-1$) and zero otherwise. The long exact sequence in homology groups corresponding to the short exact sequence of chain complexes (79) immediately implies that

$$H_j(\mathcal{E}_*(r)\otimes \mathcal{E}'_*(s)) = \begin{cases} H_{r-1}\otimes H'_{s-1} & \text{if } j=r+s-2,\\ \mathrm{Tor}(H_{r-1},H'_{s-1}) & \text{if } j=r+s-1,\\ 0 & \text{else.}\end{cases}$$

Now it is enough to assemble the homology groups of the chain complex $\mathcal{E}_*\otimes\mathcal{E}'_*$ out of the homology groups of the chain complexes $\mathcal{E}_*(s)\otimes\mathcal{E}'_*(r)$ to get the desired formula. This concludes the proof of Theorem 15.4. □

Theorem 15.5. *Let X, X' be topological spaces. Let $H^*(-)=H^*(-;\mathbf{Z})$. Then for each $q\geq 0$ there is a split exact sequence*

$$0\to \bigoplus_{r+s=q} H^r(X)\otimes H^s(X') \longrightarrow H^q(X\times X') \longrightarrow \bigoplus_{r+s=q+1}\mathrm{Tor}(H^r(X),H^s(X')) \longrightarrow 0.$$

Exercise 15.25. Outline a proof of Theorem 15.5.

Exercise 15.26. Let F be a field. Prove that

$$H_q(X\times X';F)\cong \bigoplus_{r+s=q} H_r(X;F)\otimes H_s(X';F),$$

$$H^q(X\times X';F)\cong \bigoplus_{r+s=q} H^r(X;F)\otimes H^s(X';F).$$

Exercise 15.27. Let $\beta_q(X)=\mathrm{Rank} H_q(X)$ be the Betti number of X. Prove that

$$\beta_q(X\times X') = \sum_{r+s=q}\beta_r(X)\beta_s(X').$$

Exercise 15.28. Let X, X' be such spaces that their Euler characteristics $\chi(X)$, $\chi(X')$ are finite. Prove that $\chi(X\times X')=\chi(X)\cdot\chi(X')$.

15.7. **The Eilenberg-Steenrod Axioms.** At the end of 50s, Eilenber and Steenrod suggested very simple axioms which characterize the homology theory on the category of CW-complexes. In this short section we present these axioms, however we are not going to prove that these axioms completely determine the homology theory.

First we should carefully describe what do we mean by a "homology theory". Let $\mathcal{T}op$ denote the category of pairs of topological spaces, i.e. the objects of $\mathcal{T}op$ are pairs (X, A) and the morphisms are continuous maps of pairs. Let $\mathcal{A}b_*$ be the category of graded abelian groups, i.e. the objects of $\mathcal{A}b_*$ are graded abelian groups $\mathcal{A} = \{A_q\}_{q\in\mathbf{Z}}$, and the morphisms are homomorphisms $\Phi : \mathcal{A} \longrightarrow \mathcal{B}$ given by a collection of group homomorphisms $\Phi = \{\varphi_q : A_q \longrightarrow B_{q+k}\}$. The integer k is the degree of the homorphism Φ.

A *homology theory* $(\mathcal{H}, \partial)$ consists of the following:

(1) A *covarint functor* $\mathcal{H} : \mathcal{T}op \longrightarrow \mathcal{A}b_*$, i.e. for each pair (X, A) $\mathcal{H}(X, A)$ is a graded abelian group, and for each map of pairs $f : (X, A) \longrightarrow (Y, B)$ there is a homomorphism $\mathcal{H}(f) : \mathcal{H}(X, A) \longrightarrow \mathcal{H}(Y, B)$ of degree zero.

(2) A natural transformation ∂ of the functor $\mathcal{H}$ of degree -1, i.e for any pair (X, A) there is a homomorphism $\partial : \mathcal{H}(X, A) \longrightarrow \mathcal{H}(A, \emptyset)$ of degree -1. It is natural with respect to continuous maps of pairs $f : (X, A) \longrightarrow (Y, B)$, i.e. the following diagram

$$\begin{tikzcd}
\mathcal{H}(X,A) \arrow[r, "\partial"] \arrow[d, "\mathcal{H}(f)"'] & \mathcal{H}(A,\emptyset) \arrow[d, "\mathcal{H}(f)"'] \\
\mathcal{H}(Y,B) \arrow[r, "\partial"] & \mathcal{H}(B,\emptyset)
\end{tikzcd}$$

commutes.

The functor $\mathcal{H}$ and transformation ∂ should satisfy the following axioms:

1. Homotopy Axiom. Let $f, g : (X, A) \longrightarrow (Y, B)$ be homotopic maps, then $\mathcal{H}(f) = \mathcal{H}(g)$.

2. Exactness axiom. For any pair (X, A) and the inclusions $i : (A, \emptyset) \subset (X, A)$, and $j : (X, \emptyset) \subset (X, A)$ there is an exact sequence:

$$\cdots \to \mathcal{H}(A,\emptyset) \xrightarrow{\mathcal{H}(i)} \mathcal{H}(X,\emptyset) \xrightarrow{\mathcal{H}(j)} \mathcal{H}(X,A) \xrightarrow{\partial} \mathcal{H}(A,\emptyset) \to \cdots$$

3. Excision Axiom. For any pair (X, A), and open subset $U \subset X$, such that $\overline{U} \subset \overset{o}{A}$, then the excision map $e : (X \setminus U, A \setminus U) \longrightarrow (X, A)$ induces the isomorphism

$$\mathcal{H}(e) : \mathcal{H}(X \setminus U, A \setminus U) \longrightarrow \mathcal{H}(X, A).$$

4. Dimension Axiom. Let $P = \{pt\}$. Then the coefficient group $\mathcal{H}(P, \emptyset) = \{H_q(P)\}$ is such that

$$H_q(P) = \begin{cases} \mathbf{Z}, & \text{if } q = 0, \\ 0, & \text{if } q \neq 0. \end{cases}$$

Eilenberg-Steentrod proved that the above axioms completely characterize the homology theory $(X, A) \mapsto \{H_q(X, A)\}$ in the following sense. Let $(\mathcal{H}', \partial)$ be a homology theory then on the category of pairs having a homotopy type of CW-complexes, the homology theory $(\mathcal{H}', \partial)$ coincides with the singular homology theory. The Eilenberg-Steentrod axioms have led to unexpected discoveries (in the begining of 60s). It turns out there are functors $(\mathcal{H}', \partial)$ which satisfy the first three axioms, and, in the same time, their coefficient group $\mathcal{H}(pt)$ is not concentrated just in the degree zero. The first examples were the K-*theory*, and different kind of *cobordism theories.* Now we call such homology theory a *generalized homology theory.* These days the word “generalized” dropped, since they were incorporated into major areas of mathematics.

16. Some applications

16.1. **The Lefschetz Fixed Point Theorem.** We still start with some algebraic constructions. Let A be a finitely generated abelian group. Denote $F(A)$ the free part of A, so that $A = F(A)\oplus T(A)$, where $T(A)$ is a maximum torsion subgroup of A. Let $\varphi : A \longrightarrow A$ be an endomorphism of A. We define $F(\varphi) : F(A) \longrightarrow F(A)$ by composition:

$$F(\varphi) : F(A) \xrightarrow{\text{inclusion}} A \xrightarrow{\varphi} A \xrightarrow{\text{projection}} F(A).$$

The homomorphism $F(\varphi)$ is an endomorphism of the free abelian finitely generated group $F(A)$. Hence the trace $\mathrm{Tr}(F(\varphi)) \in \mathbf{Z}$ is well-defined. We define $\mathrm{Tr}(\varphi) = \mathrm{Tr}(F(\varphi))$. Now let $\mathcal{A} = \{A_q\}_{q\geq 0}$ be a *finitely generated graded abelian group*, i.e. each group A_q is finitely generated. A homomorphism $\Phi : \mathcal{A} \longrightarrow \mathcal{B}$ of two graded abelian groups is a collection of homomorphisms $\{\varphi_q : A_q \longrightarrow B_{q-k}\}$ (the number k is the *degree* of Φ).

Now let $\mathcal{A} = \{A_q\}_{q\geq 0}$ be a finitely generated graded abelian group, and let

$$\Phi = \{\varphi_q\} : \mathcal{A} \longrightarrow \mathcal{A}$$

be an endomorphism of degree zero. We assume that $F(A_q) = 0$ for $q \geq n$ (for some n). We define the *Lefschetz number* $\mathrm{Lef}(\Phi)$ of the endomorphism Φ by the formula:

$$\mathrm{Lef}(\Phi) = \sum_{q\geq 0}(-1)^q \mathrm{Tr}(\varphi_q).$$

Clearly we have several natural examples of such endomorphisms. The main example we are going to work with is the following. Let X be a finite CW-complex, and $f : X \longrightarrow X$ be a map. Then there are the induced endomorphisms of degree zero

$$f_{\#} : \mathcal{E}_*(X) \longrightarrow \mathcal{E}_*(X), \quad f_* : H_*(X) \longrightarrow H_*(X),$$

where $\mathcal{E}_*(X) = \{\mathcal{E}_q(X)\}$, $H_*(X) = \{H_q(X)\}$ are considered as graded abelian groups.

Claim 16.1. *Let $\mathcal{C}$ be a chain complex, $\mathcal{C} = \{\mathcal{C}_q\}$, such that $\mathcal{C}_q = 0$ for $q \geq n$ (for some n). Let $\varphi : \mathcal{C} \longrightarrow \mathcal{C}$ be a chain map, and $\varphi_* : H_*\mathcal{C} \longrightarrow H_*\mathcal{C}$ be the induced homomorphism in homology groups. Then*

$$\mathrm{Lef}(\varphi) = \mathrm{Lef}(\varphi_*).$$

Exercise 16.1. Prove Claim 16.1.

Let $f : X \longrightarrow X$ be a map of finite CW-complex to itself. We define the Lefschetz number $\mathrm{Lef}(f) = \mathrm{Lef}(f_*)$, where $f_* : H_*(X) \longrightarrow H_*(X)$ is the induced homomorphism in homology groups. Clearly the Lefschetz number $\mathrm{Lef}(f)$ depends on the homotopy class of f.

Theorem 16.1. (Lefschetz Fixed Point Theorem) *Let X be a finite CW-complex and $f : X \longrightarrow X$ be a map such that $\mathrm{Lef}(f) \neq 0$. Then f has a fixed point, i.e. such a point $x_0 \in X$ that $f(x_0) = x_0$.*

Proof. First we recall that a finite CW-complex X may be embedded as a compact subspace into the Euclidian space $\mathbf{R}^n$ for some n. In particular, the metric on X (which is the induced metric from $\mathbf{R}^n$) determines the original topology on X. Let $d(x,x')$ be the distance function induced by this metric.

Assume that $f(x)\neq x$ for each point $x\in X$. Since X is a compact, there exists a positive number $\epsilon>0$ so that $d(f(x),x)>\epsilon$ for all $x\in X$. For every cell e^q of X, we use a homeomorphism $\Delta^q\cong e^q$ to define new CW-structure on X as follows. We find a barycentic subdivision of Δ^q such that

$$\mathrm{diam}(\widetilde{\Delta})<\epsilon/9, \quad \text{and} \quad \mathrm{diam}(f(\widetilde{\Delta}))<\epsilon/9$$

for each simplex $\widetilde{\Delta}$ of that barycentic subdivision. The simplices $\widetilde{\Delta}$ of this barycentic subdivision define new CW-structure on X. Let $\left\{\sigma^q_j\right\}$ be the cells of this CW-structure on X. For each cell σ^q_0 we define the subcomplex

$$E^q_0=\bigcup_{\bar{\sigma}_j\cap\bar{\sigma}^q_0\neq\emptyset}\sigma_j.$$

Notice that the diameter $\mathrm{diam}(E^q_0)<4\epsilon/9$. Indeed, let $x,x'\in E^q_0$. Choose $x_0\in\sigma^q_0$. Then $d(x,x_0)<2\epsilon/9$, and $d(x',x_0)<2\epsilon/9$. Thus

$$d(x,x')\leq d(x,x_0)+d(x',x_0)<4\epsilon/9.$$

Clearly $\mathrm{diam}(f(E^q_0))<4\epsilon/9$ as well. Now it is clear that $d(E^q_0,f(E^q_0))>\epsilon-8\epsilon/9=\epsilon/9$. Hence

$$E^q_0\cap f(E^q_0)=\emptyset.$$

Now we use the cellular approximation Theorem 5.5 where we constructed a cellular map $f'\sim f$. It is easy to see that $f'(\bar{\sigma}^q_0)\subset f(E^q_0)$ by construction we gave in the proof of Theorem 5.5. Thus $\bar{\sigma}^q_0\cap f'(\bar{\sigma}^q_0)=\emptyset$. Now consider the homomorphism $f'_{\#}:\mathcal{E}_q(X)\longrightarrow\mathcal{E}_q(X)$. We have that

$$f'_{\#}(\sigma^q_0)=\sum_i\lambda_i\sigma^q_i, \quad \text{where} \quad \sigma^q_i\neq\sigma^q_0.$$

Hence $\mathrm{Tr}(f'_{\#})=0$ for each $q\geq 0$, and

$$0=\mathrm{Lef}(f'_{\#})=\mathrm{Lef}(f_{\#})=\mathrm{Lef}(f_*)=\mathrm{Lef}(f).$$

This concludes the proof. □

Corollary 16.2. *Let X be a finite contractible CW-complex. Then any map $f:X\longrightarrow X$ has a fixed point.*

Exercise 16.2. Prove Corollary 16.2.

A continuous family $\varphi_t:X\longrightarrow X$ of maps is called a *flow* if the following conditions are satisfied:

(a) $\varphi_0=\mathrm{Id}_X$,

(b) φ_t is a homeomorphism for any $t\in\mathbf{R}$,

(c) $\varphi_{s+t}(x) = \varphi_s \circ \varphi_t(x)$.

It is convenient to treat a flow φ_t as a map $\varphi : X \times \mathbf{R} \longrightarrow \mathbf{R}$, where $\varphi(x,t) = \varphi_t(x)$. A flow is also known as *one-parameter group of homeomorphisms*. The following statement is not very hard to prove, however, it provides an important link to analysis.

Theorem 16.3. *Let X be a finite CW-complex with $\chi(X) \neq 0$, and $\varphi_t : X \longrightarrow X$ be a flow. Then there exists a point $x_0 \in X$ so that $\varphi_t(x_0) = x_0$ for all $t \in \mathbf{R}$.*

Proof. By definition, each map $\varphi_t \sim \mathrm{Id}_X$. Thus $\mathrm{Lef}(\varphi_t) = \mathrm{Lef}(\mathrm{Id}_X) = \chi(X) \neq 0$. Thus there exists a fixed point $x_0^{(t)}$ of φ_t for each t. Let

$$A_n = \left\{ x \in X \mid \varphi_{1/2^n}(x) = x \right\}.$$

Clearly $A_n \supset A_{n+1}$, and each A_n is a closed subset (as the intersection of the diagonal $\Delta(X) = \{(x,x)\} \subset X \times X$ and the graph $\Gamma(\varphi_{1/2^n}) = \left\{(x, \varphi_{1/2^n}(x))\right\} \subset X \times X$). Thus $F = \cap_n A_n$ is not empty. Let $x \in F$. Clearly x is a fixed point for any $\varphi_{m/2^n}$. Since the numbers $m/2^n$ are dense in $\mathbf{R}$, x is a fixed point for φ_t for any $t \in \mathbf{R}$. $\square$

Remark. Let $X = M^n$ be a smooth manifold, and assume a flow $\varphi : M^n \times \mathbf{R} \longrightarrow M^n$ is a smooth flow, i.e. the map φ is smooth, and φ_t is a diffeomorphism. We can even assume that the flow φ is defined only for $t \in (-\epsilon, \epsilon)$. Let $x \in M^n$, the

$$\frac{d\varphi_0(x)}{dt} = \lim_{\tau \to 0} \frac{\varphi_\tau(x) - \varphi_0(x)}{\tau} = \lim_{\tau \to 0} \frac{\varphi_\tau(x) - x}{\tau}$$

is a tangent vector to M^n at the point x, and the correspondence

$$v : x \mapsto \frac{d\varphi_0(x)}{dt}$$

defines a smooth tangent vector field $v(x)$ on M^n. Theorem 16.3 implies that if $\chi(M^n) \neq 0$, then there is no tangent vector field on M without zero points. Actually, a generic tangent vector field always has only isolated nondegenerated zero points, so that each zero point has index ± 1. The Euler-Poincarè Theorem states that the sum of those indices is exactly the Euler characteristic $\chi(M)$. [12]

Exercise 16.3. Let $f : \mathbf{RP}^{2n} \longrightarrow \mathbf{RP}^{2n}$ be a map. Prove that f always has a fixed point. Give an example that the above statement fails for a map $f : \mathbf{RP}^{2n+1} \longrightarrow \mathbf{RP}^{2n+1}$.

Exercise 16.4. Let $n \neq k$. Prove that $\mathbf{R}^n$ is not homeomorphic to $\mathbf{R}^k$.

Exercise 16.5. Let $f : S^n \longrightarrow S^n$ be a map, and $\deg(f)$ be the degree of f. Prove that $\mathrm{Lef}(f) = 1 + (-1)^n \deg(f)$.

[12] See J. Milnor, Differential topology, mimeographic notes. Princeton: Princeton University Press, 1958, for details.

Exercise 16.6. Prove that there is no tangent vector field $v(x)$ on the sphere S^{2n} such that $v(x) \neq 0$ for all $x \in S^{2n}$. (Compare with Lemma 13.9.)

16.2. **The Jordan-Brouwer Theorem.** This is a classical result about an embedded sphere $S^{n-1} \subset S^n$.

Theorem 16.4. (The Jordan-Brouwer Theorem) *Let $S^{n-1} \subset S^n$ be an embedded sphere in S^n. Then the complement $X = S^n \setminus S^{n-1}$ has two path-connected components: $X = X_1 \sqcup X_2$, where X_1, X_2 are open in S^n. Furthermore, $\partial \overline{X}_1 = \partial \overline{X}_2 = S^{n-1}$.*

First we prove a technical result.

Lemma 16.5. *Let $K \subset S^n$ be homeomorphic to the cube I^k, $0 \leq k \leq n$. Then*

$$\widetilde{H}_q(S^n \setminus K) = 0 \quad \textit{for all } q \geq 0.$$

Proof. Induction on k. The case $k = 0$ is obvious. Assume that the statement holds for all $0 \leq k \leq m-1$, and let K is homeomorphic to I^m. We choose a decomposition $K = L \times I$, where L is homeomorphic to I^{m-1}. Let $K_1 = L \times [0, \frac{1}{2}]$, and $K_2 = L \times [\frac{1}{2}, 1]$. Then $K_1 \cap K_2 = L \times \left\{\frac{1}{2}\right\} \cong I^{m-1}$. By induction,

$$\widetilde{H}_q(S^n \setminus K_1 \cap K_2) = 0 \quad \text{for all } q \geq 0.$$

We notice that the sets $S^n \setminus K_1$, $S^n \setminus K_2$ are both open in S^n. Thus we can use the Mayer-Vietoris exact sequence

$$\cdots \to \widetilde{H}_q(S^n \setminus K_1 \cup K_2) \longrightarrow \widetilde{H}_q(S^n \setminus K_1) \oplus \widetilde{H}_q(S^n \setminus K_2) \longrightarrow \widetilde{H}_q(S^n \setminus K_1 \cap K_2) \to \cdots$$

Thus we have that

$$\widetilde{H}_q(S^n \setminus K_1 \cup K_2) \cong \widetilde{H}_q(S^n \setminus K_1) \oplus \widetilde{H}_q(S^n \setminus K_2).$$

Assume that $\widetilde{H}_q(S^n \setminus K_1 \cup K_2) \neq 0$, and $z_0 \in \widetilde{H}_q(S^n \setminus K_1 \cup K_2)$, $z_0 \neq 0$. Then $z_0 = (z_0', z_0'')$, thus there exists $z_1 \neq 0$ in the group $\widetilde{H}_q(S^n \setminus K_1)$ or $\widetilde{H}_q(S^n \setminus K_2)$. Let, say, $z_1 \in \widetilde{H}_q(S^n \setminus K_1)$, $z_1 \neq 0$. Then we repeat the argument for K_1, and obtain the sequence

$$K \supset K^{(1)} \supset K^{(2)} \supset K^{(2)} \supset \cdots$$

such that

(1) $K^{(s)}$ is homeomorphic to I^m,
(2) the inclusion $i_s : S^n \setminus K \subset S^n \setminus K^{(s)}$ takes the element z to a nonzero element $z_s \in \widetilde{H}_q(S^n \setminus K^{(s)})$,
(3) the intersection $\bigcap_s K^{(s)}$ is homeomorphic to I^{m-1}.

We have that any compact subset C of $S^n \setminus \bigcap_s K^{(s)}$ lies in $S^n \setminus K^{(s)}$ for some s, we obtain that $C_q(S^n \setminus \bigcap_s K^{(s)}) = \varinjlim_s C_q(S^n \setminus K^{(s)})$, and, respectively,

$$\widetilde{H}_q(S^n \setminus \bigcap_s K^{(s)}) = \varinjlim_s \widetilde{H}_q(S^n \setminus K^{(s)}).$$

By construction, there exists an element $z_\infty \in \widetilde{H}_q(S^n \setminus \bigcap_s K^{(s)})$, $z_\infty \neq 0$. Contradiction to the inductive assumption. $\square$

Theorem 16.6. *Let $S^k \subset S^n$, $0 \leq k \leq n-1$. Then*

$$\widetilde{H}_q(S^n \setminus S^k) \cong \begin{cases} \mathbf{Z}, & \text{if } q = n-k-1, \\ 0 & \text{if } q \neq n-k-1. \end{cases} \tag{80}$$

Proof. Induction on k. If $k=0$, then $S^n \setminus S^0$ is homotopy equivalent to S^{n-1}. Thus the formula (80) holds for $k=0$. Let $k \geq 1$, then $S^k = D^k_+ \cup D^k_-$, where D^k_+, D^k_- are the south and northen hemispheres of S^k. Clearly $D^k_+ \cap D^k_- = S^{k-1}$. Notice that the sets $S^n \setminus D^k_\pm$ are open in S^n, we can use the Mayer-Vietoris exact sequence:

$$\cdots \to \widetilde{H}_{q+1}(S^n \setminus D^k_+) \oplus \widetilde{H}_{q+1}(S^n \setminus D^k_-) \to \widetilde{H}_{q+1}(S^n \setminus D^k_+ \cap D^k_-) \to$$

$$\to \widetilde{H}_q(S^n \setminus S^k) \to \widetilde{H}_q(S^n \setminus D^k_+) \oplus \widetilde{H}_q(S^n \setminus D^k_-) \to \cdots$$

The groups at the ends are equal zero by Lemma 16.5, thus

$$\widetilde{H}_q(S^n \setminus S^k) \cong \widetilde{H}_{q+1}(S^n \setminus S^{k-1})$$

since $D^k_+ \cap D^k_- = S^{k-1}$. This completes the induction. $\square$

Proof of Theorem 16.4. Theorem 16.6 gives that $\widetilde{H}_0(S^n \setminus S^{n-1}) \cong \mathbf{Z}$. Thus $X = S^n \setminus S^{n-1}$ has two path-connected components: $X = X_1 \sqcup X_2$. Notice that $S^{n-1} \subset S^n$ is closed and compact; thus its complement $S^n \setminus S^{n-1}$ is open. Hence X_1 and X_2 are open subsets of S^n. In paricular, for any point $x \in S^n \setminus S^{n-1}$ there is a small open disk which is contained completely either in X_1 or X_2. Assume that $x \in \partial X_1 := \overline{X}_1 \setminus X_1$. Then if $x \in X_2$, then there is an open ϵ-disk W centered at x, and $W \subset X_2$; on the other hand, $W \cap X_1 \neq \emptyset$ since $x \in \partial X_1$, or $X_1 \cap X_2 \neq \emptyset$. Contradiction. We conclude that $S^{n-1} \supset \partial \overline{X}_1$, $S^{n-1} \supset \partial \overline{X}_2$.

We have to prove that $S^{n-1} \subset \overline{X}_1 \cap \overline{X}_2$. It is enough to show that for any point $x \in S^{n-1}$ and any open neigborhood V of x is S^n, $U \cap \left(\overline{X}_1 \cap \overline{X}_2\right) \neq \emptyset$. Let $x \in S^{n-1}$, assume that $x \notin \partial X_1$. Then there exists an open disk V in S^n centered at x such that $V \cap X_1 = \emptyset$.

p_1

x p_2

A

S^{n-1}

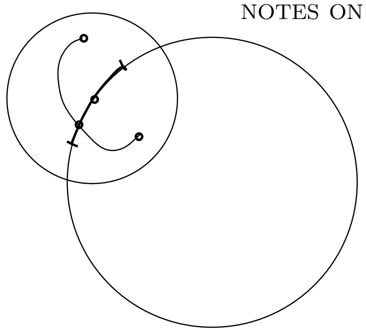


Fig. 16.1.

Let B be an open disk in S^{n-1} centered at x such that $B \subset V$. Then $A := S^{n-1} \setminus B$ is homeomorphic to D^{n-1}, and Lemma 16.5 implies that

$$\widetilde{H}_q(S^n \setminus A) = 0 \quad \text{for all } q \geq 0.$$

In particular, it means that the subspace $S^n \setminus A$ is path-connected. Then we have:

$$S^n \setminus A = X_1 \cup X_2 \cup B \subset X_1 \cup X_2 \cup V.$$

By assumption, $X_1 \cap (X_2 \cup V) = \emptyset$, thus

$$S^n \setminus A = (X_1 \cap (S^n \setminus A)) \cup ((X_2 \cup V)) \cap (S^n \setminus A))$$

is a disjoint union of two nonempty open sets. Contradiction. □

Remark. To visualize this argument, we can do the following. We just proved that for any $x \in S^{n-1}$ and any open neighborhood V of x in S^n, the intersections $V \cap X_1$ and $V \cap X_2$ are nonempty. Let $p_1 \in X_1 \cap V$, and $p_2 \in X_2 \cap V$. As we have seen above, the subspace $S^n \setminus A$ is path-connected, hence there exists a path $\gamma : I \longrightarrow S^n \setminus A$ connecting p_1 and p_2, see Fig. 16.1. Thus there exists $t \in I$ so that $\gamma(t) \in B$. Clearly $p = \gamma(t)$ belongs to $\overline{X}_1 \cap \overline{X}_2$, and $p \in S^{n-1}$. It means that $V \cap \left(\overline{X}_1 \cap \overline{X}_2\right) \neq \emptyset$. Since this is true for any open neighborhood U of x, we see one more time that $x \in \overline{X}_1 \cap \overline{X}_2$.

16.3. **The Brouwer Invariance Domain Theorem.** This is also a classical result.

Theorem 16.7. (The Brouwer Invariance Domain Theorem) *Let U and V be subsets of S^n, so that U and V are homeomorphic, and U are open in S^n. Then V is also open in S^n.*

Proof. Let $h : U \longrightarrow V$ be a homeomorphism, and $h(x) = y$. Since U is an open subset of S^n, there exsits a neighborhood A of x in U, so that A is homeomorphic to the disk D^n. Let $B = \partial A$. Denote $A' = h(A) \subset V$, $B' = h(B) \subset V$. By Lemma 16.5 the subset $S^n \setminus A'$ is path-connected, and by Theorem 16.4 the subset $S^n \setminus B'$ has two path-components. We have that

$$S^n \setminus B' = (S^n \setminus A') \cup (A' \setminus B'),$$

and the sets $S^n \setminus A'$ and $A' \setminus B'$ are path-connected, then they are the path-components of $S^n \setminus B'$. Thus $A' \setminus B'$ is open in S^n. Since $A' \setminus B' \subset V$, and $y \in V$ is an arbitrary point, the set V is open in S^n. □

16.4. **Borsuk-Ulam Theorem.** First we introduce new long exact sequence in homology which corresponds to a two-fold covering $p : T \to X$. We observe that the chain map $p_\# : \mathcal{C}(T; \mathbf{Z}/2) \to \mathcal{C}(X; \mathbf{Z}/2)$ fits into the following exact sequence of chain complexes:

$$0 \to \mathcal{C}(X; \mathbf{Z}/2) \xrightarrow{\tau} \mathcal{C}(T; \mathbf{Z}/2) \xrightarrow{p_\#} \mathcal{C}(X; \mathbf{Z}/2) \to 0. \tag{81}$$

Here the chain map $\tau : \mathcal{C}(X;\mathbf{Z}/2) \to \mathcal{C}(T;\mathbf{Z}/2)$ is defined as follows. Let $h : \Delta^q \to X$ be a generator of $C_q(X;\mathbf{Z}/2)$. Let $\Delta^q = (v_0, \dots, v_q)$, and $x_0 = h(v_0)$. Let $x_0^{(1)}, x_0^{(2)} \in T$ be two lifts of the point x_0. Then, since Δ^q is simply-connected, there exist exactly two lifts $\tilde{h}^{(i)} : \Delta^q \to T$, $i = 1, 2$ such that $\tilde{h}^{(1)}(v_0) = x_0^{(1)}$ and $\tilde{h}^{(2)}(v_0) = x_0^{(2)}$. Then

$$\tau(h : \Delta^q \to X) := (\tilde{h}^{(1)} : \Delta^q \to T) + (\tilde{h}^{(2)} : \Delta^q \to T).$$

The homomorphism τ is sometimes called a *transfer homomorphism*. On the other hand, it easy to see that the kernel of $p_\# : C_q(T;\mathbf{Z}/2) \to C_q(X;\mathbf{Z}/2)$ is generated by the sums $(\tilde{h}^{(1)} : \Delta^q \to T) + (\tilde{h}^{(2)} : \Delta^q \to T)$. Thus the short exact sequence (81) gives a long exact sequence in homology groups (with $\mathbf{Z}/2$ coefficients):

$$\cdots \to H_q(X;\mathbf{Z}/2) \xrightarrow{\tau_*} H_q(T;\mathbf{Z}/2) \xrightarrow{p_*} H_q(X;\mathbf{Z}/2) \xrightarrow{\partial} H_{q-1}(X;\mathbf{Z}/2) \to \cdots \tag{82}$$

We will use the long exact sequence (82) to prove the following result, known as Borsuk-Ulam Theorem.

Theorem 16.8. *Let $f : S^n \to S^n$ be a map such that $f(-x) = -f(x)$ (an "odd map"). Then* $\deg f$ *is odd.*

Proof. Consider the long exact sequence (82) for the covering $S^n \to \mathbf{RP}^n$:

$$0 \to H_n(\mathbf{RP}^n) \xrightarrow{\tau_*} H_n(S^n) \xrightarrow{p_*} H_n(\mathbf{RP}^n) \xrightarrow{\partial} H_{n-1}(\mathbf{RP}^n) \to 0 \to \cdots$$

$$\cdots \to 0 \to H_q(\mathbf{RP}^n) \xrightarrow{\partial} H_{q-1}(\mathbf{RP}^n) \to 0 \to \cdots$$

$$\cdots \to 0 \to H_1(\mathbf{RP}^n) \xrightarrow{\partial} H_0(\mathbf{RP}^n) \to H_0(S^n) \xrightarrow{p_*} H_0(\mathbf{RP}^n) \to 0$$

The exactness forces that the homomorphisms

$$\tau_* : H_n(\mathbf{RP}^n) \to H_n(S^n),$$

$$\partial : H_q(\mathbf{RP}^n) \to H_{q-1}(\mathbf{RP}^n), \ q = n, n-1, \dots, 1,$$

$$p_* : H_0(S^n) \to H_0(\mathbf{RP}^n)$$

to be isomorphisms, and $p_* : H_q(S^n) \to H_q(\mathbf{RP}^n)$ to be zero for $q > 0$.

Now let $f : S^n \to S^n$ be a map such that $f(-x) = -f(x)$. Then it induces a quotient map $\bar{f} : \mathbf{RP}^n \to \mathbf{RP}^n$, such that the diagram

$$\begin{tikzcd} S^n \arrow[r, "f"] \arrow[d, "p"'] & S^n \arrow[d, "p"'] \\ \mathbf{RP}^n \arrow[r, "\bar{f}"] & \mathbf{RP}^n \end{tikzcd}$$

Now we notice that $f_* : H_0(S^n) \to H_0(S^n)$ $\bar{f}_* : H_0(\mathbf{RP}^n) \to H_0(\mathbf{RP}^n)$ are isomorphisms, then we use naturality of the exact sequence (82) to get the commutative diagrams

$$\begin{tikzcd} H_q(\mathbf{RP}^n) \arrow[r, "\partial"] \arrow[d, "\bar{f}_*"'] & H_{q-1}(\mathbf{RP}^n) \arrow[d, "\bar{f}"'] \\ H_q(\mathbf{RP}^n) \arrow[r, "\partial"] & H_{q-1}(\mathbf{RP}^n) \end{tikzcd} \qquad \begin{tikzcd} H_n(\mathbf{RP}^n) \arrow[r, "\tau"] \arrow[d, "\bar{f}_*"'] & H_n(S^n) \arrow[d, "f_*"'] \\ H_n(\mathbf{RP}^n) \arrow[r, "\tau"] & H_n(S^n) \end{tikzcd} .$$

for $q = 1, \dots n$. In particular, we obtain that $f_* : H_n(S^n; \mathbf{Z}/2) \to H_n(S^n; \mathbf{Z}/2)$ is an isomorphism. On the other hand, we know that for integral homology groups

$$f_* : H_n(S^n) \to H_n(S^n)$$

is a multiplication by the degree $\deg f$. We obtain that after reduction modulo two f_* is isomorphism:

$$\begin{tikzcd} H_n(S^n; \mathbf{Z}) \arrow[r, "f_*"] \arrow[d, "pr"'] & H_n(S^n; \mathbf{Z}) \arrow[d, "pr"'] \\ H_n(S^n; \mathbf{Z}/2) \arrow[r, "\cong"] & H_n(S^n; \mathbf{Z}/2) \end{tikzcd}$$

Thus the degree $\deg f$ must be odd. □

Exercise 16.7. Let $0 \le p, q \le n-1$, and the wedge $S^p \vee S^q$ is embedded to S^n. Compute the homology groups $H_q(S^n \setminus (S^p \vee S^q))$.

Exercise 16.8. Prove that for each $n \ge 1$ there exists a space X with

$$\widetilde{H}_q(X) = \begin{cases} \mathbf{Z}/m, & \text{if } q = n, \\ 0, & \text{if } q \neq n. \end{cases}$$

Exercise 16.9. Let $\mathcal{H} = \{H_q\}$ be a graded abelian group. We assume that $H_q = 0$ for $q < 0$, and H_0 is a free abelian. Prove that there exists a space X such that $H_q(X) = H_q$ for all q. In particular, construct a space X with the homology groups:

$$\widetilde{H}_q(X) = \begin{cases} \mathbf{Z}[\frac{1}{p}], & \text{if } q = n, \\ 0, & \text{if } q \neq n. \end{cases}$$

17. Cup product in cohomology.

17.1. **Ring structure in cohomology.** The homology groups are more "geometric" than the cohomology. However, there is a natural ring structure in cohomology groups which is very useful. The Künneth formula gives natural homomorphism

$$m : H^k(X;\mathbf{Z}) \otimes H^\ell(X;\mathbf{Z}) \longrightarrow H^{k+\ell}(X\times X;\mathbf{Z})$$

Consider the diagonal map $\Delta : X \longrightarrow X\times X$ which sends x to the pair (x,x). Then we have the composition

$$H^k(X;\mathbf{Z}) \otimes H^\ell(X;\mathbf{Z}) \xrightarrow{m} H^{k+\ell}(X\times X;\mathbf{Z}) \xrightarrow{\Delta^*} H^{k+\ell}(X;\mathbf{Z}).$$

which gives the product structure in cohomology. The way we defined this product does not allow us to compute actual ring structure for particular spaces. What we are going to do is to work out this in detail starting with cup-product at the level of singular cochains.

17.2. **Definition of the cup-product.** First we need some notations. We identify a simplex Δ^q with one given by its vertices $(v_0,\ldots,v_q)$ in $\mathbf{R}^{q+1}$. Let $g:\Delta^q \longrightarrow X$ be a map. It is convenient to use symbol $(v_0,\ldots,v_q)$ to denote the singular simplex $g:\Delta^q\longrightarrow X$, and, say, $(v_0,\ldots,v_s)$ the restriction $g|_{(v_0,\ldots,v_s)}$.

Let R be a commutative ring with unit. We consider cohomology groups with coefficients in R. The actual examples we will elaborate are when $R=\mathbf{Z}$, $\mathbf{Z}/p$, $\mathbf{Q}$, $\mathbf{R}$. Let $\varphi\in C^k(X)$, $\psi \in C^\ell(X)$ be singular cochains, and $f:\Delta^{k+\ell}\longrightarrow X$ be a singular simplex. We define the cochain $\varphi\cup\psi\in C^{k+\ell}(X)$ as follows:

$$\langle \varphi\cup\psi,(v_0,\ldots,v_{k+\ell})\rangle = (\varphi\cup\psi)(v_0,\ldots,v_{k+\ell}) := \varphi(v_0,\ldots,v_k)\psi(v_k,\ldots,v_{k+\ell}).$$

To see that the cup-product at the level of cochains induces a product in cohomology groups, we have to undestand the coboundary homomorphism on $\varphi\cup\psi$.

Lemma 17.1. *Let $\varphi\in C^k(X)$, $\psi\in C^l(X)$. Then*

$$\delta(\varphi\cup\psi) = \delta\varphi\cup\psi + (-1)^k\varphi\cup\delta\psi.$$

Proof. Let $g:\Delta^{k+\ell+1}\longrightarrow X$ be a singular simplex. We compute $\langle\delta\varphi\cup\psi,g\rangle$ and $\langle\varphi\cup\delta\psi,g\rangle$:

$$\langle\delta\varphi\cup\psi,g\rangle = \sum_{j=0}^{k+1}(-1)^j\varphi(v_0,\ldots,\widehat{v}_j,\ldots,v_{k+1})\psi(v_{k+1},\ldots,v_{k+\ell+1}),$$

$$\begin{aligned}\langle\varphi\cup\delta\psi,g\rangle &= \sum_{j=k}^{k+\ell+1}(-1)^{j+k}\varphi(v_0,\ldots,v_k)\psi(v_k,\ldots,\widehat{v}_j,\ldots,v_{k+\ell+1}). \qquad (83)\\ &= (-1)^k\sum_{j=k}^{k+\ell+1}(-1)^j\varphi(v_0,\ldots,v_k)\psi(v_k,\ldots,\widehat{v}_j,\ldots,v_{k+\ell+1}).\end{aligned}$$

Consider the following terms in (83):

$$(-1)^{k+1}\varphi(v_0,\dots,\dots,v_k)\psi(v_{k+1},\dots,v_{k+l+1})$$

$$(-1)^{k}\varphi(v_0,\dots,\dots,v_k)\psi(v_{k+1},\dots,v_{k+l+1})$$

The first one corresponds to $j=k+1$ in the formula for $\langle\delta\varphi\cup\psi,g\rangle$, and the second one corresponds to $j=k$ in the formula for $\langle\varphi\cup\delta\psi,g\rangle$. Clearly they cancel each other, and we have that

$$\begin{aligned}\langle\delta\varphi\cup\psi+(-1)^k\varphi\cup\delta\psi,g\rangle &= \sum_{j=0}^{k}(-1)^j\varphi(v_0,\dots,\widehat{v}_j,\dots,v_{k+1})\psi(v_{k+1},\dots,v_{k+l+1})\\ &+ \sum_{j=k+1}^{k+l+1}(-1)^j\varphi(v_0,\dots,v_k)\psi(v_k,\dots,\widehat{v}_j,\dots,v_{k+l+1})\\ &= \sum_{j=0}^{k+l+1}(-1)^j(\varphi\cup\psi)(v_0,\dots,\widehat{v}_j,\dots,v_{k+l+1})\\ &= \langle\varphi\cup\psi,\partial g\rangle=\langle\delta(\varphi\cup\psi),g\rangle.\end{aligned} \tag{84}$$

This concludes the proof. □

Now it is clear that the cup product of two cocyles is a cocycle, and the cup-product of cocycle and coboundary is a coboundary. We conclude that the cup-product in the cochain groups induces the cup-product in cohomology:

$$\cup: H^k(X;R)\times H^l(X;R)\longrightarrow H^{k+l}(X;R),$$

where R is a commutative ring. The cup-product induces the ring structure on $H^*(X;R)$. Let $K^*=\bigoplus K^j$ be a graded R-module, with $K^0=R$ and $K^j=0$ for $j<0$. We say that K^* is a graded algebra over R if there is a product $\mu: K^*\otimes K^*\longrightarrow K^*$ so that $\mu: K^k\otimes K^\ell\longrightarrow K^{k+\ell}$, and the unit $1\in R=K^0$ is the unit of the product μ, i.e. $\mu(1\otimes a)=\mu(a\otimes 1)=a$. We say that (K^*,μ) is a graded commutative R-algebra if $\mu(a\otimes b)=(-1)^{k\ell}\mu(b\otimes a)$, where $\deg a=k$, $\deg b=\ell$.

Claim 17.1. *Let R be a commutative ring. Then $H^*(X;R)$ is a graded commutative R-algebra.*

Construction. Let $f:\Delta^q\longrightarrow X$ be a singular simplex, and the simplex Δ^q is given by its vertices $(v_0,\dots,v_q)$. Consider the singular simplex $\bar f:\bar\Delta^q\longrightarrow X$, where $\bar\Delta^q=(v_q,\dots,v_0)$, and

$$\bar f:\bar\Delta^q\xrightarrow{T}\Delta^q\xrightarrow{f}X$$

where T is given by the linear isomorphism $T : \mathbf{R}^{q+1} \longrightarrow \mathbf{R}^{q+1}$ sending vertices $(v_0, \ldots, v_q)$ to $(v_q, \ldots, v_0)$ respectively. Clearly as a linear map, T is given by the matrix

$$T = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \vdots\,\vdots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

which has determinant $\det T = (-1)^{\frac{q(q+1)}{2}}$. Thus T induces the chain map

$$t : \mathcal{C}_*(X) \longrightarrow \mathcal{C}_*(X)$$

sending a generator $f : \Delta^q \longrightarrow X$ to the generator $\bar{f} : \bar{\Delta}^q \longrightarrow X$.

Exercise 17.1 Prove that there is a chain homotopy between t and $(-1)^{\frac{q(q+1)}{2}} Id$.

The homomorphism $t : \mathcal{C}_*(X) \longrightarrow \mathcal{C}_*(X)$ induces the homomorphism

$$t^* : \mathcal{C}^*(X; R) \longrightarrow \mathcal{C}^*(X; R).$$

Clearly in cohomology the homomorphism t^* coincides with

$$(-1)^{\frac{q(q+1)}{2}} Id : H^q(X; R) \longrightarrow H^q(X; R).$$

Proof of Claim 17.1. Let $\varphi \in C^k(X; R)$, $\psi \in C^l(X; R)$, and $f : \Delta^{k+l} \longrightarrow X$ be a singular simplex. As above, we denote the singular simplex by its vertices $(v_0, \ldots, v_{k+l})$. We have:

$$\begin{aligned}
(\varphi \cup \psi)(v_0, \ldots, v_{k+l}) &= \varphi(v_0, \ldots, v_k)\psi(v_k, \ldots, v_{k+l}) \\
&= (-1)^{\frac{k(k+1)}{2}} \varphi(v_k, \ldots, v_0)(-1)^{\frac{l(l+1)}{2}} \psi(v_{k+l}, \ldots, v_k) \\
&= (-1)^{\frac{k(k+1)}{2} + \frac{l(l+1)}{2}} \psi(v_{k+l}, \ldots, v_k)\varphi(v_k, \ldots, v_0) \\
&= (-1)^{\frac{k(k+1)}{2} + \frac{l(l+1)}{2}} (\psi \cup \psi)(v_{k+l}, \ldots, v_0) \\
&= (-1)^{\frac{k(k+1)}{2} + \frac{l(l+1)}{2}} (-1)^{\frac{(k+l)(k+l+1)}{2}} (\psi \cup \psi)(v_0, \ldots, v_{k+l}) \\
&= (-1)^{kl} (\psi \cup \varphi)(v_0, \ldots, v_{k+l}).
\end{aligned}$$

Here we have identified singular simplex $\bar{g} = g \circ T$ with the map $(-1)^{\frac{q(q+1)}{2}} g$ for a singular simplex $g : \Delta^q \longrightarrow X$. $\square$

Theorem 17.2. (Properties of the cup-product) *Let X be a space and R a commutative ring with unit. Let $\gamma \in H^q(X; R)$, $\gamma' \in H^{q'}(X; R)$, $\gamma'' \in H^{q''}(X; R)$ $j = 1, 2, 3$ be any elements. Then*

(1) $\gamma \cup \gamma' = (-1)^{qq'} \gamma' \cup \gamma$;

(2) $(\gamma \cup \gamma') \cup \gamma'' = \gamma \cup (\gamma' \cup \gamma'')$;

(3) $f^*(\gamma \cup \gamma') = (f^*\gamma) \cup (f^*\gamma')$ *and* $\alpha_*(\gamma \cup \gamma') = (\alpha_*\gamma) \cup (\alpha_*\gamma')$ *for any map* $f : X \to X'$ *and ring homomorphism* $\alpha : R \to R'$.

17.3. **Example.** We compute the cup-product in cohomology $H^*(M_g^2; \mathbf{Z})$, where M_g^2 is the oriented surface of genus g. We think of M_g^2 as a $4g$-sided polygon with corresponding edges identified. We consider carefully only a part of this polygon given at Fig. 17.1. The only cup-product of interest is the product

$$H^1(M_g^2; \mathbf{Z}) \times H^1(M_g^2; \mathbf{Z}) \xrightarrow{\cup} H^2(M_g^2; \mathbf{Z}).$$

To compute this product we choose particular generators in the first homology and cohomology groups. First, we choose a simplicial (and cell) structure on M_g^2 as it is shown at Fig. 17.1.

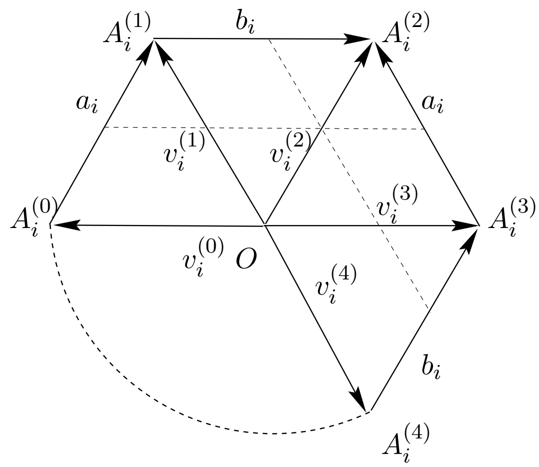


Fig. 17.1.

A basis for the homology group $H_1(M_g^2)$ is given by the 1-simplices (or 1-cells) a_i, b_i, $i = 1, \dots, g$. Then the basis of

$$H^1(M_g^2; \mathbf{Z}) = \mathrm{Hom}(H_1(M_g^2), \mathbf{Z})$$

is given by elements α_i, β_i, so that

$$\langle \alpha_i, b_j \rangle = 0, \quad \langle \alpha_i, a_j \rangle = \delta_{ij},$$
$$\langle \beta_i, b_j \rangle = \delta_{ij}, \quad \langle \beta_i, a_j \rangle = 0.$$

We choose the following cocycles φ_i, ψ_i representing α_i, β_i respectively.

We define φ_i to be equal to 1 on the adges meeting the dash-line connecting the sides a_i, and zero on all others. Similarly we define ψ_i to be on the adges meeting the dash-line connecting the sides b_i, and zero on all others. Thus

$$\langle \varphi, v_i^{(1)} \rangle = \langle \varphi, v_i^{(2)} \rangle = \langle \varphi, a_i \rangle = 1,$$
$$\langle \psi, v_i^{(2)} \rangle = \langle \psi, v_i^{(3)} \rangle = \langle \psi, b_i \rangle = 1,$$

and they are zero on all other 1-simplices. It is easy to check that $\delta\varphi_i = 0$ and $\delta\psi_i = 0$. For example, we see that

$$\langle \delta\varphi_i, (O, A_i^{(0)}, A_i^{(1)}) \rangle = \langle \varphi_i, \partial(O, A_i^{(0)}, A_i^{(1)}) \rangle = \langle \varphi_i, a_i \rangle - \langle \varphi_i, v_i^{(1)} \rangle + \langle \varphi_i, v_i^{(0)} \rangle = 0,$$

$$\langle \delta\varphi_i, (O, A_i^{(1)}, A_i^{(2)}) \rangle = \langle \varphi_i, \partial(O, A_i^{(1)}, A_i^{(2)}) \rangle = \langle \varphi_i, v_i^{(1)} \rangle + \langle \varphi_i, b_i \rangle - \langle \varphi_i, v_i^{(2)} \rangle = 0.$$

To compute the cup-product, we notice that $\varphi_i \cup \varphi_j = 0$ if $i \neq j$. Now we have:

$$\langle \varphi_i \cup \psi_i, (O, A_i^{(0)}, A_i^{(1)}) \rangle = \langle \varphi_i, v_i^{(0)} \rangle \langle \psi_i, a_i \rangle = 0,$$

$$\langle \varphi_i \cup \psi_i, (O, A_i^{(1)}, A_i^{(2)}) \rangle = \langle \varphi_i, v_i^{(1)} \rangle \langle \psi_i, b_i \rangle = 1,$$

$$\langle \varphi_i \cup \psi_i, (O, A_i^{(2)}, A_i^{(3)}) \rangle = \langle \varphi_i, v_i^{(2)} \rangle \langle \psi_i, a_i \rangle = 0,$$

$$\langle \varphi_i \cup \psi_i, (O, A_i^{(3)}, A_i^{(4)}) \rangle = \langle \varphi_i, v_i^{(3)} \rangle \langle \psi_i, b_i \rangle = 0.$$

Now we notice that the cycle generating $H_2(M_g^2)$ is represented by

$$c = \sum_{i=1}^{g} \left((O, A_i^{(0)}, A_i^{(1)}) + (O, A_i^{(1)}, A_i^{(2)}) + (O, A_i^{(2)}, A_i^{(3)}) + (O, A_i^{(3)}, A_i^{(4)}) \right).$$

Hence we have that $\langle \varphi_i \cup \psi_i, c \rangle = 1$, and the element $\varphi_i \cup \psi_i = \gamma$, where γ is a generator of $H^2(M_g^2; \mathbf{Z})$, such that $\langle \gamma, c \rangle = 1$.

Claim 17.2. *The cohomology ring $H^*(M_g^2; \mathbf{Z})$ has the following structure:*

$$\begin{array}{ll} \alpha_i \cup \beta_j = \delta_{ij}\gamma, & \\ \alpha_i \cup \alpha_j = 0, & \beta_i \cup \beta_j = 0, \\ \alpha_i \cup \beta_j = -\beta_j \cup \alpha_i. & \end{array}$$

where α_i, β_i are generators of $H^1(M_g^2; \mathbf{Z})$, and γ is a generator of $H^2(M_g^2; \mathbf{Z})$,

Exercise 17.2. Compute the cup-product for $H^*(\mathbf{RP}^2; \mathbf{Z}/2)$. Hint: Use the simplicial (or cell-decomposition) indicated on Fig. 17.2.

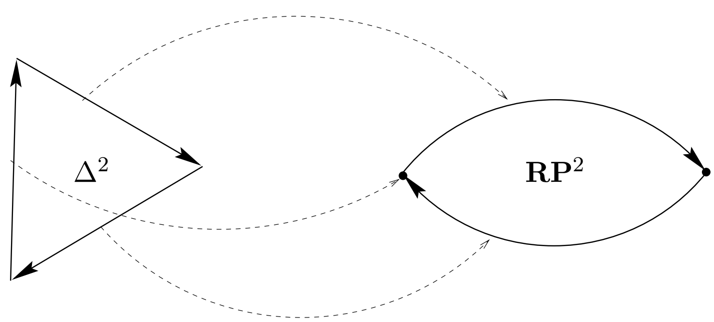


Fig. 17.2.

Exercise 17.3. Let N_g^2 be nonoriented surface of genus g, i.e. $N_g^2 = T^2 \# \cdots T^2 \# \mathbf{RP}^2$. Compute the cup-product for $H^*(N_g^2; \mathbf{Z}/2)$.

Exercise 17.4. Compute the cup product for $H^*(\mathbf{RP}^2; \mathbf{Z}/2^k)$, $k \geq 2$.

Exercise 17.5. Let $X = S^1 \cup_{f_k} e^2$, where $f_k : S^1 \longrightarrow S^1$ is a degree k map. Compute the cup product for $H^*(X; \mathbf{Z}/k)$.

17.4. **Relative case.** The same formula which defines the cup product

$$H^k(X;R) \times H^l(X;R) \longrightarrow H^{k+l}(X;R)$$

also gives the products:

$$H^k(X;R) \times H^l(X,A;R) \xrightarrow{\cup} H^{k+l}(X,A;R),$$

$$H^k(X,A;R) \times H^l(X;R) \xrightarrow{\cup} H^{k+l}(X,A;R),$$

$$H^k(X,A;R) \times H^l(X,A;R) \xrightarrow{\cup} H^{k+l}(X,A;R).$$

Furthermore, if A, B are open in X or A, B are subcomplexes of a CW-complex X, then there is a more general cup relative product

$$H^k(X,A;R) \times H^l(X,B;R) \xrightarrow{\cup} H^{k+l}(X,A \cup B;R).$$

We do not give details to define the last product. However we mention that the absolute cup product $C^k(X;R) \times C^l(X;R) \xrightarrow{\cup} C^{k+l}(X;R)$ restricts to a cup product

$$C^k(X,A;R) \times C^l(X,B;R) \xrightarrow{\cup} C^{k+l}(X,A+B;R),$$

where the group $C^q(X,A+B;R)$ consits of cochains which vanish on chains in $C_q(A)$ and in $C_q(B)$. Then one should show that the inclusion map

$$C^q(X,A+B;R) \longrightarrow C^q(X,A \cup B;R)$$

induce isomorphism in cohomology groups, in a similar way as we did proving the Excision Theorem and Mayer-Vietoris Theorem.

Exercise 17.6. Prove that the cup product is natural, i.e. if $f : X \longrightarrow Y$ is a map, and $f^* : H^*(Y;R) \longrightarrow H^*(X;R)$ is the induced homomorphism, then

$$f^*(a \cup b) = f^*(a) \cup f^*(b).$$

17.5. **External cup product.** We define an *external cup product*

$$\mu : H^*(X_1;R) \otimes_R H^*(X_2;R) \longrightarrow H^*(X_1 \times X_2;R)$$

as follows. Let $p_i : X_1 \times X_2 \longrightarrow X_i$ $(i = 1,2)$ be the projection onto X_i, i.e. $p_i(x_1,x_2) = x_i$. Then $\mu(a \otimes b) = p_1^*(a) \cup p_2^*(b)$. The above tensor product $\otimes_R$ is taken over the ring R, where $H^*(X_i;R)$ are considered as R-modules. The tensor product $H^*(X_1;R) \otimes_R H^*(X_2;R)$ has natural multiplication defined as

$$(a_1 \otimes a_2) \cdot (b_1 \otimes b_2) = (-1)^{\deg b_1 \ \deg a_2}(a_1 b_1 \otimes a_2 b_2).$$

Claim 17.3. *The external product* $H^*(X_1;R)\otimes_R H^*(X_2;R) \xrightarrow{\mu} H^*(X_1\times X_2;R)$ *is a ring homomorphism.*

Proof. Indeed, we have:

$$\begin{aligned}
\mu((a_1\otimes a_2)\cdot(b_1\otimes b_2)) &= \mu((-1)^{\deg b_1\ \deg a_2}(a_1b_1\otimes a_2b_2))\\
&= (-1)^{\deg b_1\ \deg a_2}p_1^*(a_1\cup b_1)\cup p_2^*(a_2\cup b_2)\\
&= (-1)^{\deg b_1\ \deg a_2}p_1^*(a_1)\cup p_1^*(b_1)\cup p_2^*(a_2)\cup p_2^*(b_2)\\
&= p_1^*(a_1)\cup p_2^*(a_2)\cup p_1^*(b_1)\cup p_2^*(b_2)\\
&= \mu(a_1\otimes a_2)\mu(b_1\otimes b_2). \qquad \square
\end{aligned}$$

There are many important cases when the external product μ is an isomorphism, for example for $X_1=S^{k_1}$, and $X_2=S^{k_2}$.

Theorem 17.3. *Let a space* X_2 *be such that* $H^q(X_2;R)$ *is finitely generated free* R*-module for each* q*. Then the external product*

$$H^*(X_1;R)\otimes_R H^*(X_2;R) \xrightarrow{\mu} H^*(X_1\times X_2;R)$$

is a ring isomomorphism.

Proof. First we notice that it is enough to prove Theorem 17.3 for CW-complexes X_1 and X_2 since for any space X there is a weak homotopy equivalent CW-complex X'. Next we notice that if k, l are given, then the external product $H^k(X_1;R)\otimes_R H^l(X_2;R) \xrightarrow{\mu} H^{k+l}(X_1\times X_2;R)$ is determined of finite skeletons of X_1 and X_2. Thus it is enough to prove the statement for finite CW-complexes X_1 and X_2. We need the following result.

Lemma 17.4. *Let (X, A) be a pair spaces, and Y be a space. The following diagram commutes:*

$$
\begin{tikzcd}
H^*(X,A)\otimes H^*(Y) \arrow[rr, "\beta\otimes 1"] \arrow[dd, "\mu"'] & & H^*(X)\otimes H^*(Y) \arrow[dl, "j\otimes 1"'] \arrow[dd, "\mu"] \\
& H^*(A)\otimes H^*(Y) \arrow[ul, "\delta\otimes 1"'] \arrow[dd, "\mu"] & \\
H^*(X\times Y, A\times Y) \arrow[rr, "\bar\beta"] & & H^*(X\times Y) \arrow[dl, "\bar j"] \\
& H^*(A\times Y) \arrow[ul, "\bar\delta"] &
\end{tikzcd}
\tag{85}
$$

where the above homomorphisms are from the exact sequences:

$$
\begin{tikzcd}
H^*(X,A) \arrow[rr, "\beta"] & & H^*(X) \arrow[dl, "j"] \\
& H^*(A) \arrow[ul, "\delta"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
H^*(X\times Y, A\times Y) \arrow[rr, "\bar\beta"] & & H^*(X\times Y) \arrow[dl, "\bar j"] \\
& H^*(A\times Y) \arrow[ul, "\bar\delta"] &
\end{tikzcd}
$$

Proof. The commutativity follows from the naturality of the external product and the naturality of the Künneth formula. □

We return to the proof of Theorem 17.3. Let X_1 be a zero-dimensional CW-complex, then

$$\mu : H^*(X_1) \otimes H^*(X_2) \longrightarrow H^*(X_1 \times X_2)$$

is an isomorphism since $H^q(X_1 \times X_2) \cong H^0(X_1) \otimes H^q(X_2)$. Assume Theorem 17.3 holds for all CW-complexes X_1 of dimension at most $n-1$. Consider the pair (D^n, S^{n-1}). The homomorphisms

$$\mu : H^*(D^n) \otimes H^*(X_2) \longrightarrow H^*(D^n \times X_2),$$

$$\mu : H^*(S^{n-1}) \otimes H^*(X_2) \longrightarrow H^*(S^{n-1} \times X_2)$$

are isomorphisms: the first one since $D^n \sim *$, and the second one by induction. Consider the diagram

$$
\begin{tikzcd}
H^*(D^n,S^{n-1})\otimes H^*(X_2) \arrow[rr, "\beta\otimes 1"] \arrow[dd, "\mu"'] & & H^*(D^n)\otimes H^*(X_2) \arrow[dl, "j\otimes 1"] \arrow[dd, "\mu"] \\
& H^*(S^{n-1})\otimes H^*(X_2) \arrow[ul, "\delta\otimes 1"] \arrow[dd, "\mu"] & \\
H^*(D^n\times X_2, S^{n-1}\times X_2) \arrow[rr, "\bar\beta"] & & H^*(D^n\times X_2) \arrow[dl, "\bar j"] \\
& H^*(S^{n-1}\times X_2) \arrow[ul, "\bar\delta"] &
\end{tikzcd}
\tag{86}
$$

which commutes by Lemma 17.4. Recall that $H^*(X_2)$ is a finitely generated free R-module. This implies that tensoring by over R by $H^*(X_2)$ preserves exactness. In other words, we have the implication:

$$
\begin{tikzcd}
H^*(D^n,S^{n-1}) \arrow[rr, "\beta"] & & H^*(D^n) \arrow[dl, "j"] \\
& H^*(S^{n-1}) \arrow[ul, "\delta"] &
\end{tikzcd}
\quad \Longrightarrow \quad
\begin{tikzcd}
H^*(D^n,S^{n-1})\otimes H^*(X_2) \arrow[rr, "\beta\otimes 1"] & & H^*(D^n)\otimes H^*(X_2) \arrow[dl, "j\otimes 1"] \\
& H^*(S^{n-1}\otimes H^*(X_2) \arrow[ul, "\delta\otimes 1"] &
\end{tikzcd}
$$

exact sequence $\qquad\qquad\qquad\qquad$ exact sequence

Now by 5-lemma, applied to the diagram (86), the homomorphism

$$\mu : H^*(D^n,S^{n-1})\otimes H^*(X_2) \longrightarrow H^*(D^n\times X_2, S^{n-1}\times X_2)$$

is an isomorphism. It follows now that the homomorphism

$$\mu : H^*(X,A)\otimes H^*(X_2) \longrightarrow H^*(X\times X_2, A\times X_2),$$

where $X = \bigvee_j D^n_j$, $A = \bigvee_j S^{n-1}_j \subset X$ is an isomorphism as well.

Now we prove the induction step. Consider the pair $(X^{(n)}, X^{(n-1)})$. We have the commutative diagram:

$$
\begin{tikzcd}
H^*(\bigvee_j (D^n_j, S^{n-1}_j)) \otimes H^*(X_2) \arrow[dr, "\cong"] \arrow[dddd, "\cong"'] & & & \\
& H^*(X^{(n)}, X^{(n-1)}) \otimes H^*(X_2) \arrow[rr, "\beta\otimes 1"] \arrow[dd, "\mu"'] & & H^*(X^{(n)}) \otimes H^*(X_2) \arrow[dl, "j\otimes 1"] \arrow[dd, "\mu"] \\
& & H^*(X^{(n-1)}) \otimes H^*(X_2) \arrow[ul, "\delta\otimes 1"'] \arrow[dd, "\mu"] & \\
& H^*(X^{(n)} \times X_2, X^{(n-1)} \times X_2) \arrow[rr, "\bar\beta"] & & H^*(X^{(n)} \times X_2) \arrow[dl, "\bar j"] \\
H^*(\bigvee_j D^n_j \times X_2, \bigvee S^{n-1}_j \times X_2) \arrow[ur, "\cong"] & & H^*(X^{(n-1)} \times X_2) \arrow[ul, "\bar\delta"] &
\end{tikzcd}
$$

Now 5-lemma implies that

$$\mu : H^*(X^{(n)}) \otimes H^*(X_2) \longrightarrow H^*(X^{(n)} \times X_2)$$

is an isomorphism. $\square$.

We notice that in fact we proved a relative version of Theorem 17.3:

Theorem 17.5. *Let (X, A) be any pair of spaces, and (Y, B) be such a pair that $H^q(Y, B; R)$ is finitely generated free R-module for each $q \geq 0$. Then the external product*

$$\mu : H^*(X, A; R) \otimes_R H^*(Y, B; R) \longrightarrow H^*(X \times Y, A \times Y \cup X \times B; R)$$

is an isomorphism.

Remark. Recall that we can define the product in $H^*(X)$ as the composition

$$H^k(X) \otimes H^l(X) \xrightarrow{\mu} H^{k+l}(X \times X) \xrightarrow{\Delta^*} H^{k+l}(X),$$

where $\Delta : X \longrightarrow X \times X$ is the diagonal map. Indeed, we have:

$$\Delta^* \mu(a \times b) = \Delta^*(p_1^*(a) \cup p_2^*(b)) = \Delta^*(p_1^*(a)) \cup \Delta^*(p_2^*(b)) = a \cup b.$$

Here $X \xleftarrow{p_1} X \times X \xrightarrow{p_2} X$ are the projections on the first and the second factors.

Recall that the *exterior algebra* $\Lambda_R(x_1, \dots x_n)$ *over a ring* R is given by the relations:

$$x_i x_j = -x_j x_i, \text{if } i \neq j \text{ and } x_i^2 = 0.$$

Corollary 17.6. *Let* $X = S^{2\ell_1+1} \times \cdots \times S^{2\ell_t+1} \times S^{2k_1} \times \cdots \times S^{2k_s}$. *Then*

$$H^*(X;R) \cong \Lambda_R(x_{2\ell_1+1}, \ldots, x_{2\ell_t+1}) \otimes R[x_{2k_1}, \ldots, x_{2k_s}]/x_{2k_1}^2, \ldots, x_{2k_s}^2,$$

where $\deg x_{2\ell_i+1} = 2l_i + 1$, $\deg x_{2k_j} = 2k_j$.

Example. Here is an important application of Theorem 17.5. We consider the pairs $(X, A) = (D^k, S^{k-1})$ and $(Y, B) = (D^\ell, S^{\ell-1})$. Then clearly the pair $(D^\ell, S^{\ell-1})$ satisfies the conditions of Theorem 17.5. Thus we have a ring isomorphism

$$\begin{aligned} H^*(D^k, S^{k-1}; R) \otimes_R H^*(D^\ell, S^{\ell-1}; R) &\cong H^*(D^k \times D^\ell, S^{k-1} \times D^\ell \cup (-1)^k D^k \times S^{\ell-1}; R) \\ &\cong H^*(D^{k+\ell}, S^{k+\ell-1}; R) \end{aligned}$$

since $S^{k-1} \times D^\ell \cup (-1)^k D^k \times S^{\ell-1} \cong S^{k+\ell-1}$.

18. Cap product and the Poincarè duality.

18.1. **Definition of the cap product.** Let X be a space, and R be a commutative ring. We define an R-linear cap product map $\cap : C_{k+\ell}(X;R) \times C^k(X;R) \longrightarrow C_\ell(X;R)$ as follows. Let $f : \Delta^{k+\ell} \longrightarrow X$ be a generator of $C_{k+\ell}(X;R)$, and $\varphi \in C^k(X;R)$. As before, we use the notation $(v_0, \dots, v_{k+\ell})$ for $f : \Delta^{k+\ell} \longrightarrow X$. Then

$$f \cap \varphi := \varphi(v_0, \dots, v_k)(v_k, \dots, v_{k+\ell}).$$

By linearity we define the cap product $\sigma \cap \varphi$ for any $\sigma \in C_{k+\ell}(X;R)$ and $\varphi \in C^k(X;R)$.

Let us fix a cochain $\varphi \in C^k(X;R)$, then for any cochain $\psi \in C^\ell(X;R)$, we have the composition

$$C_{k+\ell}(X;R) \xrightarrow{\cap\varphi} C_\ell(X;R) \xrightarrow{\psi} R,$$

i.e. the element $\psi(\sigma \cap \varphi) \in R$. We notice that in the case when $\sigma = (v_0, \dots, v_{k+\ell})$ is a generator $f : \Delta^{k+\ell} \longrightarrow X$, then

$$\begin{aligned} \psi(\sigma \cap \varphi) &= \psi(\varphi(v_0, \dots, v_k)(v_k, \dots, v_{k+\ell})) \\ &= \varphi(v_0, \dots, v_k)\psi(v_k, \dots, v_{k+\ell}) \\ &= (\varphi \cup \psi)\sigma. \end{aligned}$$

We write this as $\langle \psi, \sigma \cap \varphi \rangle = \langle \varphi \cup \psi, \sigma \rangle$. In particular, we use Lemma 17.1 to compute

$$\begin{aligned} \langle \psi, \partial(\sigma \cap \varphi) \rangle &= \langle \delta\psi, \sigma \cap \varphi \rangle \\ &= \langle \varphi \cup \delta\psi, \sigma \rangle \\ &= (-1)^k(\langle \delta(\varphi \cup \psi), \sigma \rangle - \langle \delta\varphi \cup \psi, \sigma \rangle) \\ &= (-1)^k(\langle \varphi \cup \psi, \partial\sigma \rangle - \langle \delta\varphi \cup \psi, \sigma \rangle) \\ &= (-1)^k(\langle \psi, \partial\sigma \cap \varphi \rangle - \langle \psi, \sigma \cap \delta\varphi \rangle). \end{aligned}$$

Since the identity holds for any cochain ψ, we obtain that

$$\partial(\sigma \cap \varphi) = (-1)^k(\partial\sigma \cap \varphi - \sigma \cap \delta\varphi) \tag{87}$$

Exercise 18.1 Prove formula (87) directly from the definition of the cap-product.

We see that the cap product of a cycle σ and a cocycle φ is a cycle. Furthermore, if $\partial\sigma = 0$ then $\partial(\sigma \cap \varphi) = \pm(\sigma \cap \delta\varphi)$. Thus the cap product of a cycle and coboundary is a boundary. Similarly if

$\delta\varphi = 0$, then $\partial(\sigma \cap \varphi) = \pm(\partial\sigma \cap \varphi)$, so we obtain that the cap product of a boundary and cocycle is a boundary. These facts imply that there is an induced cap product

$$H_{k+\ell}(X;R) \times H^k(X;R) \xrightarrow{\cap} H_\ell(X;R).$$

Using the same formulas one checks that the cap product has the relative form

$$H_{k+\ell}(X,A;R) \times H^k(X;R) \xrightarrow{\cap} H_\ell(X,A;R),$$

$$H_{k+\ell}(X,A;R) \times H^k(X,A;R) \xrightarrow{\cap} H_\ell(X;R),$$

$$H_{k+\ell}(X,A\cup B;R) \times H^k(X,A;R) \xrightarrow{\cap} H_\ell(X,B;R).$$

The last cap product is defined provided that A, B are open subsets of X or A, B are subcomplexes of X (if X is a CW-complex).

Exercise 18.2 Check that the above relative cap products are well-defined.

Claim 18.1. *Let* $f : X \longrightarrow Y$ *be a map, and*

$$f_* : H_*(X;R) \longrightarrow H_*(Y;R), \qquad f^* : H^*(Y;R) \longrightarrow H^*(X;R)$$

be the induced homomorphisms. Then

$$f_*(\sigma \cap f^*(\varphi)) = f_*(\sigma) \cap \varphi, \qquad \sigma \in H_*(X;R), \quad \varphi \in H^*(Y;R).$$

Exercise 18.3. Prove Claim 18.1.

Exercise 18.4. Let M^2_g be the oriented surface of the genus g. Let $[M^2_g] \in H_2(M^2_g;\mathbf{Z}) \cong \mathbf{Z}$ be a generator. Define the homomorphism $D : H^1(M^2_g;\mathbf{Z}) \longrightarrow H_1(M^2_g;\mathbf{Z})$ by the formula $D : \alpha \mapsto [M^2_g] \cap \alpha$. Compute the homomorphism D.

Exercise 18.5. Let N^2_g be the non-oriented surface of the genus g, i.e.

$$N^2_g = T^2 \# \cdots \# T^2 \# \mathbf{RP}^2.$$

Let $[N^2_g] \in H_2(N^2_g;\mathbf{Z}/2) \cong \mathbf{Z}/2$ be a generator. Define the homomorphism

$$D_2 : H^1(N^2_g;\mathbf{Z}/2) \longrightarrow H_1(N^2_g;\mathbf{Z}/2)$$

by the formula $D_2 : \alpha \mapsto [N^2_g] \cap \alpha$. Compute the homomorphism D_2.

Remark. The above homomorphism is the Poincarè duality isomorphism specified for 2-dimensional manifolds.

18.2. **Crash course on manifolds.** Here I will be very brief and give only necessary definitions. A manifold is a second countable Hausdorff space M so that each point $x \in M$ has an open neighborhood U homeomorphic to $\mathbf{R}^n$ or a half-space $\mathbf{R}^n_+ = \{(x_1, \ldots, x_n) \in \mathbf{R}^n \mid x_n \geq 0\}$. Then we say that $\dim M = n$, and those point of M which do not have an open neighborhood homemorphic to $\mathbf{R}^n$, form a boundary ∂M (which is also a closed manifold ∂M of dimension $(n-1)$). We have seen some examples of manifolds: $\mathbf{R}^n$, S^n, D^n (where $\partial D^n = S^{n-1}$), $\mathbf{RP}^n$, $\mathbf{CP}^n$, $\mathbf{HP}^n$, $GL_{\mathbf{R}}(n)$, $GL_{\mathbf{C}}(n)$, $SO(n)$, $U(n)$, all classical Lie groups, Grassmannian, Stiefel manifolds and so on. To work with manifolds, we should specify what do we mean by a *smooth* manifold.

Definition 18.1. *An n-dimensional smooth manifold is a second countable Hausdorff space M together with a collection of* charts, *i.e.* $\{U_\alpha\}$ *of neighborhoods and homeomorphisms $\varphi_\alpha \to \mathbf{R}^n$ (or $\varphi_\alpha \to \mathbf{R}^n_+$) such that each point $x \in M$ is in some chart U_α, and if $U_\alpha \cap U_{\alpha'} \neq \emptyset$, then the map $\varphi_{\alpha'} \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_{\alpha'}) \to \varphi_{\alpha'}(U_\alpha \cap U_{\alpha'})$ from the diagram*

$$\begin{tikzcd}
\varphi_\alpha(U_\alpha \cap U_{\alpha'}) \arrow[r, "\varphi_\alpha^{-1}"] \arrow[d, "open\ subset"'] & U_\alpha \cap U_{\alpha'} \arrow[r, "\varphi_{\alpha'}"] & \varphi_{\alpha'}(U_\alpha \cap U_{\alpha'}) \arrow[d, "open\ subset"'] \\
\mathbf{R}^n & & \mathbf{R}^n
\end{tikzcd}$$

is a diffeomorphism.

All examples mentioned above are smooth manifolds. The following fact is very important in the manifold theory.

Theorem 18.2. *Any smooth manifold M^n is diffeomorphic to a submanifold of $\mathbf{R}^{2n}$, i.e. any manifold M^n can be embedded to a finite-dimensional Euclidian space.*

Remark. I strongly recommend to read carefully few sections of Hatcher (Section 3.3-3.4) and Bredon (Sections II.1–II.4) to learn some basic facts and technique on smooth topology.

We recall that a subset $X \subset \mathbf{R}^k$ is *triangulated* (by q-simplices) if X is a union of simplices $X = \bigcup_i \Delta_i^q$ such that

- each simplex Δ_i^q is a nondegenerated simplex in $\mathbf{R}^k$;
- the intersection $\Delta_i^q \cap \Delta_j^q$ is either empty or consists of is a single joint face of the simplices Δ_i^q and Δ_j^q.

Theorem 18.2 implies the following result we need to prove the Poincaré duality.

Theorem 18.3. *Any compact smooth manifold M of dimension n is homeomorphic to a triangulated (by n-simlpices) subset of a finitely-dimensional Euclidian space.*

Remarks. (1) If $\dim M = n$, then the Euclidian space in Theorem 18.3 could be chosen to be $\mathbf{R}^{2n}$. Notice also that a triangulation of a manifold M induces a triangulation (by corresponding $(n-1)$-simplices) on its boundary ∂M.

(2) Theorem 18.3 holds also in the case when M is not compact. Then the triangulation should be infinite.

(3) We do not prove Theorems 18.2, 18.3; say, Theorem 18.2 is rather easy to prove, and its proof could be found in most classical textbooks on Algebraic Topology; Theorem 18.3 is deeper than it seems. First ad hock (and correct!) proof is due to Wittney (end of 30's). A transperent version of that proof is given by Munkres in his "Lectures on Differential Topology".

Exercise 18.6. Construct an embedding of the projective spaces $\mathbf{RP}^n$, $\mathbf{CP}^n$, $\mathbf{HP}^n$ into Euclidian space.

Exercise 18.7. Let $M^n \subset \mathbf{R}^k$ be a triangulated (by n-simplices) manifold, $M = \bigcup \Delta_i^n$, with possibly non-empty boundary ∂M. Consider any $(n-1)$-face Δ^{n-1} of a simplex Δ_i^n. Prove that if Δ^{n-1} does not belong to the induced triangulation of its boundary, then there exists a unique simplex Δ_j^n, $j \neq i$, which also has the simplex Δ^{n-1} as a face.

Consider the case when a manifold $M^n \subset \mathbf{R}^k$ is a compact closed (i.e. $\partial M^n = \emptyset$) manifold. Then we can assume that the triangulation $M^n = \bigcup_i \Delta_i^n$ is finite. In particular, the triangulation $M^n = \bigcup_i \Delta_i^n$ gives a CW-decomposition of the manifold M where n-cells e_i^n are identified with the enterior of the simplex Δ_i^n, and $\bar{e}_i^n = \Delta_i^n$. A triangulated manifold M^n is said to be *orientable* (over a ring R) if there is a choice of orientations on each simplex Δ_i^n, such that the chain

$$\sum_i e_i^n \qquad \text{(where the summation is taken over all indices } i\text{)} \tag{88}$$

is a cycle in the chain complex $\mathcal{E}_*(M)$. Once we fix an orientations, we call the manifold M *oriented*.

Remark. If $R = \mathbf{Z}/2$, then any closed compact manifold has "orientation", and its unique. In that case one can see that the chain (88) is always a cycle.

We state the following result which summarizes our observations.

Theorem 18.4. *Let M^n be a smooth compact manifold. Then*

$$H_n(M;\mathbf{Z}) = \begin{cases} \mathbf{Z}, & \text{if } M \text{ is closed and oriented,} \\ 0, & \text{else} \end{cases}$$

$$H_n(M;\mathbf{Z}/2) = \begin{cases} \mathbf{Z}/2, & \text{if } M \text{ is closed,} \\ 0, & \text{else} \end{cases}$$

Remarks. **(1)** It is easy to see that if a manifold M^n is *oriented over* $\mathbf{Z}$, then it is oriented over any ring R. The converse is not true. It is also easy to see that any manifold M^n is oriented over $\mathbf{Z}/2$ (Prove it!). A cohomology class defined by the cycle (88) is denoted by $[M^n] \in H_n(M^n; R)$ and is called the *fundamental class* of M^n

(2) An example of a non-oriented manifold is $\mathbf{RP}^{2n}$: $H_{2n}(\mathbf{RP}^{2n}; \mathbf{Z}) = 0$; however, we have the fundamental class $[\mathbf{RP}^{2n}] \in H_{2n}(\mathbf{RP}^{2n}; \mathbf{Z}/2)$.

(3) We say that a homology class $\alpha \in H_k(M^n; R)$ is *represented by a submanifold* $N^k \subset M^n$ if $i_*([N^k]) = \alpha$, where $i : N^k \to M^n$ is the inclusion map. For example, a generator $\alpha_k \in H_k(\mathbf{RP}^n)$ is represented by $\mathbf{RP}^k \subset \mathbf{RP}^n$; as well as a generator $\beta_j \in H_{2j}(\mathbf{CP}^n; \mathbf{Z})$ is represented by $\mathbf{CP}^j \subset \mathbf{CP}^n$. It turns out that not every homology class of a smooth manifold could be represented by a submanifold: this was discovered by Rene Thom in 1954.

18.3. **Poincaré isomorphism.** Let M^n be a closed manifold. We define a homomorphism

$$D : H^q(M; \mathbf{Z}) \longrightarrow H_{n-q}(M; \mathbf{Z}) \quad \alpha \mapsto [M] \cap \alpha \quad \text{if } M \text{ is oriented}$$

$$D : H^q(M; \mathbf{Z}/2) \longrightarrow H_{n-q}(M; \mathbf{Z}/2) \quad \alpha \mapsto [M] \cap \alpha \quad \text{if } M \text{ is not oriented}$$

Theorem 18.5. (Poincaré isomorphism Theorem) *Let M^n be a closed compact manifold. Then the homorphism*

$$D : H^q(M; \mathbf{Z}/2) \longrightarrow H_{n-q}(M; \mathbf{Z}/2)$$

is an isomorphism for each q. If, in addition, M is oriented manifold, then the homomorphism

$$D : H^q(M; \mathbf{Z}) \longrightarrow H_{n-q}(M; \mathbf{Z})$$

is an isomorphism for each q.

Remark. There are several different ways to prove Theorem 18.5. In particular, nice proof is given in the book by Hatcher (Sections 3.3-3.4). Here we will present a geometric proof which is rather close to an original idea due to Poincaré.

Construction. Consider a triangulation $\mathcal{T} = \{\Delta_i^n\}$ of an open disk $B^n(r) \subset \mathbf{R}^n$ of radius r. Here it means that $B^n(r) \subset \bigcup_i \Delta_i^n$, and the intersection $\Delta_i^n \cap \Delta_j^n$ is either empty or consists of is a single joint face of the simplices Δ_i^n and Δ_j^n. We assume that the triangulation is fine enough, say, if $\Delta_i^n \cap B^n(r/2) \neq \emptyset$, then $\Delta_i^n \subset B^n(r)$. In other words, this triangulation is a good local model of a neighborhood near a point on a manifold equipped with a triangulation.

Let $\Delta^q \subset \Delta_i^n \in \mathcal{T}$ be a subsimplex with barycenter x_0 at the center of the ball $B^n(r)$. Now let $\beta\mathcal{T}$ be the barycentric subdivision of our triangultion. We define a *barycentric star* $S(\Delta^q)$ as the following union (see Fig. 18.1):

$$S(\Delta^q) := \bigcup_{\substack{\Delta \subset \Delta^n \in \beta\mathcal{T} \\ \Delta \cap \Delta^q = \{x_0\}}} \Delta .$$

Notice that all subsimplices Δ with those properties have dimension $(n-q)$, moreover, $S(\Delta^q) \subset B^n(r)$ is homeomorphic to a disk D^{n-q} decomposed into $(n-q)$-simplices, see Fig. 18.2.

Proof of Theorem 18.5. Let $\mathcal{T}$ be a triangulation of a closed oriented manifold M^n. In particular, the triangulation $\mathcal{T}$ determines a CW-decomposition of M^n, where all q-cells are given by q-simplices $\{\Delta_i^q\}$ of $\mathcal{T}$. We notice that the stars $S(\Delta^q)$ determine an alternative "dual" CW-structure of M^n. Let $\mathcal{E}_*(M^n)$ be a chain complex determined by the first CW-decomposition, and $\bar{\mathcal{E}}_*(M^n)$ the chain complex determined by the dual CW-structure.

In particular, generators of the chain group $\bar{\mathcal{E}}_{n-q}(M^n)$ are the stars $S(\Delta_i^q)$. Also, let $\mathcal{E}^*(M^n) = \mathrm{Hom}(\mathcal{E}_*(M^n), \mathbf{Z})$ be the corresponding cochain complex. We define a homomorphism $\bar{D} : \mathcal{E}^q(M^n) \to \bar{\mathcal{E}}_{n-q}(M^n)$ as follows. For a cochain $\varphi \in \mathcal{E}^q(M^n)$, $\varphi : \Delta_i^q \mapsto \lambda_i$, we define

$$\bar{D}(\varphi) := \sum_i \lambda_i S(\Delta_i^q) \in \bar{\mathcal{E}}_{n-q}(M^n).$$

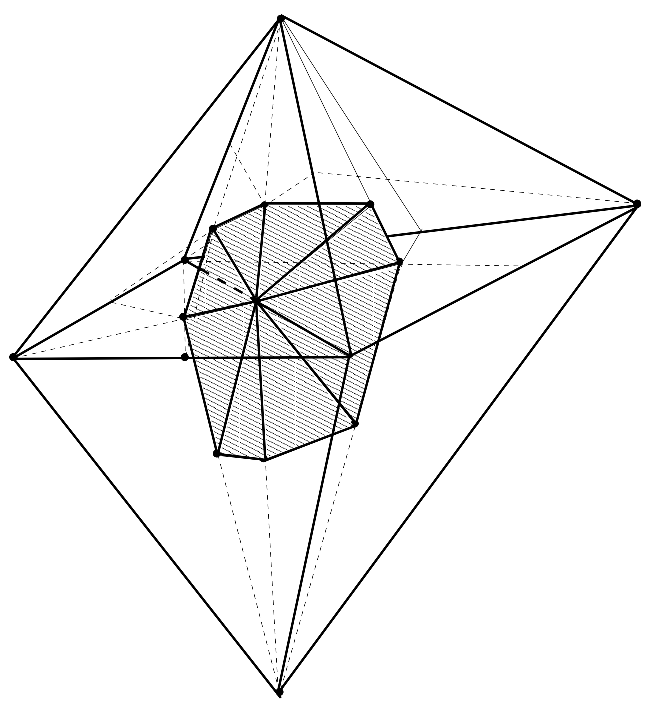


Fig. 18.1. A barycentric star in $\mathbf{R}^n$.

It is easy to check that $\bar{D}\delta\varphi = \pm\partial\bar{D}\varphi$ (we do not specify the sign here). Thus we have the following commutative diagram:

$$
\begin{tikzcd}
\bar{\mathcal{E}}_n(M^n) \arrow[r, "\partial_n"] & \bar{\mathcal{E}}_{n-1}(M^n) \arrow[r, "\partial_{n-1}"] & \cdots \arrow[r, "\partial_2"] & \bar{\mathcal{E}}_1(M^n) \arrow[r, "\partial_1"] & \bar{\mathcal{E}}_0(M^n) \\
\mathcal{E}^0(M^n) \arrow[u, "\bar{D}"] \arrow[r, "\delta_1"] & \mathcal{E}^1(M^n) \arrow[u, "\bar{D}"] \arrow[r, "\delta_2"] & \cdots \arrow[r, "\delta_{n-2}"] & \mathcal{E}^{n-1}(M^n) \arrow[u, "\bar{D}"] \arrow[r, "\delta_{n-1}"] & \mathcal{E}^n(M^n) \arrow[u, "\bar{D}"]
\end{tikzcd}
$$

Thus we have that $\bar{D}$ is an isomorphism for each q and, in fact, the above complexes $\mathcal{E}^*(M^n)$ and $\bar{\mathcal{E}}_*(M^n)$ are identical via the chain map $\bar{D}$. Hence we have that $H^q(M^n;\mathbf{Z}) \cong H_{n-q}(M^n;\mathbf{Z})$.

Exercise 18.9. Show that the duality isomomorphism $\bar{D}$ induces the map as

$$D : H^q(M^n;\mathbf{Z}) \xrightarrow{[M^n]\cap} H_{n-q}(M^n;\mathbf{Z}).$$

Hint: replace the cochain complex $\mathcal{E}^*(M^n)$ by the the cochain complex given by the barycentic subdivision $\beta\mathcal{T}$.

This conclude our proof of Theorem 18.5. $\square$

Corollary 18.6. *Let M^n be a closed compact manifold of odd dimension n. Then $\chi(M^n) = 0$.*

Exercise 18.10. Prove Corollary 18.6. Notice that M^n is not necessarily an oriented manifold.

18.4. **Some computations.** Recall that for the cap-product

$$H_{k+\ell}(X;R) \times H^k(X;R) \xrightarrow{\cap} H_\ell(X;R).$$

we have the identity $\langle\psi, \sigma\cap\varphi\rangle = \langle\varphi\cup\psi, \sigma\rangle$. For a closed oriented manifold M^n we consider the pairing

$$H^q(M^n;R) \times H^{n-q}(M^n;R) \to R, \quad (\varphi,\psi) := \langle\varphi\cup\psi, [M^n]\rangle. \tag{89}$$

A bilinear pairing $\mu : A\times B \to R$, (where A and B are R-modules) is *nonsingular* if the maps

$$A \to \mathrm{Hom}_R(B,R), \quad a \mapsto \mu(a,\cdot) \in \mathrm{Hom}_R(B,R), \quad \text{and}$$

$$B \to \mathrm{Hom}_R(A,R), \quad b \mapsto \mu(\cdot,b) \in \mathrm{Hom}_R(A,R)$$

are both isomorphisms.

Lemma 18.7. *Let M^n be an oriented manifold (over R). Then the pairing (89) is nonsingular provided that R is a field. Furthermore, if $R = \mathbf{Z}$, then the induced pairing*

$$(H^q(M^n;\mathbf{Z})/\mathrm{Tor}) \times \big(H^{n-q}(M^n;\mathbf{Z})/\mathrm{Tor}\big) \to \mathbf{Z}, \quad (\varphi,\psi) := \langle\varphi\cup\psi, [M^n]\rangle. \tag{90}$$

is nonsingular.

Exercise 18.11. Prove Lemma 18.7. Hint: Make use of the universal coefficient Theorem and Poincaré duality.

Corollary 18.8. *Let M^n be an oriented manifold. Then for each element of infinite order $\alpha \in H^q(M^n;\mathbf{Z})$, there exists an element $\beta \in H^{n-q}(M^n;\mathbf{Z})$ of infinite order such that $\langle \alpha\cup\beta, [M^n]\rangle = 1$, i.e. the element $\alpha\cup\beta$ is a generator of the group $H^n(M^n;\mathbf{Z})$.*

Exercise 18.12. Prove Corollary 18.8.

Theorem 18.9. *Let R be any ring. Then*

- **(1)** $H^*(\mathbf{RP}^n;\mathbf{Z}/2) \cong \mathbf{Z}/2[x]/x^{n+1}$, *where $x \in H^1(\mathbf{RP}^n;\mathbf{Z}/2)$ is a generator;*
- **(2)** $H^*(\mathbf{CP}^n;R) \cong R[y]/y^{n+1}$, *where $y \in H^2(\mathbf{CP}^n;R)$ is a generator;*
- **(3)** $H^*(\mathbf{HP}^n;R) \cong R[z]/z^{n+1}$, *where $z \in H^4(\mathbf{HP}^n;R)$ is a generator.*

Proof. We prove (2). Induction on n. Clearly $H^*(\mathbf{CP}^1;R) \cong R[y]/y^2$. Induction step. The inclusion $i : \mathbf{CP}^{n-1} \to \mathbf{CP}^n$ induces an isomorphism

$$i^* : H^q(\mathbf{CP}^n;\mathbf{Z}) \to H^q(\mathbf{CP}^{n-1};\mathbf{Z})$$

for $q \leq n-1$. In particular, the groups $H^{2j}(\mathbf{CP}^{n-1};\mathbf{Z})$ are generated by y^j for $j \leq n-1$.

By Corollary 18.7, there exists an integer m such that the element $y^{n-1}\cup my = my^n$ generates the group $H^{2n}(\mathbf{CP}^{n-1};\mathbf{Z}) \cong \mathbf{Z}$. Thus we obtain that $m = \pm 1$, and $H^*(\mathbf{CP}^n;R) \cong R[y]/y^{n+1}$. $\square$

Corollary 18.10. *Let R be any ring. Then*

- **(1)** $H^*(\mathbf{RP}^\infty;\mathbf{Z}/2) \cong \mathbf{Z}/2[x]$, *where $x \in H^1(\mathbf{RP}^n;\mathbf{Z}/2)$ is a generator;*
- **(2)** $H^*(\mathbf{CP}^\infty;R) \cong R[y]$, *where $y \in H^2(\mathbf{CP}^n;R)$ is a generator;*
- **(3)** $H^*(\mathbf{HP}^\infty;R) \cong R[z]$, *where $z \in H^4(\mathbf{HP}^n;R)$ is a generator.*

19. Hopf Invariant

19.1. **Whitehead product.** Here we remind the Whitehead product: for any elements $\alpha \in \pi_m(X)$, $\beta \in \pi_n(X)$ we construct the element $[\alpha, \beta] \in \pi_{m+n-1}(X)$.

First we consider the product $S^m \times S^n$. The cell structure of $S^m \times S^n$ is obvious: we have cells σ^0, σ^m, σ^n, σ^{m+n}. A union of the cells σ^0, σ^m, σ^n is the space $S^m \vee S^n$. Let $w : S^{m+n-1} \longrightarrow S^m \vee S^n$ be an attaching map of the cell σ^{m+n}, i.e.

$$S^m \times S^n = (S^m \vee S^n) \cup_w D^{m+n-1}.$$

Now let $f : S^m \longrightarrow X$, $g : S^n \longrightarrow X$ be representatives of elements $\alpha \in \pi_m(X)$, $\beta \in \pi_n(X)$. The composition

$$S^{m+n-1} \xrightarrow{w} S^m \vee S^n \xrightarrow{f \vee g} X$$

gives an element of $\pi_{m+n-1}(X)$. By definition,

$$[\alpha, \beta] = \{\text{the homotopy class of } (f \vee g) \circ w\}.$$

The construction above does depend on a choice of the attaching map w.

Let ι_{2n} be a generator of the group $\pi_{2n}(S^{2n})$. We have proved "geometrically" the following result.

Theorem 19.1. *The group $\pi_{4n-1}(S^{2n})$ is infinite for any $n \geq 1$; the element $[\iota_{2n}, \iota_{2n}] \in \pi_{4n-1}(S^{2n})$ has infinite order.*

Next, we introduce an invariant, known as Hopf invariant to give another proof of Theorem 19.1.

19.2. **Hopf invariant.** Before proving the theorem we define the **Hopf invariant.** Let $\varphi \in \pi_{4n-1}(S^{2n})$, and let $f : S^{4n-1} \longrightarrow S^{2n}$ be a representative of φ. Let $X_\varphi = S^{2n} \cup_f D^{4n}$. Compute the cohomology groups of X_φ:

$$H^q(X_\varphi; \mathbf{Z}) = \begin{cases} \mathbf{Z}, & q = 0, 2n, 4n, \\ 0, & \text{otherwise.} \end{cases}$$

Let $a \in H^{2n}(X_\varphi; \mathbf{Z})$, $b \in H^{4n}(X_\varphi; \mathbf{Z})$ be generators. Since $a^2 = a \cup a \in H^{4n}(X_\varphi; \mathbf{Z})$, then $a^2 = hb$, where $h \in \mathbf{Z}$. The number $h(\varphi) = h$ is the **Hopf invariant** of the element $\varphi \in \pi_{4n-1}(S^{2n})$.

Examples. Let $h : S^3 \to \mathbf{CP}^1 = S^2$ and $H : S^7 \to \mathbf{HP}^1 = S^4$ be the Hopf maps. Notice that $X_h = \mathbf{CP}^2$ and $X_H = \mathbf{HP}^2$. As we have computed,

$$H^*(\mathbf{CP}^2; \mathbf{Z}) = \mathbf{Z}[y]/y^3, \quad y \in H^2(\mathbf{CP}^2; \mathbf{Z}),$$

$$H^*(\mathbf{HP}^2; \mathbf{Z}) = \mathbf{Z}[z]/z^3, \quad z \in H^4(\mathbf{HP}^4; \mathbf{Z}).$$

Thus $h(h) = 1$ and $h(H) = 1$. There is one more case when this is true. Let **Ca** be the Calley algebra; this is the algebra defined on $\mathbf{R}^8$. Furthermore, there exists a projective line $\mathbf{CaP}^1 \cong S^8$ and a projective plane $\mathbf{CaP}^2$ with

$$H^*(\mathbf{CaP}^2; \mathbf{Z}) = \mathbf{Z}[\sigma]/\sigma^3, \qquad \sigma \in H^8(\mathbf{CaP}^2; \mathbf{Z}).$$

The attaching map $\mathcal{H} : S^{15} \to S^8$ for the cell e^{16} also has $h(\mathcal{H}) = 1$.

Lemma 19.2. $h(\varphi_1) + h(\varphi_2) = h(\varphi_1 + \varphi_2)$.

Lemma 19.3. *The Hopf invariant is not trivial, in particular,*

$$h([\iota_{2n}, \iota_{2n}]) = 2.$$

Proof of Lemma 19.2. For given elements $\varphi_1, \varphi_2 \in \pi_{4n-1}(S^{2n})$ we choose representatives $f_1 : S^{4n-1} \longrightarrow S^{2n}$, $f_2 : S^{4n-1} \longrightarrow S^{2n}$ and consider the spaces X_{φ_1}, X_{φ_2}, $X_{\varphi_1+\varphi_2}$. Also we construct the following space:

$$Y_{\varphi_1,\varphi_2} = (S^{2n} \cup_{f_1} D^{4n}) \cup_{f_2} D^{4n} = S^{2n} \cup_{f_1 \vee f_2} (D^{4n} \vee D^{4n}),$$

where $[f_1] = \varphi_1$, $[f_2] = \varphi_2$. We compute the cohomology groups of Y_{φ_1,φ_2}:

$$H^q(Y_{\varphi_1,\varphi_2}; \mathbf{Z}) = \begin{cases} \mathbf{Z}, & q = 0, 2n, \\ \mathbf{Z} \oplus \mathbf{Z} & q = 4n, \\ 0, & \text{otherwise.} \end{cases}$$

Let $a' \in H^{2n}(Y_{\varphi_1,\varphi_2}; \mathbf{Z})$, $b'_1, b'_2 \in H^{4n}(Y_{\varphi_1,\varphi_2}; \mathbf{Z})$ be generators. We have natural maps:

$$i_1 : X_{\varphi_1} \longrightarrow Y_{\varphi_1,\varphi_2},$$

$$i_2 : X_{\varphi_2} \longrightarrow Y_{\varphi_1,\varphi_2},$$

where i_1, i_2 are cell-inclusion maps:

$$S^{2n} \cup_{f_1} D^{4n} \longrightarrow S^{2n} \cup_{f_1 \vee f_2} (D^{4n} \vee D^{4n}),$$

$$S^{2n} \cup_{f_2} D^{4n} \longrightarrow S^{2n} \cup_{f_1 \vee f_2} (D^{4n} \vee D^{4n}).$$

We choose generators

$$a_1 \in H^{2n}(X_{\varphi_1}; \mathbf{Z}), \qquad b_1 \in H^{4n}(X_{\varphi_1}; \mathbf{Z}),$$

$$a_2 \in H^{2n}(X_{\varphi_2}; \mathbf{Z}), \qquad b_2 \in H^{4n}(X_{\varphi_2}; \mathbf{Z})$$

in such way that

$$i_1^*(a') = a_1, \quad i_1^*(b'_1) = b_1, \quad i_1^*(b'_2) = 0,$$

$$i_2^*(a') = a_2, \quad i_2^*(b'_2) = b_2, \quad i_1^*(b'_1) = 0.$$

Now we construct a map

$$j : X_{\varphi_1+\varphi_2} \longrightarrow Y_{\varphi_1,\varphi_2}$$

as follows. Recall that

$$X_{\varphi_1+\varphi_2} = S^{2n} \cup_f D^{4n},$$

where f is the composition:

$$S^{4n-1} \longrightarrow S^{4n-1} \vee S^{4n-1} \xrightarrow{f_1 \vee f_2} S^{2n}.$$

Now we send the sphere $S^{2n} \subset X_{\varphi_1+\varphi_2}$, $S^{2n} \xrightarrow{Id} S^{2n} \subset Y_{\varphi_1,\varphi_2}$ identically, and $j : D^{4n} \longrightarrow D^{4n} \vee D^{4n}$, where we contract the equator disk D^{4n-1}:

$$D^{4n} \longrightarrow D^{4n} \vee D^{4n}$$

A restriction of j on the sphere S^{4n-1} gives the map

$$S^{4n-1} \longrightarrow S^{4n-1} \vee S^{4n-1}$$

Note that the diagram of maps

$$\begin{tikzcd} S^{4n-1} \arrow[rr, "j|_{S^{4n-1}}"] \arrow[dr, "f_1+f_2"'] & & S^{4n-1} \vee S^{4n-1} \arrow[dl, "f_1\vee f_2"] \\ & S^{2n} & \end{tikzcd}$$

commutes by definition of the addition operation in homotopy groups. In particular, the following diagram commutes as well:

$$\begin{tikzcd} D^{4n} \arrow[rr, "j"] & & D^{4n} \vee D^{4n} \\ S^{4n-1} \arrow[u] \arrow[rr, "j|_{S^{4n-1}}"] \arrow[dr, "f_1+f_2"'] & & S^{4n-1} \vee S^{4n-1} \arrow[u] \arrow[dl, "f_1\vee f_2"] \\ & S^{2n} & \end{tikzcd}$$

The construction above defines the map

$$j : X_{\varphi_1+\varphi_2} \longrightarrow Y_{\varphi_1,\varphi_2}.$$

Now we compute the homomorphisms i_1^*, i_2^* and j^* in cohomology:

$$i_1^* : H^q(Y_{\varphi_1,\varphi_2}) \longrightarrow H^q(X_{\varphi_1}),$$

$$i_2^* : H^q(Y_{\varphi_1,\varphi_2}) \longrightarrow H^q(X_{\varphi_2}).$$

We have that

$$i_1^*(a') = a_1, \quad i_1^*(b_1') = b_1, \quad i_1^*(b_2') = 0,$$

$$i_2^*(a') = a_2, \quad i_2^*(b_1') = 0, \quad i_1^*(b_2') = b_2,$$

The homomorphism

$$j^* : H^q(Y_{\varphi_1,\varphi_2}) \longrightarrow H^q(X_{\varphi_1+\varphi_2})$$

sends

$$j^*(a') = a, \quad j^*(b_1') = b, \quad j^*(b_2') = b.$$

The element

$$(a')^2 \in H^{4n}(Y_{\varphi_1,\varphi_2})$$

is equal to $(a')^2 = \mu_1 b_1' + \mu_2 b_2'$. Since $i_1^*((a')^2) = a_1^2 = h(\varphi_1) b_1$, and $i_1^*(b_1') = b_1$, then $\mu_1 = h(\varphi_1)$. The same reason gives that $\mu_2 = h(\varphi_2)$. Note that $a^2 = h(\varphi_1 + \varphi_2) b$, and since $j^*(a') = a$, $j^*(b_1') = b$, $j^*(b_2') = b$, we conclude that $h(\varphi_1 + \varphi_2) = h(\varphi_1) + h(\varphi_2)$. □

Before we prove Lemma 19.3, we compute the cohomology (together with a product structure) $H^q(S^{2n} \times S^{2n})$. First compute the cohomology groups:

$$H^q(S^{2n} \times S^{2n}; \mathbf{Z}) = \begin{cases} \mathbf{Z}, & q = 0, 4n, \\ \mathbf{Z} \oplus \mathbf{Z}, & q = 2n \\ 0, & \text{otherwise.} \end{cases}$$

Let $c_1, c_2 \in H^{2n}(S^{2n} \times S^{2n})$ be such generators that the homomorphisms

$$p_1^* : H^{2n}(S_1^{2n}) \longrightarrow H^{2n}(S_1^{2n} \times S_2^{2n}),$$

$$p_2^* : H^{2n}(S_2^{2n}) \longrightarrow H^{2n}(S_1^{2n} \times S_2^{2n}),$$

induced by the projections

$$S_1^{2n} \times S_2^{2n} \xrightarrow{p_1} S_1^{2n}, \quad S_1^{2n} \times S_2^{2n} \xrightarrow{p_2} S_2^{2n}$$

send the generators c_1 and c_2 to the generators of the groups $H^{2n}(S_1^{2n})$, $H^{2n}(S_2^{2n})$. Let $d \in H^{4n}(S^{2n} \times S^{2n})$ be a generator. It follows from Corollary 17.6 that

$$c_1 c_2 = d.$$

We also note that $c_1^2 = 0$ and $c_2^2 = 0$ since by naturality $p_1^*(c_1)^2 = 0$ and $p_2^*(c_2)^2 = 0$. So we have that the ring $H^*(S_1^{2n} \times S_2^{2n})$ is generated over $\mathbf{Z}$ by the elements 1, c_1, c_2 with the relations $c_1^2 = 0$, $c_2^2 = 0$. In particular, we have:

$$(c_1 + c_2)^2 = c_1^2 + 2c_1c_2 + c_2^2 = 2d.$$

Proof of Lemma 19.3. We consider the factor space

$$X = S^{2n} \times S^{2n} / \sim,$$

where we identify the points $(x, x_0) = (x_0, x)$, where x_0 is the base point of S^{2n}.

Claim 19.1. *The space* $X = S^{2n} \times S^{2n}/\sim$ *is homeomorphic to the space* $S^{2n} \cup_f D^{4n}$*, where* f *is the map defining the Whitehead product* $[\iota_{2n}, \iota_{2n}]$*.*

Proof of Claim 19.1. Recall that $S^{2n} \times S^{2n} = (S^{2n} \vee S^{2n}) \cup_w D^{4n}$, where w is the map we described above. The generator ι_{2n} is represented by the identical map $S^{2n} \longrightarrow S^{2n}$. The composition

$$S^{4n-1} \xrightarrow{w} S^{2n} \vee S^{2n} \xrightarrow{Id \vee Id} S^{2n}$$

represents the element $[\iota_{2n}, \iota_{2n}]$. It exatly means that the identification $(S^{2n}, x_0) = (x_0, S^{2n})$ we just did in the space $S^{2n} \times S^{2n}$ is the same as to attach D^{4n} with the attaching map $(Id \vee Id) \circ w$.

Compute the cohomology of X:

$$H^q(X; \mathbf{Z}) = \begin{cases} \mathbf{Z}, & q = 0, 2n, 4n, \\ 0, & \text{otherwise.} \end{cases}$$

We note that the projection $S^{2n} \times S^{2n} \longrightarrow X$ sends the generator $c \in H^{2n}(X)$ to $c_1 + c_2$. Besides the generator d maps to a generator of $H^{4n}(X; \mathbf{Z})$ (we denote it also by d). So we have: $c^2 = 2d$, or $h([\iota_{2n}, \iota_{2n}]) = 2$. $\square$

This concludes our proof of Theorem 19.1. $\square$

Remarks. **(1)** In fact, it is true that $\pi_{4n-1}(S^{2n}) = \mathbf{Z} \oplus \{\text{finite abelian group}\}$, in particular, as we know, $\pi_3(S^2) = \mathbf{Z}$, $\pi_7(S^4) = \mathbf{Z} \oplus \mathbf{Z}/12$, $\pi_{11}(S^6) = \mathbf{Z}$, $\pi_{15}(S^8) = \mathbf{Z} \oplus \mathbf{Z}/120$. Moreover, all homotopy groups of the spheres are finite with the exception of $\pi_n(S^n) = \mathbf{Z}$ and $\pi_{4n-1}(S^{2n}) = \mathbf{Z} \oplus \{\text{finite abelian group}\}$.

(2) We proved that the image of the Hopf invariant $h : \pi_{4n-1}(S^{2n}) \longrightarrow \mathbf{Z}$ either all group $\mathbf{Z}$ or $2\mathbf{Z}$.

Problem: Does there exists an element in $\pi_{4n-1}(S^{2n})$ with the Hopf invariant 1?

This problem has several remarkable reformulations. One of them is the following: for which n does the vector space $\mathbf{R}^{n+1}$ admit a structure of real division algebra with a unit. Frank Adams (1960) proved that there are elements with the Hopf invariant one **only** in the groups $\pi_3(S^2)$ $\pi_6(S^4)$, $\pi_{15}(S^8)$. Thus there are only the following real division algebra with a unit: $\mathbf{R}^2 \cong \mathbf{C}$, $\mathbf{R}^4 \cong \mathbf{H}$, and $\mathbf{R}^8 \cong \mathbf{Ca}$.